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NON-STANDARD SEQUENCES

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The paper shows the existence of a previously unknown relationship between the theory of monoids and category theory. A non-standard mathematical method based on terms from non-standard (not always existing) sequences is proposed. In the article in particular are proved the following statement: Every category is a complete model of some monoid with an associative zero. Conversely, any such monoid completely models some category. Category theory is logically equivalent to the theory of monoids with an associative zero. Both are non-essential extensions of each other. (Theorem 1)

The concepts of category theory proved to be sufficiently productive in modern mathematics. The simplest category is the monoid. And its concept is the first thing that belongs simultaneously to both worlds, both algebraic and categorical. And the latter, in turn, is a very successful fusion of algebra, geometry, topology, logic, set theory, you can't list everything. Moreover, monoid homomorphisms are functors between the corresponding categories. In this paper, we consider another interesting property of monoids and categories.

A sequence of variables in its termal, logical sense is a certain subset of the Cartesian degree of an individual area for which a certain predicate can be performed. Empirically (linguistically), this is a linear notation generated in a certain alphabet — which is what it is in the theory of Markov algorithms. A sequence can be defined algebraically without resorting to other construction. It is striking that this product is in a certain free monoid, the unit of which is an empty symbol. One of the implementations of this approach is modern language constructs. But this definition is too narrow, leaving no room for other interesting varieties of what might be called a sequence. Let us try to proceed from the main "working" properties of sequences. One of the properties that make them such a versatile tool of thinking. Sequences are what we call words. However, the meaningful word does not always exist.

Definition 1. Let there be a monoid $\boldsymbol{M}$ with unit $\boldsymbol{E}$ and set $\boldsymbol{Q}$, for which:

$$\forall a\ (a \neq E)\ \forall b\ (b \neq E)\ \forall c\ (c \neq E)\colon (abc \in Q) \Rightarrow [(ab \in Q) \vee (bc \in Q)], \tag{1}$$

$$\forall x\ \forall y\ : (x \in Q) \Rightarrow [(xy \in Q) \wedge (yx \in Q)]. \tag{2}$$

Any element from $\boldsymbol{(M\text{-}Q)}$ that is not equal to one, we will call a chain (in this case, some chains can be equal, or consist of only one element "chain link"). The set $\boldsymbol{Q}$ is called the big zero. If $\boldsymbol{Q}$ is one-element, then we call it associative zero and denote it as $\boldsymbol{0}$. Note: in quantifier expressions like: $\forall a\ (a \neq E)$, we mean the statement: "for all a not equal to $\boldsymbol{E}$".

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Let me explain the meaning (the second of the conditions simply says that **Q** is a two-sided ideal). It is intuitively clear that if in a sequence of three elements one of the pairs does not form a chain, then the whole consecutive triplet will not be a chain either (well, the "hooks" are different and do not hook). Moreover, if an element is not a chain, then any of its possible "linking" with another element will also not be a chain. The result of a binary operation is unique for any pair of elements. An associative binary operation is unique for any sequence of elements (including single ones). A sequence (in its traditional sense) is formally characterized by uniqueness and associativity; there is one more, the main of its properties: any sequence exists if and only if any of its subsequences (any of its "continuous pieces") exist. It is easy to see that for our definition of a chain, all properties of sequences are satisfied by induction: the existence of a finite sequence is equivalent to the existence of all binary sequences included in it (according to Markov). But ... a simple feature — so defined (chain) sequences do not always exist. But the existence of a certain finite chain implies the existence of all its subchains (in the order of writing). And if we abstract from the fourth fundamental property of sequences, from the existence of a sequence of any chains of symbols, then we get our definition of a chain.

So, belonging to $\boldsymbol{Q}$ means "not being a sequence". A chain is a non-standard definition of a sequence. $\boldsymbol{Q}$ is a collection of "non-strings", non-existent sequences. And what is the big zero $\boldsymbol{Q}$ in the monoid? This is a two-sided *associative* (as one would like to call it) ideal (see the first property in Definition 1). Which is an approximation to some prime ideal $\boldsymbol{P}$ (in the latter:

$$(\forall x)(\forall y)\colon (xy \in P) \Leftrightarrow [(x \in P) \vee (y \in P)],$$

where the right implication is the definition of a *completely divisible* set). Since the first property in Definition 1 is a certain weakening of the simplicity condition (divisibility of the product of *triples* of factors).

The following lemma is true.

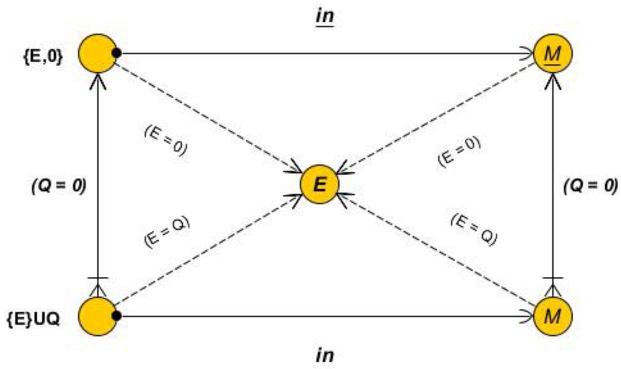
Lemma 1. *Every prime ideal is associative. The reverse is not true. The definition of an associative ideal is consistent.*

Proof. By the definition of the simplicity of an ideal,

$$(\forall a)(\forall b)(\forall c)\colon [(ab \in P) \vee (bc \in P)] \Longleftrightarrow (abc \in P).$$

Let us give an example of an associative, but not a simple, ideal. Consider a monoid in which there are no commuting products without the participation of unity or the power of the element. And let this monoid have a set of generators. Then all elements that are multiples of some generatrix constitute a simple ideal, not necessarily an associative one. However, elements that are multiples of the product of two generators will constitute an associative, but not a simple, ideal (the product of this very pair of generators is indivisible within this set). However, we will see other examples later. □

When the monoid is factorized by the identity relation for its associative ideal: ***(Q=0)*** (we mean the identification of all elements of the ideal), we get a monoid $\underline{\boldsymbol{M}}$ with an associative zero $(abc = 0) \Rightarrow [(ab = 0) \vee (bc = 0)]$, $x0 = 0x = 0$, onto which $\boldsymbol{M}$ is mapped epimorphically, but the entire structure of non-zero elements remains unchanged. At the same time, the identification: ***(Q = E)*** will lead to the triviality of the entire monoid: ***(M = E)***. More precisely, the following co-cartezian square (it is commutative and the pair of arrows in the upper right corner is an amalgam of the pair of arrows from the lower left):



Here:
E is the null object of the category of monoids (isomorphic to the unity of all monoids);
$(\{E\} \cup Q)$ is the monoid consisting of the ideal ***Q*** and the identity ***E***;
$(\{E\} \cup 0)$ is a monoid consisting of ***0*** and ***1*** (similarly);
the relations ***(E=Q)***, ***(E=0)*** and ***(Q=0)*** represent the corresponding epimorphisms;
in and ***in*** are the monomorphisms induced by these relations.

Now that we have a little understanding of the position of associative ideals in monoid theory, let us move on to studying the relationship they have to category theory.

The categorical composition of arrows does not always exist, but is always associative.

Earlier we talked about the fundamental properties of sequences:

- unambiguity (i.e., associativity of the operation of joining sequences); the existence of those, and only those sequences that have all their subsequences (the possibility of arbitrary contraction on the left and right);
- the presence of an empty sequence (the result of assigning to which does not change anything). In a monoid, the role of an empty sequence is played by a unit. Single arrows (objects) are in the category.
- the existence of an element is equivalent to the existence of its sequence with unit length.

In the monoid ***M***, we consider as existing any element that does not belong to the given associative ideal. Then a model of a certain existence of sequences is obtained — all three properties are satisfied: the only difference is that not all sequences exist. Indeed, let us set the predicate: $ex(x) \equiv (x \notin Q)$. Then, according to Definition 1 we get

$$[ex(ab) \wedge ex(bc)] \Leftrightarrow ex(abc)$$

the predicate ***ex*** for any existing string is satisfied if and only if it is satisfied for all its binary substrings, which is proved by induction; there are chains of unit length, because the ideal does not include all the elements of the monoid; moreover, ***E*** should be identified with the empty string.

Therefore, an ex-predicate defined on a free monoid has all the properties of a partial existence quantifier for sequences. And in a free monoid, products are one-to-one with sequences of elements. At the same time, all relations between elements of any monoid correspond one-to-one to relations between category arrows.

Theorem 1. *Every category is a complete model of some monoid with an associative zero. Conversely, any such monoid completely models some category. Category theory is logically equivalent to the theory of monoids with an associative zero. Both are non-essential extensions of each other.*

Prof. From a category to a monoid. We identify the objects of the category with their identity arrows: $A \longleftrightarrow 1_A$; this is a bijective correspondence, which is an inessential modification of category theory: thus, a theory equivalent to the original one arises — the differences are only in the original systems of definitions.

Let xy be the product of arrows in a category. We introduce the constant **0** and define a *well-defined* law of arrow composition everywhere: $x \cdot y \equiv xy$, if $\exists z\colon z = xy$; otherwise we assume: $x \cdot y \equiv 0$. We assume that: $\forall x\colon x \cdot 0 \equiv 0 \cdot x \equiv 0$. As a result, we again obtain an inessential extension of category theory — all non-zero elements correspond one-to-one with the arrows of the original category, and the law of composition coincides with the original law wherever the product of arrows exists.

This situation will not change if we introduce "our own, individual" **0** for each pair of arrows with a product that does not exist in the category. That is, if we define a constant $0_{x,y}$ as follows

$$(\forall x)(\forall y)\ (\neg\exists z\colon z = xy) \Longrightarrow (\exists t)\ (t = x \cdot y = 0_{x,y}) \wedge (\forall s)\ (s \cdot 0_{x,y} = 0_{x,y} \cdot s = 0_{x,y}).$$

Then the constant 0 defined earlier applies to the case when the collection of constants $0_{x,y}$ contains only a single element

$$(\forall x)(\forall y)\ (\exists t\colon t = 0_{x,y}) \Longrightarrow (0_{x,y} = 0).$$

The difference between these cases does not affect the course of further reasoning (when applying the theorem in practice, one or the other case can be used).The difference between these cases does not affect the course of further reasoning (when applying the theorem in practice, one or the other case can be used).

We also introduce a constant **E** (which corresponds to the identity functor of the category): $\forall x\colon x \cdot E = E \cdot x = x$ — the introduction of this constant also does not lead to an inessential extension of the theory.

The composition law "$\cdot$" is everywhere defined and associative. And **E** is the unit of this law. Thus, the original category is immersed in a monoid, the theory of which is an inessential extension of the theory of the original category. The set $\{0_{x,y}\}$ is an associative ideal of this monoid (in the case of a single constant **0**, this ideal is the zero element defined by us earlier).

From a monoid to a category. Consider a **pair (M,Q)** consisting of a monoid and its associative ideal. On the set of elements **(M–Q)** we define the operation "$\circ$" of partial composition:

$$x \cdot y \notin Q \Longrightarrow x \circ y \equiv x \cdot y, \quad x \cdot y \in Q \Longrightarrow x \circ y \equiv \neg\exists t\colon x \cdot y = t,$$

where "$\cdot$" is the law of composition of the monoid **M**. Since **Q** is an associative ideal, the composition "$\circ$" is associative wherever it exists. Moreover, its very existence is also "associative":

$$(\exists t\colon (x \circ y = t) \wedge \exists t\colon (y \circ z = t)) \Longleftrightarrow \exists! t\colon (x \circ y \circ z = t)$$

Now consider the set **(M-Q-E)** (**M** without the associative ideal **Q** and the monoid unit **E** and write down the axiom of existence of the unit for the law "$\circ$":

$$\exists \varepsilon \forall x\colon ((\exists t\colon \varepsilon \circ x = t) \Longrightarrow (\varepsilon \circ x = x)) \wedge ((\exists t\colon x \circ \varepsilon = t) \Longrightarrow (x \circ \varepsilon = x));$$

ε differs from the usual monoid unit in that this axiom does not imply the equality of all units.

But this axiom should be clarified. The law "$\circ$" differs from the monoid law "$\cdot$" in that this composition does not always exist, and we clearly define the domain of its existence. The composition "$\circ$" is local in the sense that its domain of existence is determined by a specific pair of multiplied elements. Therefore, ε-units must have the same locality with respect to multiplication. Therefore, if we want the axiom of the existence of a unit to have a "maximal" domain of truth, we must find the "maximal domain of existence" of ε-units — they must exist locally everywhere, for each **x**. Finally, we arrive at the following axiom of existence of small units:

$$\forall x \exists \varepsilon\colon (x \circ \varepsilon = x) \wedge \forall y\colon [(\exists t\colon \varepsilon \circ y = t) \Rightarrow (\varepsilon \circ y = y)] \wedge [(\exists t\colon y \circ \varepsilon = t) \Rightarrow (y \circ \varepsilon = y)]$$
$$\forall x \exists \varepsilon\colon (\varepsilon \circ x = x) \wedge \forall y\colon [(\exists t\colon \varepsilon \circ y = t) \Rightarrow (\varepsilon \circ y = y)] \wedge [(\exists t\colon y \circ \varepsilon = t) \Rightarrow (y \circ \varepsilon = y)]$$

It should be noted that for the original monoid **M** this axiom is an inessential extension of its theory. The axiom states that there exist some constants ε with some limit properties.

So, similarly to how we did it in the first part of the proof (for the category), we obtain the following construction. Obviously, $\{\varepsilon\} \in (M - Q - \{E\})$ defines a set of arrows of the category with the composition "$\circ$". $\{\varepsilon\}$ is the set of its unit arrows; the elements of $\{\varepsilon\}$ corresponding to each element $x \in (M - Q)$, (from the right or from the left) correspond to the objects of the category (i.e. — **cod x** or **dom x**); **Q** is a set of pairs of arrows for which there is no composition (all of **Q** can be identified with **0**, — the zero element, as we have already said); **E** corresponds to the identity functor of the category. □

Next follows a somewhat more detailed study of the procedures and properties used in the proof of Theorem 1.

Objects of an arbitrary category are identified with single arrows. And then we will look at the category as a set of arrows with a partially defined composition. Let us make the composition completely defined by adding a certain object ***0*** to the category, to which we will equate all non-existent compositions of arrows. We will also introduce the unit functor of the category into our construction as ***E***, the identity of the monoid. Since categorical composition is associative where it exists, we indeed include our category isomorphically in a minimal monoid with a large zero. Or, if you like, into the semigroup of such a monoid.

And vice versa, non-trivial (not ***E*** and not ***0***) elements of any monoid with an associative zero can be identified with arrows of a certain category by declaring zero products non-existent. And by associating ***E*** with its identity functor. This will also lead to an isomorphic immersion of the monoid into the minimal category.

As for logical equivalence, such artifacts as ***E*** and category objects (unlike its arrows) do not carry new information and are only insignificant extensions of the corresponding theories (and in category theory, ***0*** will not be a significant extension). We will describe the process in more details.

Let us pay attention to the fact that the existence of a composition of arrows (or elements of a monoid), being partial, is localized with respect to factors. And localization is performed at a more fundamental level – on sequences of elements. Let us start with the operation, which we call the *localization of the identity of the monoid with respect to the associative ideal **Q***.

Definition 2. An element e of our monoid is called *small*, or *local units*, if:

$$L(e) \equiv \forall x\colon [(ex \notin Q) \Rightarrow (ex = x)] \wedge [(xe \notin Q) \Rightarrow (xe = x)]$$

Let us add an (insignificant!) couple of axioms:

$$\forall x (x \notin Q) \exists e\colon L(e) \wedge (ex \notin Q), \qquad \forall x (x \notin Q) \exists e\colon L(e) \wedge (xe \notin Q)$$

(in exactly the same way we will designate and call (***e*** and ***L(e)***) unit arrows in the category, i.e. its objects (after all, they have the same properties); allowing "liberty of speech", we will also use the notation: "$e \in L$", meaning belonging to a collection of local units; now and in the future, by the word: "collection" we will mean a set or a class — in the problems we are studying, this is not important now).

This is how above we connected the objects (single arrows) of the category and the elements of the monoid. Actually, the unit of the monoid becomes a set of small units when it "transforms" into a category. And the set of objects of the category becomes the unit of the monoid in the reverse process. If we identify the elements with the operations that they perform (due to multiplication) on other elements, then the "big" unit $\boldsymbol{E}$ takes the element $\boldsymbol{x}$ from $\boldsymbol{M}$ into itself when acting from the left and right. When mapped onto some small unit (i.e., onto a single arrow, object $\boldsymbol{K}$), it will translate into itself a certain arrow (acting from the left or right). This arrow will reappear at $\boldsymbol{x}$. We get a commutative diagram.

In what follows, by $\boldsymbol{M1}$ we mean the monoid M extended by the collection of its small units, i.e., $M1 = M \cup L$.

By $\boldsymbol{K1}$ we mean the category $\boldsymbol{K}$ supplemented by two objects: $\boldsymbol{o}$ is the erasing functor of our $\boldsymbol{K}$ into the empty category, and $\boldsymbol{I}$ is the identity functor of $\boldsymbol{K}$ (its "big unit"). Those, is $K1 \equiv (I \cup K \cup o)$, together with two "outer" arrows, the "actions" of this pair of operators on $\boldsymbol{K}$.

More precisely, there is a homomorphic relation $\boldsymbol{R}$ between an arbitrary category and some monoid with an associative ideal. In other words, it is the same as with the usual homomorphism, but without the condition of functional uniqueness:

$$R \subseteq (M1 \times K1), \qquad \forall a(a \in M)\exists x(x \in K1)\colon R(a, x), \qquad [R(a, x) \wedge R(b, x)] \Rightarrow R(ab, xy).$$

Now let us define the properties:

$$R(E, I) \wedge \exists! x(x \in K1)\colon R(E, x), \qquad R(Q, o) \wedge \exists! x(x \in K1)\colon R(Q, x)$$

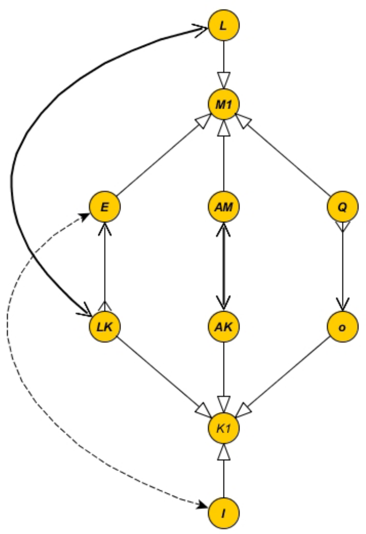
— the unit of the monoid always corresponds to the "large unit" of the category; the associative ideal is always completely transformed into the element $\boldsymbol{o}$.

$$\forall a(a \in (M1 - E, Q))\exists! x(x \in K)\colon R(a, x), \qquad \forall x(x \in K)\exists! a(a \in (M1 - E, Q))\colon R(a, x)$$

— an extended monoid without unity and zero ideal, is in one-to-one correspondence with category arrows;

$$\forall e(e \in M1)\colon L(e) \Rightarrow L(R(L(e))), \qquad \forall e(e \in K)\colon L(e) \Rightarrow L(R(L(e))),$$

— small units of the monoid and the category, in turn, also one-to-one correspond to each other. The structure of relation $\boldsymbol{R}$ is as follows:



where two-sided arrows denote isomorphisms, arrows with transparent "tips" denote inclusions in the direct sum (combinations of collections), arrows with a branched base are epimorphic mappings into a single element.

And these correspondences preserve the operation of composition where they exist. Moreover, if we also introduce a set of non-existent categorical compositions into consideration, we get something isomorphic to the zero ideal. Based on this ideal, it is possible to create certain structures that, in a certain sense, are additional to categorical ones. The category theory itself can be immersed in the category of monoids.

The proof is over.

This is the actual identification of categories with some type of monoids. And not always obvious immersion of the theory of monoids in the theory of categories. I think that such a representation can be more productive than the trivial identification of a monoid with a category of one object.

Statement. *Any mathematical theory can be constructed in a logic in which terms are localized with respect to individuals. The existence of sequences and the results of operations on them, in this case, is modeled non-standard.*

This proposition (quite obvious after considering the example of monoids and categories) most likely pertains to the meta-logical domain. It requires concrete implementations in other theories, not just in relation to monoids and categories. However, such implementations are possible as a result of further, independent research.

We have seen that the very existence of any sequence of elements can be localized with respect to each of its elements. And this leads to a transformation of any mathematical theory that uses terms, to a change in functions, operations, and even relations. The radical nature of such changes is clearly seen in the example of monoids and categories.

The category is a locally thermal monoid. Locally thermal theories are non-standard theories, with non-standard sequences and non-standard existence. Which is very similar to non-standard according to Robinson. All this uses certain modifications of ultrafilters and compactness theorems at various metatheoretical levels. And it allows you to somehow simulate contradictions and approach the solution of paradoxes.

The article "Predicates and terms from non-standard sequences", with a somewhat similar theme, was published by the author on "arXiv".

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