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METRIC CHARACTERIZATIONS OF SOME SUBSETS OF THE REAL LINE

I. Banakh, T. Banakh, M. Kolinko, A. Ravsky. *Metric characterizations of some subsets of the real line*, Mat. Stud. **59** (2023), 205–214.

A metric space (X, d) is called a *subline* if every 3-element subset T of X can be written as $T = \{x, y, z\}$ for some points x, y, z such that $\mathsf{d}(x, z) = \mathsf{d}(x, y) + \mathsf{d}(y, z)$. By a classical result of Menger, every subline of cardinality $\neq 4$ is isometric to a subspace of the real line. A subline (X, d) is called an *n-subline* for a natural number n if for every $c \in X$ and positive real number $r \in \mathsf{d}[X^2]$, the sphere $\mathsf{S}(c; r) := \{x \in X : \mathsf{d}(x, c) = r\}$ contains at least n points. We prove that every 2-subline is isometric to some additive subgroup of the real line. Moreover, for every subgroup $G \subseteq \mathbb{R}$, a metric space (X, d) is isometric to G if and only if X is a 2-subline with $\mathsf{d}[X^2] = G_+ := G \cap [0, \infty)$. A metric space (X, d) is called a *ray* if X is a 1-subline and X contains a point $o \in X$ such that for every $r \in \mathsf{d}[X^2]$ the sphere $\mathsf{S}(o; r)$ is a singleton. We prove that for a subgroup $G \subseteq \mathbb{Q}$, a metric space (X, d) is isometric to the ray G_+ if and only if X is a ray with $\mathsf{d}[X^2] = G_+$. A metric space X is isometric to the ray $\mathbb{R}_+$ if and only if X is a complete ray such that $\mathbb{Q}_+ \subseteq \mathsf{d}[X^2]$. On the other hand, the real line contains a dense ray $X \subseteq \mathbb{R}$ such that $\mathsf{d}[X^2] = \mathbb{R}_+$.

1. Introduction and main results. In this paper we discuss characterizations of metric spaces which are isometric to some important subspaces of the real line, in particular, to the spaces $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ of natural, integer, rational, real numbers, respectively. The space of real numbers $\mathbb{R}$ and its subspaces $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are endowed with the standard Euclidean metric $\mathsf{d}(x, y) = |x - y|$. For a subset $X \subseteq \mathbb{R}$, let $X_+ := \{x \in X : x \geq 0\}$.

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are subgroups of the real line, and $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ are submonoids of $\mathbb{R}$.

A set $X \subseteq \mathbb{R}$ is called

- a *submonoid* of $\mathbb{R}$ if $0 \in X$ and $x + y \in X$ for all $x, y \in X$;
- a *subgroup* of $\mathbb{R}$ if X is a submonoid of $\mathbb{R}$ such that $-x \in X$ for every $x \in X$.

For a metric space X, we denote by d_X (or just by d if X is clear from the context) the metric of the space X. For two points x, y of a metric space X, the real number $\mathsf{d}_X(x, y)$ will be denoted by xy.

Two metric spaces X and Y are *isometric* if there exists a bijective function $f : X \to Y$ such that $\mathsf{d}_Y(f(x), f(y)) = \mathsf{d}_X(x, y)$ for all $x, y \in X$. A metric space X is defined to *embed* into a metric space Y if X is isometric to some subspace of Y.

Observe that the space $\mathbb{N} := \mathbb{Z}_+ \setminus \{0\}$ is isometric to $\mathbb{Z}_+$.

Definition 1. A metric space X is called a *subline* if any 3-element subset $T \subseteq X$ embeds into the real line. This happens if and only if any points $x, y, z \in X$ satisfy the following property called the *Triangle Equality*: $yz = yx + xz \ \vee \ xz = xy + yz \ \vee \ xy = xz + zy$.

2020 *Mathematics Subject Classification:* 51F99, 54E35, 54E40.
Keywords: metric space; triangle equality; subline; ray.
doi:10.30970/ms.59.2.205-214

According to an old result of Menger [7] (see also [4]), a subline X embeds into a real line if and only if it is not an ℓ_1-rectangle.

Definition 2. A metric space (X, d) is called an *ℓ_1-rectangle* if $X = \{a, b, c, d\}$ for some pairwise distinct points a, b, c, d such that $ab = cd$, $bc = ad$ and $ac = ab + bc = bd$.

Example 1. Let $\mathbb{R}^2$ be the real plane endowed with the ℓ_1-metric

$$\mathsf{d}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad \mathsf{d}\colon \big((x, y), (u, v)\big) \mapsto |x - u| + |y - v|.$$

For any positive real numbers a, b, the subset

$$\square_a^b \coloneqq \{(a, b), (a, -b), (-a, b), (-a, -b)\}$$

of $\mathbb{R}^2$ is an ℓ_1-rectangle. Moreover, every ℓ_1-rectangle is isometric to the ℓ_1-rectangle $\square_a^b$ for unique positive real numbers $a \le b$.

The following metric characterization of subspaces of the real line was surely known to Karl Menger [7] and was also mentioned (without proof) in [4].

Theorem 1. *A metric space X embeds into the real line if and only if X is a subline and X is not an ℓ_1-rectangle.*

Theorem 1 has the following corollary.

Corollary 1. *A metric space X of cardinality $|X| \ne 4$ embeds into the real line if and only if it is a subline.*

Corollary 1 is a partial case of the following characterization that was proved by Karl Menger [7] in general terms of congruence relations and reproved by John Bowers and Philip Bowers [4] for metric spaces.

Theorem 2. *For every natural number n, a metric space X of cardinality $|X| \ne n + 3$ embeds into the Euclidean space $\mathbb{R}^n$ if and only if every subspace $A \subseteq X$ of cardinality $|A| \le n + 2$ embeds into $\mathbb{R}^n$.*

The paper [4] contains a decription of metric spaces of cardinality $n + 3$ that do not embed into $\mathbb{R}^n$ but whose all proper subsets do embed into $\mathbb{R}^n$. For $n = 1$ such metric spaces are exactly ℓ_1-rectangles.

Theorem 1 will be applied in the metric characterizations of the spaces $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$. Those characterizations involve the following definition.

Definition 3. Let κ be a cardinal number. A metric space X is called

- *κ-spherical* if for every $r \in \mathsf{d}[X^2] \setminus \{0\}$ and $c \in X$ the sphere $\mathsf{S}(c; r) \coloneqq \{x \in X : xc = r\}$ contains at least κ points;
- a *κ-subline* if X is a κ-spherical subline.

For $\kappa > 2$, the definition of a κ-subline is vacuous: indeed, assuming that some sphere $\mathsf{S}(c; r)$ in a subline contains three pairwise distinct points x, y, z, we can apply the Triangle Equality and conclude that $xy = xc + cy = 2r = xz = yz$, witnessing that the Triangle Equality fails for the points x, y, z.

Therefore, for every metric space we have the implications

$$\text{2-subline} \Rightarrow \text{1-subline} \Rightarrow \text{0-subline} \Leftrightarrow \text{subline}.$$

Theorem 3. *Every nonempty 2-subline is isometric to a subgroup of the real line. Moreover, a metric space X is isometric to a subgroup G of $\mathbb{R}$ if and only if X is a 2-subline such that $\mathsf{d}[X^2] = G_+$.*

A metric space X is called *Banakh* if for every $c \in X$ and $r \in \mathsf{d}[X^2]$, there exist points $x, y \in X$ such that $\mathsf{S}(c; r) = \{x, y\}$ and $\mathsf{d}(x, y) = 2r$. It is easy to see that every 2-subline is a Banakh space. Theorem 3 can be compared with the following metric characterizations of subgroups of $\mathbb{Q}$, proved in [1].

Theorem 4. *A metric space X is isometric to a subgroup G of the group $\mathbb{Q}$ if and only if X is a Banakh space with $\mathsf{d}[X^2] = G_+$.*

Theorems 3 and 4 imply the following characterizations of the metric spaces $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Corollary 2. *A metric space X is isometric to $\mathbb{Z}$ if and only if X is a 2-subline with $\mathsf{d}[X^2] = \mathbb{Z}_+$ if and only if X is a Banakh space with $\mathsf{d}[X^2] = \mathbb{Z}_+$.*

Corollary 3. *A metric space X is isometric to $\mathbb{Q}$ if and only if X is a 2-subline with $\mathsf{d}[X^2] = \mathbb{Q}_+$ if and only if X is a Banakh space with $\mathsf{d}[X^2] = \mathbb{Q}_+$.*

Corollary 4. *A metric space (X, d) is isometric to $\mathbb{R}$ if and only if X is a 2-subline such that $\mathsf{d}[X^2] = \mathbb{R}_+$.*

Corollary 4 can be compared with the following metric characterization of the real line, proved by Will Brian [5] (see also [1, 1.7]).

Theorem 5. *A metric space X is isometric to the real line if and only if X is a complete Banakh space with $\mathbb{Q}_+ \subseteq \mathsf{d}[X^2]$.*

We recall that a metric space X is *complete* if every Cauchy sequence in X is convergent.

Metric characterizations of the spaces $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ are based on the notion of a ray.

Definition 4. A metric space (X, d) is called a *ray* if X is a 1-subline containing a point $o \in X$ such that for every $r \in \mathsf{d}[X^2]$ the sphere $\mathsf{S}(o; r)$ is a singleton.

Observe that no ray is a 2-subline.

Theorem 6. *Let G be a subgroup of the additive group $\mathbb{Q}$ of rational numbers. A metric space X is isometric to G_+ if and only if X is a ray with $\mathsf{d}[X^2] = G_+$.*

Corollary 5. *A metric space X is isometric to $\mathbb{Z}_+$ if and only if X is a ray with $\mathsf{d}[X^2] = \mathbb{Z}_+$.*

Corollary 6. *A metric space X is isometric to $\mathbb{Q}_+$ if and only if X is a ray with $\mathsf{d}[X^2] = \mathbb{Q}_+$.*

Theorem 7. *A metric space X is isometric to $\mathbb{R}_+$ if and only if X is a complete ray with $\mathbb{Q}_+ \subseteq \mathsf{d}[X^2]$.*

The completeness cannot be removed from Theorem 7 as shown by the following example.

Example 2. For every subgroup $G \subseteq \mathbb{R}$ containing nonzero elements $a, b \in G$ such that $b \notin \mathbb{Q}{\cdot}a \subseteq G$, there exists a dense submonoid X of $\mathbb{R}$ such that X is a ray with $d[X^2] = G_+$ and X is not isometric to G_+.

Example 2 shows that (in contrast to Theorem 6) Theorem 7 does not hold for arbitrary subgroups of the real line. The submonoid X in Example 2 is the image of G_+ under a suitable additive bijective function $\Phi\colon G \to G$. A function $\Phi\colon G \to G$ on a group G is *additive* if $\Phi(x+y) = \Phi(x) + \Phi(y)$ for all $x, y \in G$. Example 2 suggests the following open

Problem 1. *Is every ray X with $\mathsf{d}[X^2] = \mathbb{R}_+$ isometric to the metric subspace $\Phi[\mathbb{R}_+]$ of $\mathbb{R}$ for some injective additive function $\Phi\colon \mathbb{R} \to \mathbb{R}$?*

By Theorem 3, every 2-subline is isometric to a subgroup of the real line. In this context it would be interesting to know a classification of 1-sublines. Observe that a metric subspace X of the real line is a 1-subline if and only if X is 1-spherical if and only if X is semiaffine in the group $\mathbb{R}$. A subset X of an Abelian group G is called *semiaffine* if for every $x, y, z \in X$ the doubleton $\{x + y - z, x - y + z\}$ intersects X. Semiaffine sets in Abelian groups were characterized in [3] as follows.

Theorem 8. *A subset X of an Abelian group G is semiaffine if and only if one of the following conditions holds:*

1. *$X = (H + a) \cup (H + b)$ for some subgroup H of G and some elements $a, b \in X$;*
2. *$X = (H \setminus C) + g$ for some $g \in G$, some subgroup $H \subseteq G$ and some midconvex set C in H.*

A subset X of a group G is called *midconvex* in G if for every $x, y \in X$ the set

$$\frac{x+y}{2} := \{z \in G\colon 2z = x + y\}$$

is a subset of X.

The following characterization of 1-sublines follows from Theorems 1 and 8.

Theorem 9. *A metric space X is a 1-subline if and only if X is isometric to one of the following metric spaces:*

1. *the ℓ_1-rectangle $\square_a^b$ for some positive real numbers a, b;*
2. *$(H + a) \cup (H + b)$ for some subgroup H of $\mathbb{R}$ and some real numbers a, b;*
3. *$H \setminus C$ for some subgroup H of $\mathbb{R}$ and some midconvex set C in the group H.*

Midconvex sets in Abelian groups were characterized in [2] as follows.

Theorem 10. *A subset X of an Abelian group G is midconvex if and only if for every $g \in G$ and $x \in X$, the set $\{n \in \mathbb{Z}\colon x + ng \in X\}$ is equal to $C \cap H$ for some order-convex set $C \subseteq \mathbb{Z}$ and some subgroup $H \subseteq \mathbb{Z}$ such that the quotient group $\mathbb{Z}/H$ has no elements of even order.*

A subset C of a subgroup H of $\mathbb{R}$ is called *order-convex* in H if for any $x, y \in C$, the order interval $\{z \in H\colon x \le y \le z\}$ is a subset of C.

Midconvex sets in subgroups of the group $\mathbb{Q}$ were characterized in [2] as follows.

Theorem 11. *Let H be a subgroup of $\mathbb{Q}$. A nonempty set $X \subseteq H$ is midconvex in H if and only if $X = C \cap (P + x)$ for some order-convex set $C \subseteq H$, some $x \in X$ and some subgroup P of H such that the quotient group H/P contains no elements of even order.*

The necessary information on metric spaces can be found in [6, Ch.4]; for basic notions of group theory, we refer the reader to the textbook [8].

2. Proof of Theorem 1. Since we have found no published proof of Theorem 1, we present the detailed proof of this theorem in this section. Bowers and Bowers write in [4] that Theorem 1 "can be proved by chasing around betweenness relations among four points of X". This indeed can be done with the help of the following lemma.

Lemma 1. *If a finite subline X is not an ℓ_1-rectangle, then X is isometric to a subspace of the real line.*

Proof. If $|X| \leq 1$, then X is isometric to a subspace of any nonempty metric space, including the real line. So, we assume that $|X| > 1$. Since X is finite, there exist points $a, b \in X$ such that $ab = D := \max\{xy \colon x, y \in X\}$. For every point $x \in X$, the maximality of $ab = D$ and the Triangle Equality for the points $\{a, x, b\}$ ensure that

$$ab = ax + xb. \tag{1}$$

We claim that the function $f \colon X \to \mathbb{R}$, $x \overset{f}{\mapsto} ax$, is an isometric embedding of X into the real line.

Indeed, otherwise there exist points $x, y \in X$ such that $xy \neq |ax - ay|$. Then the points x, a, y are pairwise distinct and the Triangle Equality for the points x, y, a implies that $xa + ay = xy$. Then $yx + xb + by = ya + ax + xb + by = 2D$, so the longest side of the triangle $\{x, y, b\}$ has length D. Taking into account that $x \neq a \neq y$, $xb = ab - ax < D$ and $yb = ab - ay < D$, we conclude that $xy = D$. Then $xa = xy - ay = xy + yb - ab = yb$ and $xb = xy - by = xy - xa = ay$, which means that $\{x, a, y, b\}$ is an ℓ_1-rectangle. It follows from $ax + xb = ab = ay + yb$ and $\{x, y\} \cap \{a, b\} = \varnothing$ that

$$\max\{ax, xb, ay, yb\} < D. \tag{2}$$

Since X is not an ℓ_1-rectangle, there exists a point $z \in X \setminus \{x, a, y, b\}$. The Triangle Equality for the triangles $\{x, y, z\}$ and $\{a, z, b\}$ implies $xz + zy = xy = D = ab = az + bz$. Consequently,

$$\max\{xz, zy, az, bz\} < D. \tag{3}$$

By the strict inequalities (2) and (3) and the Triangle Equality, no side of the triangles $\{x, z, b\}$, $\{y, z, a\}$, and $\{y, z, b\}$ has length D. Then, by (1) and the equalities $xz + yz = xy = xa + ay$,

$$\begin{aligned}
xa + az - xz = xa + az + zb + bx - (xz + zb + bx) > 2ab - 2D = 0,\\
ax + xz - az = ax + xz + yz + ay - (yz + za + ay) > 2xy - 2D = 0,\\
az + zx - xa = az + zx - xa + yz + zb + by - (yz + zb + by) =\\
= az + zx + yz + zb - (yz + zb + by) > ab + xy - 2D = 0.
\end{aligned}$$

This contradicts the Triangle Equality for the points a, x, z. □

Now we are able to present a proof of Theorem 1. Given a metric space X, we need to prove that X is isometric to a subspace of the real line if and only if X is a subline and X is not an ℓ_1-rectangle.

The "only if" part of this characterization is trivial. To prove the "if" part, assume that a metric space X is a subline and X is not an ℓ_1-rectangle. If X is finite, then X is isometric to a subspace of the real line, by Lemma 1. It remains to consider the case of infinite metric space X. Pick any distinct points $a, b \in X$. Let $\mathcal{F}$ be the family of all finite subsets of X containing a and b. By Lemma 1, every set $F \in \mathcal{F}$ is isometric to a subspace of the real line. Therefore there exists an isometry $f_F\colon F \to \mathbb{R}$ such that $f_F(a) = 0$ and $f_F(b) = ab$. For each point $x \in F$ the image $f_F(x)$ is a unique point of $\mathbb{R}$ such that $|f_F(x)| = |f_F(x) - f_F(a)| = xa$ and $|f_F(x) - ab| = |f_F(x) - f_F(b)| = xb$. So, $f_F(x)$ is uniquely determined by the distances xa and xb, and we can define a function $f\colon X \to \mathbb{R}$ assigning to every $x \in X$ the real number $f_F(x)$, where F is an arbitrary set in $\mathcal{F}$ that contains x.

Given any points $x, y \in X$, we can take any set $F \in \mathcal{F}$ with $x, y \in F$ and conclude that $|f(x) - f(y)| = |f_F(x) - f_F(y)| = xy$, which means that f is an isometric embedding of X into the real line.

3. Proof of Theorem 3. We divide the proof of Theorem 3 into two lemmas.

Lemma 2. *Every nonempty 2-subline X is isometric to a subgroup $G \subseteq \mathbb{R}$ such that $\mathsf{d}[X^2] = G_+$.*

Proof. Observe that the space X is infinite. Indeed, otherwise there exist points $a, b \in X$ such that $ab = D \coloneqq \max\{xy\colon x, y \in X\}$. The Triangle Equality implies that the sphere $\mathsf{S}(a, D)$ coincides with the singleton $\{b\}$, witnessing that X is not a 2-subline.

By Theorem 1, the subline X is isometric to a subspace G of the real line. Being an isometric copy of the nonempty 2-subline X, the space G is a nonempty 2-subline, too. Without loss of generality, we can assume that $0 \in G \subseteq \mathbb{R}$.

For every numbers $x \in G$ and $y \in G \setminus \{0\}$, we have $|y| \in \mathsf{d}[G^2] = \mathsf{d}[X^2]$. Since G is a 2-subline, the sphere $\mathsf{S}(x, |y|) = G \cap \{x - |y|, x + |y|\}$ contains at least two points, which implies $\{x - y, x + y\} = \{x - |y|, x + |y|\} \subseteq G$. Consequently, G is a subgroup of $\mathbb{R}$ with $\mathsf{d}[X^2] = \mathsf{d}[G^2] = G_+$. □

Lemma 3. *Let G be a subgroup of the real line. A metric space X is isometric to G if and only if X is a 2-subline with $\mathsf{d}[X^2] = G_+$.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that X is a 2-subline with $\mathsf{d}[X^2] = G_+$. By Lemma 2, X is isometric to some subgroup $H \subseteq \mathbb{R}$ with $\mathsf{d}[X^2] = H_+$. Then $H_+ = G_+$ and hence $H = H_+ \cup \{-x\colon x \in H_+\} = G_+ \cup \{-x\colon x \in G_+\} = G$. Therefore, the metric space X is isometric to the group G. □

4. A metric characterization of the space $\mathbb{Z}_+$.

Theorem 12. *Let a be a positive real number. A metric space X is isometric to the metric space $a\mathbb{Z}_+ \coloneqq \{an\colon n \in \mathbb{Z}_+\}$ if and only if X is a ray such that $\{a, 2a\} \subseteq \mathsf{d}[X^2] \subseteq a\mathbb{Z}_+$ and X is not an ℓ_1-rectangle.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that X is a ray such that $\{a, 2a\} \subseteq \mathsf{d}[X^2] \subseteq \mathbb{Z}$ and X is not an ℓ_1-rectangle. By Theorem 1, X is isometric to a subspace of the real line. So, we lose no generality assuming that $X \subseteq \mathbb{R}$. Since X is a ray, there exists a point $o \in X$ such that for every $r \in \mathsf{d}[X^2]$, the sphere $\mathsf{S}(o, r)$ is a singleton. We lose no generality assuming that $0 = o \in X \subseteq \mathbb{R}$. Then X is antisymmetric in the sense that for every $r \in \mathsf{d}[X^2] \setminus \{0\}$ we have $r \in X$ if and only if $-r \notin X$.

It follows from $0 \in X$ and $\mathsf{d}[X^2] \subseteq a\mathbb{Z}_+$ that $X \subseteq a\mathbb{Z}$. Since the sphere $\mathsf{S}(o;a)$ is a singleton, $a \in X$ or $-a \in X$. We lose no generality assuming that $a \in X$ and hence $-a \notin X$.

We claim that $2a \in X$. To derive a contradiction, assume that $2a \notin X$ and hence $-2a \in X$, by the inclusion $2a \in \mathsf{d}[X^2]$ and the antisymmetry of X. Taking into account that X is a 1-subline with $-a \notin X$, $-2a \in X$ and $a \in \mathsf{d}[X^2]$, we conclude that $-3a \in X$ and $3a \notin X$, by the antisymmetry of X. Taking into account that X is a 1-subline with $a \in X$, $-a \notin X$, and $2a \in \mathsf{d}[X^2]$, we conclude that $a+2a = 3a \in X$, which is a desired contradiction showing that $2a \in X$ and $-2a \notin X$, by the antisymmetry of X. The following lemma and the antisymmetry of X complete the proof of the theorem. □

Lemma 4. *Let a be a positive real number. If $X \subset a\mathbb{Z}$ is a 1-subline with $0, a, 2a \in X$ and $-2a, -a \notin X$, then $a\mathbb{Z}_+ \subseteq X$.*

Proof. By induction we shall prove that for every $i \in \mathbb{N}$ the set $\{0, a, \dots, ia\}$ is a subset of X. This is so for $i \in \{1, 2\}$. Assume that for some positive number $i \geq 2$ we know that $\{0, a, \dots, ia\} \subseteq X$.

If i is even, consider the number $c = \frac{1}{2}ia$ and observe that $c+a = \frac{1}{2}ai+a \leq \frac{1}{2}ia+\frac{1}{2}ia = ia$ and hence $c, c+a \in \mathsf{d}[X^2]$. Since X is a 1-subline and $c - (c+a) = -a \notin X$, the number $c + (c+a) = ia + a$ belongs to X, witnessing that $\{0, a, \dots, ia, (i+1)a\} \subseteq X$.

If i is odd, consider the number $c = \frac{1}{2}(i-1)a$ and observe that $c + 2a = \frac{1}{2}ia + \frac{3}{2}a \leq \frac{1}{2}ia+\frac{1}{2}ia = ia$ and hence $c, c+2a \in \mathsf{d}[X^2]$. Since X is a 1-subline and $c-(c+2a) = -2a \notin X$, the number $c+(c+2a) = (i+1)a$ belongs to X, witnessing that $\{0, a, \dots, ia, (i+1)a\} \subseteq X$.

This completes the inductive step. By the Principle of Mathematical Induction, $\{0, a, \dots, ia\} \subseteq X$ and hence $a\mathbb{Z}_+ \subseteq X$. □

Remark 1. The "if" part of Theorem 12 can be also derived from the properties of semiaffine and midconvex sets established in Theorems 9 and 11. Indeed, assume that a metric space X is a ray such that $\{a, 2a\} \subseteq \mathsf{d}[X^2] \subseteq a\mathbb{Z}_+$ and X is not an ℓ_1-rectangle. Applying Theorem 9, we can prove that X is isometric to $H \setminus C$ for some subgroup H of $\mathbb{R}$, some midconvex set C in the group H. Composing the isometry with a shift on the group H we can assume that $0 \in H \setminus C$ and for each $r \in \mathsf{d}[X^2]$, the sphere $\mathsf{S}(0, r)$ is a singleton. Since $\mathsf{d}[X^2] \subseteq a\mathbb{Z}_+$, replacing H by $H \cap a\mathbb{Z}$ and C by $C \cap a\mathbb{Z}$, if needed, we can suppose that $H = a\mathbb{Z}$. By Theorem 11, $C = C' \cap (P + x)$ for some order-convex set $C' \subseteq H = a\mathbb{Z}$, some $x \in C$ and some subgroup P of H. Since for every $r \in \mathsf{d}[X^2]$ the sphere $\mathsf{S}(0, r)$ is a singleton, the coset $P + x$ equals H, and $C' \cap (P + x)$ equals $H \cap a\mathbb{N}$ or $H \cap (-a\mathbb{N})$.

6. Proof of Theorem 6. Let G be a subgroup of the additive group $\mathbb{Q}$ of rational numbers. Given a metric space X, we should prove that X is isometric to the monoid $G_+ := \{x \in G \colon x \geq 0\}$ if and only if X is a ray with $\mathsf{d}[X^2] = G_+$. The "only if" part of this characterization is trivial. To prove the "if" part, assume that X is a ray with $\mathsf{d}[X^2] = G_+$. If the group G is trivial, then $\mathsf{d}[X^2] = G_+ = \{0\}$ and hence X is a singleton, isometric to the singleton $G_+ = \{0\}$. So, assume that the group G is not trivial. Then G is infinite and so is the set G_+. Since $\mathsf{d}[X^2] = G_+$, the metric space X is infinite and hence is not an ℓ_1-rectangle.

If the group G is finitely generated, then G is cyclic (being a subgroup of a cyclic subgroup of $\mathbb{Q}$) and hence $G = a\mathbb{Z}$ for some positive number $a \in G \subseteq \mathbb{Q}$. By Theorem 12, X is isometric to $a\mathbb{Z}_+ = G_+$.

If the group G is infinitely generated, then $G = \bigcup_{n\in\omega} G_n$ for a strictly increasing sequence $(G_n)_{n\in\omega}$ of non-trivial finitely-generated subgroups G_n of $G \subseteq \mathbb{Q}$. Every group G_n is cyclic, being a subgroup of a suitable cyclic subgroup of the group $\mathbb{Q}$. Then $G_n = a_n\mathbb{Z}$ for some positive rational number a_n.

Being a ray, the space X contains a point $o \in X$ such that for every $r \in \mathsf{d}[X^2]$, the sphere $\mathsf{S}(o;r)$ is a singleton. By Theorem 1, the infinite subline X is isometric to a subspace of the real line. We lose no generality assuming that $0 = o \in X \subseteq \mathbb{R}$. It follows from $\mathsf{d}[X^2] = G_+ \subseteq \mathbb{Q}$ that $X \subseteq \mathbb{Q}$. It is easy to see that for every $n \in \mathbb{N}$ the intersection $X_n = X \cap G_n$ is a ray such that $\mathsf{d}[X_n^2] = (G_n)_+ = a_n\mathbb{Z}_+$. By Theorem 12, the space X_n is isometric to the space $a_n\mathbb{Z}_+$ and hence $X_n = a_n\mathbb{Z}_+$ or $X_n = -a_n\mathbb{Z}_+$. We lose no generality assuming that $X_0 = a_0\mathbb{Z}_+$. Then $X_n = a_n\mathbb{Z}_+$ for all $n \in \omega$ and hence

$$X = \bigcup_{n\in\omega} X_n = \bigcup_{n\in\omega} a_n\mathbb{Z}_+ = \bigcup_{n\in\omega}(G+n)_+ = G_+.$$

7. Proof of Theorem 7. Given a metric space X, we should prove that X is isometric to the half-line $\mathbb{R}_+$ if and only if X is a complete ray such that $\mathbb{Q}_+ \subseteq \mathsf{d}[X^2]$. The “only if” part of this characterization is trivial. To prove the “if” part, assume that X is a complete ray such that $\mathbb{Q}_+ \subseteq \mathsf{d}[X^2]$. Then X is infinite and hence is isometric to a subspace of the real line. Being a ray, the space X contains a point $o \in X$ such that for every $r \in \mathsf{d}[X^2]$ the sphere $\mathsf{S}(o;r)$ is a singleton. We lose no generality assuming that $0 = o \in X \subseteq \mathbb{R}$.

It is easy to see that the subspace $Y = \{x \in X : xo \in \mathbb{Q}_+\}$ of X is a ray with $\mathsf{d}[Y^2] = \mathbb{Q}_+$. By Theorem 6, Y is isometric to $\mathbb{Q}_+$. Then $Y = \mathbb{Q}_+$ or $Y = -\mathbb{Q}_+$. Replacing the set $X \subseteq \mathbb{R}$ by $-X$, if necessary, we can assume that $Y = \mathbb{Q}_+$. Being complete, the metric space X contains the completion of its subspace Y, which coincides with $\mathbb{R}_+$. Then $\mathbb{R}_+ \subseteq X$ and $\mathsf{d}[X^2] = \mathbb{R}_+$. Assuming that $X \neq \mathbb{R}_+$, we can find a point $x \in X \setminus \mathbb{R}_+$ and conclude that the sphere $\mathsf{S}(o;|x|)$ contains two distinct points x and $-x$, which contradicts the choice of the point o.

8. Constructing Example 2. In this section we elaborate the construction of the ray X from Example 2.

This construction exploits the following algebraic characterization of rays in the real line.

Lemma 5. *A subset X of the real line is a ray if and only if it has two properties:*

1. $\forall x \in X\ \forall r \in X - X\ \ (\{x-r, x+r\} \cap X \neq \varnothing)$;
2. $\exists o \in X\ \forall r \in X - X\ \ (|\{o-r, o+r\} \cap X| \le 1)$.

Proof. To see that this characterization indeed holds, observe that $\mathsf{d}[X^2] = (X-X)_+$ and $\mathsf{S}(x;r) = \{x-r, x+r\} \cap X$ for any $x \in X$ and $r \in \mathbb{R}_+$. □

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *additive* if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Lemma 6. *For every ray $X \subseteq \mathbb{R}$ and every injective additive function $f\colon \mathbb{R} \to \mathbb{R}$, the metric subspace $f[X]$ of $\mathbb{R}$ is a ray.*

Proof. To show that $f[X]$ is a ray, it suffices to check that $f[X]$ satisfies the algebraic conditions of Lemma 5.

1. Fix any numbers $y \in f[X]$ and $r \in f[X] - f[X]$. The additivity of f ensures that $f[X] - f[X] = f[X-X]$ and hence $r = f(s)$ for some $s \in X - X$. Since X is a ray, for the element $x = f^{-1}(y) \in X$, the set $X \cap \{x-s, x+s\}$ contains some point $z = x \pm s$. The

additivity of f ensures that $f(z) = f(x \pm s) = f(x) \pm f(s) = y \pm r \in \{y - r, y + r\} \cap f[X]$ and hence the set $\{y - r, y + r\} \cap f[X]$ is not empty.

2. Since X is a ray, there exists a point o such that for every $s \in \mathsf{d}[X^2] = (X - X)_+$ the sphere $\{x \in X : |o - x| = s\}$ is a singleton. We claim that the point $f(o)$ has the property required in condition (2) of Lemma 5. Assuming that this condition does not hold, we can find a real number $r \in f[X] - f[X]$ such that $\{f(o) - r, f(o) + r\} \cap f[X]$ is a doubleton. Then $f(o) - r = f(x)$ and $f(o) + r = f(y)$ for some distinct real numbers $x, y \in X$. Since $r \in f[X] - f[X] = f[X - X]$, the real number $s = f^{-1}(r)$ belongs to the set $X - X$ and hence $|s| \in \mathsf{d}[X^2]$.

Since the function f is additive, $f(o - s) = f(o) - f(s) = f(o) - r = f(x)$ and hence $o - s = x$ by the injectivity of f. By analogy we can show that $o + s = y$. Then $\{x, y\} = \{o - s, o + s\} = \{o - |s|, o + |s|\}$ is a doubleton in X, which contradicts the choice of the point o. This contradiction completes the proof of the second condition of Lemma 5 for the set $f[X]$.

By Lemma 5, the metric subspace $f[X]$ of the real line is a ray. □

Now we are able to justify Example 2. Let G be any subgroup of $\mathbb{R}$ containing two nonzero elements $a, b \in G$ such that $b \notin \mathbb{Q} \cdot a \subseteq G$. We lose no generality assuming that $a, b > 0$. Consider the real line as a vector space $\mathbb{Q}$ over the field of rational numbers. The conditions $b \notin \mathbb{Q}{\cdot}a$ and $a \neq 0$ imply that the vectors a, b are linearly independent over the field $\mathbb{Q}$. Using the Kuratowski–Zorn Lemma, choose a maximal subset $B \subseteq \mathbb{R}$ such that $\{a, b\} \subseteq B$ and B is linearly independent over $\mathbb{Q}$. The maximality of B guarantees that B is an algebraic basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Consider the function $f\colon B \to \mathbb{R}$ such that $f(a) = -a$ and $f(x) = x$ for all $x \in B \setminus \{a\}$. By the linear independence of B, the function admits a unique extension to an additive function $\bar{f}\colon \mathbb{R} \to \mathbb{R}$. Since the set $f[B] = \{-a\} \cup (B \setminus \{a\})$ is linearly independent over the field $\mathbb{Q}$, the additive function $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ is injective.

Claim. $f[G] = G$.

Proof. Since B is a basis of the $\mathbb{Q}$-vector space $\mathbb{R}$, for every element $x \in G$, there exist a finite set $F \subseteq B$ and a function $\lambda\colon F \to \mathbb{Q} \setminus \{0\}$ such that $x = \sum_{e \in F} \lambda(e) \cdot e$. If $a \notin F$, then $\bar{f}(x) = \sum_{e \in F} \lambda(e) f(e) = \sum_{e \in F} \lambda(e) e = x \in G$. If $a \in F$, then

$$\begin{aligned}\bar{f}(x) = \lambda(a) f(a) + \sum_{e \in F \setminus \{a\}} \lambda(e) f(e) = -\lambda(a) a + \sum_{e \in F \setminus \{a\}} \lambda(e) e = \\ = -2\lambda(a) a + \sum_{e \in F} \lambda(e) e = -2\lambda(a) a + x \in G\end{aligned}$$

because $\mathbb{Q}{\cdot}a \subseteq G$.

Therefore, $\bar{f}[G] \subseteq G$. Since $f \circ f$ is the identity map of the algebraic basis B of the $\mathbb{Q}$-vector space $\mathbb{R}$, the composition $\bar{f} \circ \bar{f}$ is the idenity function of $\mathbb{R}$. Then $\bar{f}[G] \subseteq G$ implies $G = \bar{f}[\bar{f}[G]] \subseteq \bar{f}[G]$ and hence $\bar{f}[G] = G$. □

By Lemma 6, the metric subspace $X = f[G_+]$ of the real line is a ray. Since f is additive, $X - X = f[G_+] - f[G_+] = f[G_+ - G_+] = f[G] = G$, see Claim . Then $\mathsf{d}[X^2] = (X - X)_+ = G_+$.

Taking into account that the numbers a, b are positive and $\bar{f}(a) = f(a) = -a$, $\bar{f}(b) = f(b) = b$, we conclude that the set $\mathbb{Z}_+ b - \mathbb{Q}_+ a \subseteq f[G_+] + f[G_+] = f[G_+ + G_+] = f[G_+] = X$ is dense in $\mathbb{R}$.

Since the set X is dense in $\mathbb{R}$, the completion of the metric space X coincides with $\mathbb{R}$. On the other hand, the completion of the space $G_+ \supseteq \mathbb{Q}_+ \cdot a$ coincides with the half-line $\mathbb{R}_+$, which is not isometric to $\mathbb{R}$. This implies that the ray X is not isometric to the ray G_+.

Acknowledgement. The second author express his sincere thanks to Mamuka Jibladze for pointing out the references [4] and [7] to Menger's Theorems 1 and 2, and also to Pavlo Dzikovskyi for fruitful discussions on the Triangle Equality in metric spaces.

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Received 23.01.2023
Revised 12.06.2023