HANKEL AND TOEPLITZ DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS


Let the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ be locally univalent for $z \in \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ and $0 \leq \alpha < 1$. Then, $f \in M(\alpha)$ if and only if
\[ \Re \left( \frac{(1 - z^2) f(z)}{z} \right) > \alpha, \quad z \in \mathbb{D}. \]

Due to their geometrical characteristics, this class has a significant impact on the theory of geometric functions. In the article we obtain sharp bounds for the second Hankel determinant $|H_2(2)(f)| = |a_2 a_4 - a_3^2|$ and some Toeplitz determinants $|T_3(1)(f)| = |1 - 2a_2^2 + 2a_2^2 a_3 - a_5^2|$, $|T_3(2)(f)| = |a_3^2 - 2a_2 a_3^2 + 2a_3 a_4 - a_2 a_4^2|$ of a subclass of analytic functions $M(\alpha)$ in the open unit disk $\mathbb{D}$.

1. Introduction and definitions. Let $H$ be the class of analytic functions in the unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$, and let $A$ be the subclass normalized by $f(0) = f'(0) - 1 = 0$, that is, the functions of the form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}, \]
for $a_0 = 0, a_1 = 1$. Let $S$ be the subclass of $A$ that consists of univalent (one-to-one) functions. A function $f \in A$ is said to be starlike (with respect to the origin) if $f(\mathbb{D})$ is starlike with respect to the origin, and convex if $f(\mathbb{D})$ is convex. Let $S^*(\alpha)$ and $C(\alpha)$ denote, respectively, the classes of starlike and convex functions of order $\alpha$ ($0 \leq \alpha < 1$) in $S$. It is well known that a function $f \in A$ belongs to $S^*(\alpha)$ if and only if,
\[ \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{D}, \]
and that $f \in C(\alpha)$ if and only if,
\[ \Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{D}. \]

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Note that $S^s(0) = S^s$ and $C(0) = C$.

Let $f \in A$ and be locally univalent for $z \in \mathbb{D}$, and $0 \leq \alpha < 1$. Then, $f \in M(\alpha)$ if and only if

$$\text{Re}\left(\left(1 - z^2\right)\frac{f(z)}{z}\right) > \alpha, \quad z \in \mathbb{D}. \quad (2)$$

Due to their geometrical characteristics, this class has a significant impact on the theory of geometric functions. A function $f \in M(\alpha)$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ convex in the direction of the imaginary axis, i.e., for $w_1, w_2 \in f(\mathbb{D})$ such that $\text{Re}(w_1) = \text{Re}(w_2)$ the line segment $[w_1, w_2]$ lies in $f(\mathbb{D})$, with the additional property that there exist two points $w_1, w_2$ on the boundary of $f(\mathbb{D})$ for which $\{w_1 + it: t > 0\} \subset C \setminus f(\mathbb{D})$ and $\{w_2 - it: t > 0\} \subset C \setminus f(\mathbb{D})$ (see, e.g., [7, p.199]).

In this study, we find the sharp bound for the second Hankel determinant as well as the sharp bounds for a number of the Toeplitz determinants defined below, whose constituent coefficients are functions in $M(\alpha)$.

We begin by outlining the meanings of the Hankel and Toeplitz determinants for $f \in A$.

Let $f \in A$ be of form (1). The $qth$ Hankel determinant is defined by

$$H_q(r)(f) = \begin{vmatrix} a_r & a_{r+1} & \ldots & a_{r+q-1} \\ a_{r+1} & a_{r+2} & \ldots & a_{r+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+q-1} & a_{r+q} & \ldots & a_{r+2q-2} \end{vmatrix}$$

for $q \geq 1$ and $r \geq 0$. In particular, $H_2(2)(f) = a_2a_4 - a_3^2$.

Hankel matrices naturally occur in a variety of applications in science, engineering, and other related fields such as signal processing, image processing, and control theory. The reader is referred to [8, 9] and the references therein for a study of Hankel matrices and polynomials.

Finding sharp bounds for the Hankel determinants of functions in $A$ has been the subject of numerous papers in recent years. Many results about the second Hankel determinant $H_2(2)(f) = a_2a_4 - a_3^2$ when $f \in S$ and its subclasses are known, and in [2, 3, 4, 12], a summary of some of the more significant findings can be found.

Let $f \in A$ be given by (1). Then, the $qth$ Toeplitz determinant is defined by

$$T_q(r)(f) = \begin{vmatrix} a_r & a_{r+1} & \ldots & a_{r+q-1} \\ a_{r+1} & a_r & \ldots & a_{r+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+q-1} & a_{r+q-2} & \ldots & a_r \end{vmatrix}$$

for $q \geq 1$ and $r \geq 0$. In particular, $T_3(1)(f) = 1 - 2a_2^2 + 2a_3^2a_4 - a_3^2$ and $T_3(2)(f) = a_2^3 - 2a_2a_3^2 + 2a_3^2a_4 - a_2a_4^2$.

Toeplitz matrices and their determinants play an important role in several branches of mathematics and have many applications [13]. For information on applications of Toeplitz matrices to several areas of pure and applied mathematics, we refer to the survey article by Ye and Lim ([14]). However, research on Toeplitz determinants was only recently published in [?, 2].
The following results will be used for functions $p \in P$, the class of functions with positive real part in $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{k=1}^{\infty} d_k z^k. \quad (3)$$

Because the coefficients $a_2, a_3,$ and $a_4$ will be our main focus, we also need Lemma 4, which can easily be deduced from (1), (2) and (3).

**Lemma 1** ([5]). Let $p \in P$ be given by (3), then $|d_k| \leq 2$, when $k \geq 2$. Also

$$|d_2 - \frac{v}{2} d_1^2| \leq \max \{2, 2|v - 1|\} = \begin{cases} 2, & 0 \leq v \leq 2; \\ 2|v - 1|, & \text{elsewhere.} \end{cases} \quad (4)$$

**Lemma 2** ([6]). If $p \in P$ is given by (3), then

$$|d_r - v d_k d_{r-k}| \leq 2 \max \{1, |2v - 1|\}$$

for $v \in \mathbb{C}$ and $1 \leq k \leq r$.

**Lemma 3** ([11]). Assume that $p \in P$, with coefficients given by (3), and $d_1 \geq 0$. Then, for some complex valued $\zeta$ with $|\zeta| \leq 1$ and some complex-valued $y$ with $|y| \leq 1$

$$2d_2 = d_1^2 + y (4 - d_1^2), \quad 4d_3 = d_1^2 + 2 (4 - d_1^2) d_1 y - d_1 (4 - d_1^2) y^2 + 2 (4 - d_1^2) (1 - |y|^2) \zeta.$$ 

**Lemma 4.** Assume that $f \in M(\alpha)$, and is given by (1). Then

$$a_2 = (1 - \alpha) d_1, \quad (5)$$

$$a_3 = (1 - \alpha) d_2 + 1, \quad (6)$$

$$a_4 = (1 - \alpha) (d_3 + d_1), \quad (7)$$

$$a_5 = (1 - \alpha) (d_2 + d_4) + 1, \quad (8)$$

where $d_1$, $d_2$, and $d_3$, $d_4$ are given by (3).

**Proof.** By (2) there exists $p \in P$ of the form (3) such that

$$(1 - z^2) \frac{f(z)}{z} = p(z) (1 - \alpha) + \alpha \quad (z \in \mathbb{D}). \quad (9)$$

Substituting the series (1) and (3) into (9) by equating the coefficients we obtain (5)--(8). \qed

2. The second Hankel determinant $H_2(2)(f)$. For the second Hankel determinant of $f \in M(\alpha)$, we will present the sharp bound.

**Theorem 1.** If $f \in M(\alpha), 0 \leq \alpha < 1$, then

$$|H_2(2)(f)| \leq \frac{4 (1 - \alpha) (64 - 37\alpha) + 27}{27}.$$ 

This inequality is sharp.
Proof. Firstly, note that from (2) we can write
\[ (1 - z^2) \frac{f(z)}{z} = p(z) (1 - \alpha) + \alpha \quad (z \in \mathbb{D}). \] (10)

Thus, from Lemma 4 we have
\[ a_2a_4 - a_3^2 = (1 - \alpha)^2 d_1^2 + (1 - \alpha)^2 d_1 d_3 - 2 (1 - \alpha) d_2 - (1 - \alpha)^2 d_2^2 - 1. \] (11)

It should be noted that both the class \( M(\alpha) \) and the functional \( H_2(2)(f) \) are rotationally invariant, we now use Lemma 3 to express the coefficients \( d_3 \) and \( d_2 \) in terms of \( d_1 \), and write \( u := d_1 \) to get with \( 0 \leq u \leq 2 \)
\[
a_2a_4 - a_3^2 = -\alpha (1 - \alpha) u^2 - \frac{(1 - \alpha)^2}{4} u^2 (4 - u^2) y^2 - \frac{(1 - \alpha)^2}{4} (4 - u^2) y^2 -
- (1 - \alpha) (4 - u^2) y + \frac{(1 - \alpha)^2}{2} u (4 - u^2) (1 - |y|^2) \zeta - 1.
\]

We now take the modulus to obtain
\[
|H_2(2)(f)| \leq \alpha (1 - \alpha) u^2 + (1 - \alpha) (4 - u^2) |y| + 
+ \frac{(1 - \alpha)^2}{2} (4 - u^2) (u + 2) |y|^2 + \frac{(1 - \alpha)^2}{2} (4 - u^2) u + 1 = \varphi(u, |y|).
\]

For \( u = 2 \), we have \( |H_2(2)(f)| = 4\alpha (1 - \alpha) + 1 \leq 4 (1 - \alpha) (2 - \alpha) + 1. \)

Since \( 0 \leq u < 2 \), the function \( [0, 1] \ni |y| \to \varphi(u, |y|) \) is easily seen to be increasing, so
\[
|H_2(2)(f)| \leq \varphi(u, |y|) \leq \varphi(u, 1) = 
= (1 - \alpha) \left[ -(1 - \alpha) u^3 - 2 (1 - \alpha) u^2 + 4 (1 - \alpha) u + 8 - 4\alpha \right] + 1.
\]

Hence, the function \( [0, 2] \ni u \to \varphi(u, 1) \) has critical points at \( u = -2 \) and \( u = \frac{2}{3} = u_0 \) with values \( 4\alpha (1 - \alpha) + 1 \), and \( \frac{4(1-\alpha)(64-37\alpha)}{27} + 1 \), respectively, and since
\[
4\alpha (1 - \alpha) + 1 \leq \frac{4 (1 - \alpha) (64 - 37\alpha)}{27} + 1
\]
when \( 0 \leq \alpha < 1 \).

So, the proof of theorem is completed.

To see that the inequality is sharp, take a function
\[
p(z) = \frac{1 - z^2}{1 - t_0 z + z^2}, \quad z \in \mathbb{D},
\]

for which \( d_1 = \frac{2}{3}, \ d_2 = -\frac{14}{9} \) and \( d_3 = \frac{26}{27} \).

Choosing \( \alpha = \frac{1}{2} \), we arrive at the following sharp inequality.

Corollary 1. If \( f \in M(\frac{1}{2}) \), then
\[
|H_2(2)(f)| \leq \frac{118}{27} \approx 4.3703.
\]

This inequality is sharp.
3. Toeplitz determinants. We will give the sharp bounds for various Toeplitz determinants of $f \in M(\alpha)$.

Theorem 2. If $f \in M(\alpha)$, $0 \leq \alpha < 1$, then

$$|T_3(1)(f)| \leq 2 \left(15 - 8\alpha^3 + 30\alpha^2 - 36\alpha\right).$$

Proof. We first note that

$$|T_3(1)(f)| = |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \leq 1 + 2|a_2|^2 + |a_3| |a_3 - 2a_2^2| \leq 1 + 8(1 - \alpha)^2 + (3 - 2\alpha) |a_3 - 2a_2^2|$$

(12)

where we have used Lemmas 1 and 4.

As a result, it is necessary to estimate $|a_3 - 2a_2^2|$.

Note first that

$$a_3 - 2a_2^2 = (1 - \alpha) \left(d_2 - 2 (1 - \alpha) d_1^2\right) + 1.$$ 

Thus, taking $v = 4 (1 - \alpha)$, we derived from Lemma 1 that

$$|a_3 - 2a_2^2| \leq 2 (1 - \alpha) (3 - 4\alpha) + 1,$$

and so, from (12), we obtain

$$|T_3(1)(f)| \leq 2 \left(15 - 8\alpha^3 + 30\alpha^2 - 36\alpha\right).$$

Choosing $\alpha = \frac{1}{2}$, we arrive at the following sharp inequality.

Corollary 2. If $f \in M\left(\frac{1}{2}\right)$, $0 \leq \alpha < 1$, then

$$|T_3(1)(f)| \leq 7.$$

Theorem 3. If $f \in M(\alpha)$, $0 \leq \alpha < 1$, then

$$|T_3(2)(f)| \leq 6 (1 - \alpha) \left(12\alpha^2 - 32\alpha + 22\right).$$

Proof. We first note that

$$|T_3(2)(f)| = \left|(a_2 - a_4)(a_2^2 + a_2a_4 - 2a_3^2)\right|,$$

and since $|a_2 - a_4| \leq |a_2| + |a_4|$, we have

$$|a_2 - a_4| \leq 6(1 - \alpha).$$

Thus, it remains to estimate $|a_2^2 + a_2a_4 - 2a_3^2|$.

Using Lemma 4, we obtain

$$a_2^2 + a_2a_4 - 2a_3^2 = 2(1 - \alpha)^2 d_1^2 + (1 - \alpha)^2 d_1 d_3 - 2(1 - \alpha) d_2^2 - 4 (1 - \alpha) d_2 - 2.$$

Taking the modulus, we obtain

$$\left|a_2^2 + a_2a_4 - 2a_3^2\right| \leq 12(1 - \alpha)^2 + 2 + 4 (1 - \alpha) \left|d_2 - \frac{(1 - \alpha) d_1^2}{2}\right|.$$

Since Lemma 1 gives $|d_2 - \frac{(1 - \alpha) d_1^2}{2}| \leq 2$, we obtain $|a_2^2 + a_2a_4 - 2a_3^2| \leq 12\alpha^2 - 32\alpha + 22$, and so

$$|T_3(2)(f)| \leq 6 (1 - \alpha) \left(12\alpha^2 - 32\alpha + 22\right)$$

as required.
Choosing \( \alpha = \frac{1}{2} \), we arrive at the following sharp inequality.

**Corollary 3.** If \( f \in M(\frac{1}{2}) \), then

\[
|T_3(2)(f)| \leq 27.
\]

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**REFERENCES**


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