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A NOTE ON n-JORDAN HOMOMORPHISMS

M. El Azhari. *A note on n-Jordan homomorphisms*, Mat. Stud. **62** (2024), 77–80.

Let A, B be two rings and $n \geqslant 2$ be an integer. An additive map $h\colon A \to B$ is called an n-Jordan homomorphism if $h(x^n) = h(x)^n$ for all $x \in A$; h is called an n-homomorphism or an anti-n-homomorphism if $h(\prod_{i=1}^{n} x_i) = \prod_{i=1}^{n} h(x_i)$ or $h(\prod_{i=1}^{n} x_i) = \prod_{i=0}^{n-1} h(x_{n-i})$, respectively, for all $x_1, ..., x_n \in A$.

We give the following variation of a theorem on n-Jordan homomorphisms due to I.N. Herstein: Let $n \geq 2$ be an integer and h be an $n-$Jordan homomorphism from a ring A into a ring B of characteristic greater than n. Suppose further that A has a unit e, then $h = h(e)\tau$, where $h(e)$ is in the centralizer of $h(A)$ and τ is a Jordan homomorphism.

By using this variation, we deduce the following result of G. An: Let A and B be two rings, where A has a unit and B is of characteristic greater than an integer $n \geq 2$. If every Jordan homomorphism from A into B is a homomorphism (anti-homomorphism), then every $n-$Jordan homomorphism from A into B is an n-homomorphism (anti-n-homomorphism). As a consequence of an appropriate lemma, we also obtain the following result of E. Gselmann: Let A, B be two commutative rings and B is of characteristic greater than an integer $n \geq 2$. Then every n-Jordan homomorphism from A into B is an $n-$homomorphism.

1. Preliminaries. Let A, B be two rings and $n \geqslant 2$ be an integer. An additive map $h\colon A \to B$ is called an n-Jordan homomorphism if $h(x^n) = h(x)^n$ for all $x \in A$. Also, an additive map $h\colon A \to B$ is called an n-homomorphism or an anti-n-homomorphism if

$$h\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} h(x_i) \quad \text{or} \quad h\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=0}^{n-1} h(x_{n-i}),$$

respectively, for all $x_1, ..., x_n \in A$. In the usual sense, a 2-Jordan homomorphism is a Jordan homomorphism, a 2-homomorphism is a homomorphism and an anti-2-homomorphism is an anti-homomorphism. It is obvious that n-homomorphisms are n-Jordan homomorphisms. Conversely, under certain conditions, n-Jordan homomorphisms are n-homomorphisms. We say that a ring A is of characteristic greater than n (char $(B) > n$) if $n!x = 0$ implies $x = 0$ for all $x \in A$.

2. Results.

Lemma 1 ([4], Lemma 1)**.** *Let A, B be two rings, $n \geqslant 2$ be an integer and $f : A^n \to B$ be a multi-additive map such that $f(x, x, ..., x) = 0$ for all x in A. Then*

$$\sum_{\sigma \in S_n} f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = 0$$

for all $x_1, ..., x_n \in A$, where S_n is the set of all permutations of $\{1, \ldots, n\}$.

2020 *Mathematics Subject Classification:* 46H15.
Keywords: Jordan homomorphism; homomorphism; n-Jordan homomorphism; n-homomorphism.
doi:10.30970/ms.62.1.77-80

By using Lemma 1, we have the following lemma.

Lemma 2. *Let A, B be two rings, $n \geqslant 2$ be an integer and $h\colon A \to B$ be an n-Jordan homomorphism. Then*

$$\sum_{\sigma\in S_n}\Big(h\big(\prod_{i=1}^{n} x_{\sigma(i)}\big) - \prod_{i=1}^{n} h(x_{\sigma(i)})\Big) = 0$$

for all $x_1, ..., x_n \in A$, where S_n is the set of all permutations on $\{1, ..., n\}$.

Proof. Consider the map $f\colon A^n \to B$, such that

$$f(x_1, x_2, \dots, x_n) = h\bigg(\prod_{i=1}^{n} x_i\bigg) - \prod_{i=1}^{n} h(x_i)$$

The map f is clearly multi-additive and

$$f(x, x, \dots, x) = h(x^n) - h(x)^n = 0 \text{ for all } x \in A.$$

By Lemma 1,

$$\sum_{\sigma\in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \sum_{\sigma\in S_n}\left(h\big(\prod_{i=1}^{n} x_{\sigma(i)}\big) - \prod_{i=1}^{n} h(x_{\sigma(i)})\right) = 0$$

for all $x_1, \dots, x_n \in A$. □

It was shown in [3] that if $n \geq 2$, A, B are commutative rings, char $(B) > n$ and $h\colon A \to B$ is an n-Jordan homomorphism, then h is an n-homomorphism. The same was also proved for algebras in [2] and [5]. Here we obtain this result as a consequence of Lemma 2.

Theorem 1. *Let A, B be two commutative rings and* char $(B) > n$. *Then every n-Jordan homomorphism from A into B is an n-homomorphism.*

Proof. Let $h\colon A \to B$ be an n-Jordan homomorphism. By Lemma 2 and since A, B are commutative,

$$\sum_{\sigma\in S_n}\left(h\bigg(\prod_{i=1}^{n} x_{\sigma(i)}\bigg) - \prod_{i=1}^{n} h(x_{\sigma(i)})\right) = n!\left(h\bigg(\prod_{i=1}^{n} x_i\bigg) - \prod_{i=1}^{n} h(x_i)\right) = 0$$

for all $x_1, ..., x_n \in A$. Since char $(B) > n$, h is an n-homomorphism □

Now we give the following variation of a theorem on n-Jordan homomorphisms due to Herstein [4, Theorem K].

Theorem 2. *Let h be an n-Jordan homomorphism from a ring A into a ring B with* char $(B) > n$. *Suppose further that A has a unit e, then $h = h(e)\tau$ where $h(e)$ is in the centralizer of $h(A)$ and τ is a Jordan homomorphism.*

Proof. Since h is an n-Jordan homomorphism, $h(e) = h(e^n) = h(e)^n$. By Lemma 2 and putting $x_1 = x$, $x_2 = x_3 = \cdots = x_n = e$, we get

$$n!h(x) = (n-1)!(h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \cdots + h(x)h(e)^{n-1}).$$

Since char $(B) > n$, we have

$$nh(x) = h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \cdots + h(x)h(e)^{n-1}. \tag{1}$$

By multiplying on the right by $f(e)$ both sides of the equality (1), one has

$$nh(x)h(e) = h(e)^{n-1}h(x)h(e) + h(e)^{n-2}h(x)h(e)^2 + \cdots + h(x)h(e). \tag{2}$$

Also, by multiplying on the left by $f(e)$ both sides of the equality (1), we get

$$nh(e)h(x) = h(e)h(x) + h(e)^{n-1}h(x)h(e) + \cdots + h(e)h(x)h(e)^{n-1}. \tag{3}$$

By (2) and (3), $(n-1)h(x)h(e) = (n-1)h(e)h(x)$. Since char $(B) > n$, we have

$$h(x)h(e) = h(e)h(x). \tag{4}$$

Then $h(e)$ is in the centralizer of $h(A)$. By (1) and (4), $nh(x) = nh(e)^{n-1}h(x)$. Since char $(B) > n$,

$$h(x) = h(e)^{n-1}h(x). \tag{5}$$

By Lemma 2, (4), putting $x_1 = x_2 = x$, $x_3 = \ldots = x_n = e$, one has

$$n!(h(x^2) - h(e)^{n-2}h(x)^2) = 0,$$

hence

$$h(x^2) = h(e)^{n-2}h(x)^2, \tag{6}$$

because char $(B) > n$. Consider the map $\tau\colon A \to B$, $\tau(x) = h(e)^{n-2}h(x)$. The map τ is clearly additive. By (5),

$$h(x) = h(e)^{n-1}h(x) = h(e)h(e)^{n-2}h(x) = h(e)\tau(x) \text{ for all } x \in A.$$

By (6),

$$\tau(x^2) = h(e)^{n-2}h(x^2) = h(e)^{2(n-2)}h(x)^2 = (h(e)^{(n-2)}h(x))^2 = \tau(x)^2 \text{ for all } x \in A,$$

thus τ is a Jordan homomorphism. □

As a consequence, we obtain the following result of G. An [1].

Corollary 1 ([1], Theorem 2.4)**.** *Let A and B be two rings where A has a unit e and char $(B) > n$. If every Jordan homomorphism from A into B is a homomorphism (anti-homomorphism), then every n-Jordan homomorphism from A into B is an n-homomorphism (anti-n-homomorphism).*

Proof. Let $h\colon A \to B$ be an n-Jordan homomorphism. By Theorem 2, $h = h(e)\tau$, where $h(e)$ is in the centralizer of $h(A)$ and τ is a Jordan homomorphism. Therefore,

$$\begin{aligned} h(x_1x_2\cdot\ldots\cdot x_n) &= h(e)\tau(x_1x_2\cdot\ldots\cdot x_n) = h(e)\tau(x_1)\tau(x_2)\cdots\tau(x_n) = \\ &= h(e)^n\tau(x_1)\tau(x_2)\cdots\tau(x_n)\ \big[\text{since } \ h(e) = h(e^n) = h(e)^n\big] = \\ &= h(e)\tau(x_1)h(e)\tau(x_2)\cdots h(e)\tau(x_n)\ \big[\text{since } \ h(e) \text{ commutes with each } \tau(x)\big] = \\ &= h(x_1)h(x_2)\cdot\ldots\cdot h(x_n) \end{aligned}$$

for all $x_1, \ldots, x_n \in A$. Thus, h is an n-homomorphism.

Similarly, if τ is an anti-homomorphism, then h is an anti-n-homomorphism. □

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Received 13.07.2023
Revised 23.07.2024