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HANKEL AND TOEPLITZ DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Let the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ be locally univalent for $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $0 \le \alpha < 1$. Then, $f \in M(\alpha)$ if and only if

$$\mathrm{Re}\left(\left(1 - z^2\right) \frac{f(z)}{z} \right) > \alpha, \quad z \in \mathbb{D}.$$

Due to their geometrical characteristics, this class has a significant impact on the theory of geometric functions. In the article we obtain sharp bounds for the second Hankel determinant

$$|H_2(2)(f)| = \left|a_2 a_4 - a_3^2\right|$$

and some Toeplitz determinants

$$|T_3(1)(f)| = \left|1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2\right|, \quad |T_3(2)(f)| = \left|a_2^3 - 2a_2 a_3^2 + 2a_3^2 a_4 - a_2 a_4^2\right|$$

of a subclass of analytic functions $M(\alpha)$ in the open unit disk $\mathbb{D}$.

1. Introduction and definitions. Let H be the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$, and let A be the subclass normalized by $f(0) = f'(0) - 1 = 0$, that is, the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}, \tag{1}$$

$a_0 = 0, a_1 = 1$. Let S be the subclass of A that consists of univalent (one-to-one) functions. A function $f \in A$ is said to be starlike (with respect to the origin) if $f(\mathbb{D})$ is starlike with respect to the origin, and convex if $f(\mathbb{D})$ is convex. Let $S^*(\alpha)$ and $C(\alpha)$ denote, respectively, the classes of starlike and convex functions of order α $(0 \le \alpha < 1)$ in S. It is well known that a function $f \in A$ belongs to $S^*(\alpha)$ if and only if,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}),$$

and that $f \in C(\alpha)$ if and only if,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

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Note that $S^*(0) =: S^*$and $C(0) =: C$.

Let $f \in A$ and be locally univalent for $z \in \mathbb{D}$, and $0 \leq \alpha < 1$. Then, $f \in M(\alpha)$ if and only if

$$\operatorname{Re}\left(\left(1 - z^2\right) \frac{f(z)}{z} \right) > \alpha, \quad z \in \mathbb{D}. \tag{2}$$

Due to their geometrical characteristics, this class has a significant impact on the theory of geometric functions. A function $f \in M(\alpha)$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ convex in the direction of the imaginary axis, i.e., for $w_1, w_2 \in f(\mathbb{D})$ such that $\operatorname{Re}(w_1) = \operatorname{Re}(w_2)$ the line segment $[w_1, w_2]$ lies in $f(\mathbb{D})$, with the additional property that there exist two points w_1, w_2 on the boundary of $f(\mathbb{D})$ for which $\{w_1 + it\colon t > 0\} \subset \mathbb{C}\backslash f(\mathbb{D})$ and $\{w_2 - it\colon t > 0\} \subset \mathbb{C}\backslash f(\mathbb{D})$ (see, e.g., [7, p.199]).

In this study, we find the sharp bound for the second Hankel determinant as well as the sharp bounds for a number of the Toeplitz determinants defined below, whose constituent coefficients are functions in $M(\alpha)$.

We begin by outlining the meanings of the Hankel and Toeplitz determinants for $f \in A$.

Let $f \in A$ be of form (1). The *qth Hankel determinant* is defined by

$$H_q(r)(f) = \begin{vmatrix} a_r & a_{r+1} & \dots & a_{r+q-1} \\ a_{r+1} & a_{r+2} & \dots & a_{r+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r+q-1} & a_{r+q} & \dots & a_{r+2q-2} \end{vmatrix}$$

for $q \geq 1$ and $r \geq 0$. In particular, $H_2(2)(f) = a_2a_4 - a_3^2$.

Hankel matrices naturally occur in a variety of applications in science, engineering, and other related fields such as signal processing, image processing, and control theory. The reader is referred to [8, 9] and the references therein for a study of Hankel matrices and polynomials.

Finding sharp bounds for the Hankel determinants of functions in A has been the subject of numerous papers in recent years. Many results about the second Hankel determinant $H_2(2)(f) = a_2a_4 - a_3^2$ when $f \in S$ and its subclasses are known, and in [2, 3, 4, 12], a summary of some of the more significant findings can be found.

Let $f \in A$ be given by (1). Then, the *qth Toeplitz determinant* is defined by

$$T_q(r)(f) = \begin{vmatrix} a_r & a_{r+1} & \dots & a_{r+q-1} \\ a_{r+1} & a_r & \dots & a_{r+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r+q-1} & a_{r+q-2} & \dots & a_r \end{vmatrix}$$

for $q \geq 1$ and $r \geq 0$. In particular, $T_3(1)(f) = 1 - 2a_2^2 + 2a_2^2a_3 - a_3^2$ and

$$T_3(2)(f) = a_2^3 - 2a_2a_3^2 + 2a_3^2a_4 - a_2a_4^2.$$

Toeplitz matrices and their determinants play an important role in several branches of mathematics and have many applications [13]. For information on applications of Toeplitz matrices to several areas of pure and applied mathematics, we refer to the survey article by Ye and Lim ([14]). However, research on Toeplitz determinants was only recently published in [?, 2].

The following results will be used for functions $p \in P$, the class of functions with positive real part in $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{k=1}^{\infty} d_k z^k. \tag{3}$$

Because the coefficients a_2, a_3, and a_4 will be our main focus, we also need Lemma 4, which can easily be deduced from (1), (2) and (3).

Lemma 1 ([5]). *Let $p \in P$ be given by (3), then $|d_k| \leq 2$, when $k \geq 2$. Also*

$$\left|d_2 - \frac{\upsilon}{2} d_1^2\right| \leq \max\{2,\ 2|\upsilon - 1|\} = \begin{cases} 2, & 0 \leq \upsilon \leq 2; \\ 2|\upsilon - 1|, & \textit{elsewhere.} \end{cases} \tag{4}$$

Lemma 2 ([6]). *If $p \in P$ is given by (3), then*

$$|d_r - \upsilon d_k d_{r-k}| \leq 2 \max\{1, |2\upsilon - 1|\}$$

for $\upsilon \in \mathbb{C}$ and $1 \leq k \leq r$.

Lemma 3 ([11]). *Assume that $p \in P$, with coefficients given by (3), and $d_1 \geq 0$. Then, for some complex valued ζ with $|\zeta| \leq 1$ and some complex-valued y with $|y| \leq 1$*

$$2d_2 = d_1^2 + y\left(4 - d_1^2\right), \quad 4d_3 = d_1^3 + 2\left(4 - d_1^2\right) d_1 y - d_1\left(4 - d_1^2\right) y^2 + 2\left(4 - d_1^2\right)\left(1 - |y|^2\right)\zeta.$$

Lemma 4. *Assume that $f \in M(\alpha)$, and is given by (1). Then*

$$a_2 = (1 - \alpha) d_1, \tag{5}$$
$$a_3 = (1 - \alpha) d_2 + 1, \tag{6}$$
$$a_4 = (1 - \alpha)(d_3 + d_1), \tag{7}$$
$$a_5 = (1 - \alpha)(d_2 + d_4) + 1, \tag{8}$$

where $d_1,\ d_2$, and $d_3,\ d_4$ are given by (3).

Proof. By (2) there exists $p \in P$ of the form (3) such that

$$\left(1 - z^2\right) \frac{f(z)}{z} = p(z)(1 - \alpha) + \alpha \qquad (z \in \mathbb{D}). \tag{9}$$

Substituting the series (1) and (3) into (9) by equating the coefficients we obtain (5)–(8). □

2. The second Hankel determinant $\mathbf{H_2(2)(f)}$. For the second Hankel determinant of $f \in M(\alpha)$, we will present the sharp bound.

Theorem 1. *If $f \in M(\alpha)$, $0 \leq \alpha < 1$, then*

$$|H_2(2)(f)| \leq \frac{4(1 - \alpha)(64 - 37\alpha) + 27}{27}.$$

This inequality is sharp.

Proof. Firstly, note that from (2) we can write

$$\left(1-z^2\right)\frac{f(z)}{z}=p\left(z\right)\left(1-\alpha\right)+\alpha\qquad\left(z\in\mathbb{D}\right). \tag{10}$$

Thus, from Lemma 4 we have

$$a_2a_4-a_3^2=\left(1-\alpha\right)^2d_1^2+\left(1-\alpha\right)_1^2d_1d_3-2\left(1-\alpha\right)d_2-\left(1-\alpha\right)^2d_2^2-1. \tag{11}$$

It should be noted that both the class $M(\alpha)$ and the functional $H_2(2)(f)$ are rotationally invariant, we now use Lemma 3 to express the coefficients d_3 and d_2 in terms of d_1, and write $u:=d_1$ to get with $0\le u\le 2$

$$a_2a_4-a_3^2=-\alpha\left(1-\alpha\right)u^2-\frac{\left(1-\alpha\right)^2}{4}u^2\left(4-u^2\right)y^2-\frac{\left(1-\alpha\right)^2}{4}\left(4-u^2\right)^2y^2-$$
$$-\left(1-\alpha\right)\left(4-u^2\right)y+\frac{\left(1-\alpha\right)^2}{2}u\left(4-u^2\right)\left(1-|y|^2\right)\zeta-1.$$

We now take the modulus to obtain

$$|H_2\left(2\right)\left(f\right)|\le\alpha\left(1-\alpha\right)u^2+\left(1-\alpha\right)\left(4-u^2\right)|y|+$$
$$+\frac{\left(1-\alpha\right)^2}{2}\left(4-u^2\right)\left(u+2\right)|y|^2+\frac{\left(1-\alpha\right)^2}{2}\left(4-u^2\right)u+1\ =\varphi\left(u,|y|\right).$$

For $u=2$, we have $|H_2\left(2\right)\left(f\right)|=4\alpha\left(1-\alpha\right)+1\ \le4\left(1-\alpha\right)\left(2-\alpha\right)+1$.

Since $0\le u<2$, the function $[0,1]\ni|y|\to\varphi(u,|y|)$ is easily seen to be increasing, so

$$|H_2\left(2\right)\left(f\right)|\le\varphi\left(u,|y|\right)\le\varphi\left(u,1\right)=$$
$$=\left(1-\alpha\right)\left[-\left(1-\alpha\right)u^3-2\left(1-\alpha\right)u^2+4\left(1-\alpha\right)u+8-4\alpha\right]+1.$$

Hence, the function $[0,2]\ni u\to\varphi(u,1)$ has critical points at $u=-2$ and $u=\frac{2}{3}=u_0$ with values $4\alpha\left(1-\alpha\right)+1$, and $\frac{4(1-\alpha)(64-37\alpha)}{27}+1$, respectively, and since

$$4\alpha\left(1-\alpha\right)+1\le\frac{4\left(1-\alpha\right)\left(64-37\alpha\right)}{27}+1$$

when $0\le\alpha<1$.

So, the proof of theorem is completed.

To see that the inequality is sharp, take a function

$$p\left(z\right)=\frac{1-z^2}{1-t_0z+z^2},\quad z\in\mathbb{D},$$

for which $d_1=\frac{2}{3}$, $d_2=-\frac{14}{9}$ and $d_3=\frac{26}{27}$. □

Choosing $\alpha=\frac{1}{2}$, we arrive at the following sharp inequality.

Corollary 1. *If $f\in M(\frac{1}{2})$, then*

$$|H_2\left(2\right)\left(f\right)|\le\frac{118}{27}\cong4,3703.$$

This inequality is sharp.

3. Toeplitz determinants. We will give the sharp bounds for various Toeplitz determinants of $f \in M(\alpha)$.

Theorem 2. *If* $f \in M(\alpha)$, $0 \le \alpha < 1$, *then*

$$|T_3(1)(f)| \le 2\left(15 - 8\alpha^3 + 30\alpha^2 - 36\alpha\right).$$

Proof. We first note that

$$\begin{aligned}|T_3(1)(f)| &= \left|1 - 2a_2^2 + 2a_2^2a_3 - a_3^2\right| \le 1 + 2|a_2|^2 + |a_3|\left|a_3 - 2a_2^2\right| \le \\ &\le 1 + 8(1-\alpha)^2 + (3 - 2\alpha)\left|a_3 - 2a_2^2\right| \qquad (12)\end{aligned}$$

where we have used Lemmas 1 and 4.

As a result, it is necessary to estimate $|a_3 - 2a_2^2|$.

Note first that

$$a_3 - 2a_2^2 = (1-\alpha)\left(d_2 - 2(1-\alpha)d_1^2\right) + 1.$$

Thus, taking $v = 4(1-\alpha)$, we derived from Lemma 1 that

$$\left|a_3 - 2a_2^2\right| \le 2(1-\alpha)(3 - 4\alpha) + 1,$$

and so, from (12), we obtain

$$|T_3(1)(f)| \le 2\left(15 - 8\alpha^3 + 30\alpha^2 - 36\alpha\right).$$

□

Choosing $\alpha = \frac{1}{2}$, we arrive at the following sharp inequality.

Corollary 2. *If* $f \in M(\frac{1}{2})$, $0 \le \alpha < 1$, *then*

$$|T_3(1)(f)| \le 7.$$

Theorem 3. *If* $f \in M(\alpha)$, $0 \le \alpha < 1$, *then*

$$|T_3(2)(f)| \le 6(1-\alpha)\left(12\alpha^2 - 32\alpha + 22\right).$$

Proof. We first note that

$$|T_3(2)(f)| = \left|(a_2 - a_4)({a_2}^2 + a_2a_4 - 2{a_3}^2)\right|,$$

and since $|a_2 - a_4| \le |a_2| + |a_4|$, we have

$$|a_2 - a_4| \le 6(1-\alpha).$$

Thus, it remains to estimate $|a_2^2 + a_2a_4 - 2a_3^2|$.

Using Lemma 4, we obtain

$$a_2^2 + a_2a_4 - 2a_3^2 = 2(1-\alpha)^2d_1^2 + (1-\alpha)_1^2d_1d_3 - 2(1-\alpha)^2d_2^2 - 4(1-\alpha)d_2 - 2.$$

Taking the modulus, we obtain

$$\left|a_2^2 + a_2a_4 - 2a_3^2\right| \le 12(1-\alpha)^2 + 2 + 4(1-\alpha)\left|d_2 - \frac{(1-\alpha)}{2}d_1^2\right|.$$

Since Lemma 1 gives $\left|d_2 - \frac{(1-\alpha)}{2}d_1^2\right| \le 2$, we obtain $\left|a_2^2 + a_2a_4 - 2a_3^2\right| \le 12\alpha^2 - 32\alpha + 22$, and so

$$|T_3(2)(f)| \le 6(1-\alpha)\left(12\alpha^2 - 32\alpha + 22\right)$$

as required. □

Choosing $\alpha = \frac{1}{2}$, we arrive at the following sharp inequality.

Corollary 3. *If* $f \in M(\frac{1}{2})$, *then*

$$|T_3(2)(f)| \leq 27.$$

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