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AN EXACT ESTIMATE OF THE THIRD HANKEL DETERMINANTS FOR FUNCTIONS INVERSE TO CONVEX FUNCTIONS

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Invesigation of bounds for Hankel determinant of analytic univalent functions has been interesting for many researchers since early twentieth century to study geometric properties. Many authors obtained non sharp upper bounds of the third Hankel determinat for different subclasses of analytic univalent functions until Kwon et al. [5] obtained exact estimation of the fourth coefficient of Carathéodory class. Recently the authors using an exact estimation of the fourth coefficient, well known second and third coefficient of Carathéodory class have obtained a sharp bound for the third Hankel determinant associated with subclasses of analytic univalent functions.

Let $w = f(z) = z + a_2z^2 + \cdots$ be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the subclass of normalized univalent functions with $f(0) = 0$, and $f'(0) = 1$. Let $z = f^{-1}$ be the inverse function of f, given by $f^{-1}(w) = w + t_2w^2 + \cdots$ for some $|w| < r_o(f)$. Let $\mathcal{S}^c \subset \mathcal{S}$ be the subset of convex functions in $\mathbb{D}$. In this paper, we estimate the best possible upper bound for the third Hankel determinant for the inverse function $z = f^{-1}$ when $f \in \mathcal{S}^c$.

Let $\mathcal{S}^c$ be the class of convex functions. We prove the following statement (Theorem 1): If $f \in \mathcal{S}^c$, then

$$\left|H_{3,1}(f^{-1})\right| \leq \frac{1}{36}$$

and the inequality is attained for $p_0(z) = (1 + z^3)/(1 - z^3)$.

1. Introduction. Let us denote by $\mathcal{H}$ the family of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and by $\mathcal{A}$ the subfamily of functions f normilized by the conditions $f(0) = f'(0) - 1 = 0$, i.e. of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \ a_1 := 1. \tag{1}$$

Let $\mathcal{S}$ be the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. For $f \in \mathcal{S}$, the inverse f^{-1} is given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, \quad |w| < r_o(f); \left(r_o(f) \geq \frac{1}{4}\right). \tag{2}$$

A typical problem in the geometric function theory is to study some functionals. Each time the appearance of such functionals is dictated by the need to study the geometric

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properties of conformal mappings. Estimates of such functionals are obtained by expressing them explicitly or implicitly through Taylor coefficients. Hankel determinants are one of such functionals. For the positive integers $r,\ n$, Pommerenke [10] characterized the r^{th}-Hankel determinant of n^{th}-order for f given in (1) defined as follows

$$H_{r,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{3}$$

The problem of finding sharp estimates of the third Hankel determinant (at $r=3$ and $n=1$ in (3)), is technically much tough than that at $r=n=2$.

In recent years, many authors are working on obtaining the sharp upper bound to $|H_{3,1}(f)|$ for certain subclasses of analytic functions (see [2, 3, 4, 6, 12, 14]).

Ali [1] estimated sharp bounds for the first four coefficients and the Fekete-Szegő coefficient functional of the inverse functions which belong to the class of strongly starlike functions denoted by $\mathcal{SS}^*(\alpha)$ defined as $|\arg\left(zf'(z)/f(z)\right)|<\pi\alpha/2,\ (0<\alpha\leq 1)$. Sim et al. [13] investigated a sharp bound of $|H_{2,2}(f^{-1})|$ for the class of strongly Ozaki functions $\mathcal{F}_o(\lambda)$ defined as $\operatorname{Re}\left\{1+\left(zf''(z)/f'(z)\right)\right\}<\left(1-2\lambda\right)/2,\ (1/2\leq\lambda\leq 1)$.

Recently Lecko et al. [3] obtained the sharp bound for the class of convex functions $\mathcal{S}^c$, defined by

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0. \tag{4}$$

Motivated by these results, in this paper we obtain the sharp estimate for $H_{3,1}(f^{-1})$ when $f\in\mathcal{S}^c$ as $1/36$.

Let $\mathcal{P}$ be the class of all functions p having a positive real part in $\mathbb{D}$,

$$p(z)=1+\sum_{t=1}^{\infty}c_t z^t. \tag{5}$$

Every such function is called a Carathéodory function. In view of (4) and (5), the coefficients of the functions in $\mathcal{S}^c$ can be expressed in terms of coefficients of functions in $\mathcal{P}$. We then obtain the upper bound of $|H_{3,1}(f^{-1})|$, buliding our analysis on the familiar formulas for coefficients c_2 (see, [9, p. 166]), c_3 (see [7, 8]) and c_4 (can be found in [11]).

The foundation for proofs of our main results is the following lemma and we adopt the procedure framed through Libera and Zlotkiewicz [8].

Lemma 1 ([11]). *If $p\in\mathcal{P}$ is of the form (5) with $c_1\geq 0$ such that $c_1\in[0,2]$ then*

$$\begin{gathered} 2c_2=c_1^2+\nu\mu,\quad 4c_3=c_1^3+2c_1\nu\mu-c_1\nu\mu^2+2\nu\left(1-|\mu|^2\right)\rho, \\ 8c_4=c_1^4+3c_1^2\nu\mu+\left(4-3c_1^2\right)\nu\mu^2+c_1^2\nu\mu^3+4\nu\left(1-|\mu|^2\right)\left(1-|\rho|^2\right)\psi+ \\ +4\nu\left(1-|\mu|^2\right)\left(c_1\rho-c\mu\rho-\bar{\mu}\rho^2\right), \end{gathered}$$

where $\nu:=4-c_1^2$ for some μ, ρ and ψ such that $|\mu|\leq 1$, $|\rho|\leq 1$ and $|\psi|\leq 1$.

2. Main result. We now prove the main theorem of this paper.

Theorem 1. *If $f \in \mathcal{S}^c$, then*

$$\left|H_{3,1}(f^{-1})\right| \leq \frac{1}{36},$$

and the inequality is attained for $p_0(z) = (1+z^3)/(1-z^3)$.

Proof. For $f \in \mathcal{S}^c$, there exists a holomorphic function $p \in \mathcal{P}$ such that

$$\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = p(z) \Leftrightarrow \{f'(z) + zf''(z)\} = p(z)f'(z). \tag{6}$$

Using the series representation for f and p in (6), a simple calculation gives

$$\begin{aligned} a_2 = \frac{c_1}{2},\ a_3 = \frac{c_1^2 + c_2}{6},\ a_4 = \frac{1}{12}\left[\frac{1}{2}c_1^3 + \frac{3}{2}c_1c_2 + c_3\right], \\ a_5 = \frac{1}{20}\left[\frac{1}{6}c_1^4 + c_1^2c_2 + \frac{1}{2}c_2^2 + \frac{4}{3}c_1c_3 + c_4\right]. \end{aligned} \tag{7}$$

Now from the definition (2), we have

$$w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n(f^{-1}(w))^n. \tag{8}$$

Further, we have

$$w = f(f^{-1}) = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n\left(w + \sum_{n=2}^{\infty} t_n w^n\right)^n. \tag{9}$$

Upon simplification, we obtain

$$\begin{aligned} (t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + \\ +(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \ldots\ldots = 0. \end{aligned} \tag{10}$$

Equating the coefficients at the same powers in (10), upon simplification, we obtain

$$\begin{aligned} t_2 = -a_2;\ t_3 = \{-a_3 + 2a_2^2\};\ t_4 = \{-a_4 + 5a_2a_3 - 5a_2^3\}; \\ t_5 = \{-a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4\}. \end{aligned} \tag{11}$$

Using the values of $a_n (n = 2, 3, 4, 5)$ from (7) in (11), upon simplification, we obtain

$$\begin{aligned} t_2 = -\frac{c_1}{2},\ t_3 = \frac{1}{6}\left(2c_1^2 - c_2\right),\ t_4 = \frac{1}{24}\left(-6c_1^3 + 7c_1c_2 - 2c_3\right), \\ t_5 = \frac{1}{120}\left(-6c_4 + 22c_1c_3 - 46c_1^2c_2 + 7c_2^2 + 24c_1^4\right). \end{aligned} \tag{12}$$

Now,

$$H_{3,1}(f^{-1}) = \begin{vmatrix} t_1 = 1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix}. \tag{13}$$

Using the values of t_j, $(j=2,3,4,5)$ from (12) in (13), it simplifies to give

$$H_{3,1}(f^{-1}) = \frac{1}{8640}\left[4c_1^6 - 24c_1^4c_2 + 12c_1^3c_3 + 39c_1^2c_2^2 - 44c_2^3 + 36c_1c_2c_3 - \right.$$
$$\left. -36c_1^2c_4 - 60c_3^2 + 72c_2c_4\right]. \tag{14}$$

In view of (14), using the values of c_2, c_3 and c_4 from Lemma 1 we obtain

$$\begin{aligned}
24c_1^4c_2 &= 12\left[c_1^6 + c_1^4\nu\mu\right];\\
12c_1^3c_3 &= 3\left[c_1^6 + 2c_1^4\nu\mu - c_1^4\nu\mu^2 + 2c_1^3\nu(1-|\mu|^2)\rho\right];\\
44c_2^3 &= \frac{11}{2}\left[c_1^6 + 3c_1^4\nu\mu + 3c_1^2\nu^2\mu^2 + \nu^3\mu^3\right];\\
39c_1^2c_2^2 &= \frac{39}{4}\left[c_1^6 + 2c_1^4\nu\mu + c_1^2\nu^2\mu^2\right];\\
36c_1c_2c_3 &= \frac{9}{2}\left[c_1^6 + 3c_1^4\nu\mu + 2c_1^2\nu^2\mu^2 - c_1^4\nu\mu^2 - c_1^2\nu^2\mu^3 + 2\nu\left(c_1^3 + c_1\nu\mu\right)\left(1-|\mu|^2\right)\rho\right];\\
60c_3^2 &= \frac{15}{4}\left[c^6 + 4c^4\nu\mu + 4c^4\nu^2\mu^2 - 2c^4\nu\mu^2 - 4c^2\nu^2\mu^3 + c^2\nu^2\mu^4 + \right.\\
&\quad \left. +4\nu(c^3 + 2c\nu\mu - c\nu\mu^2)(1-|\mu|^2)\rho + 4\nu^2(1-|\mu|^2)^2\rho^2\right];\\
72c_2c_4 - 36c_1^2c_4 &= \frac{9}{2}[c_1^4\nu\mu + 3c_1^2\nu^2\mu^2 + (4-3c_1^2)\nu^2\mu^3 + c_1^2\nu^2\mu^4 + \\
&\quad +4\nu^2c_1\mu(1-\mu)(1-|\mu|^2)\rho - 4\nu^2(1-|\mu|^2)|\mu|^2\rho^2 + 4\nu^2(1-|\mu|^2)(1-|\rho|^2)\mu\psi].
\end{aligned} \tag{15}$$

Inputting the values from (15) in the expression (14), after simplification, we get

$$H_{3,1}(f^{-1}) = \frac{1}{8640}\left[\frac{3}{4}c_1^2\nu^2\mu^2 - 3c_1^2\nu^2\mu^3 + \frac{3}{4}c_1^2\nu^2\mu^4 - \frac{11}{2}\nu^3\mu^3 + 18\nu^2\mu^3 - \right.$$
$$-\left(3c_1\nu^2\mu + 3c_1\nu^2\mu^2\right)\left(1-|\mu|^2\right)\rho - 3\nu^2\left(5+|\mu|^2\right)\left(1-|\mu|^2\right)\rho^2 + \tag{16}$$
$$\left. +18\nu^2\mu\left(1-|\mu|^2\right)\left(1-|\rho|^2\right)\psi\right].$$

Putting $c := c_1$ and taking $\nu = (4-c^2)$ in (16), we obtain

$$H_{3,1}(f^{-1}) = \frac{(4-c^2)^2}{8640}\left[\frac{3}{4}c^2\mu^2 + \frac{3}{2}c^2\mu^3 + \frac{3}{4}c^2\mu^4 - (4-c^2)\mu^3 - \right.$$
$$\left. -3c\mu(1+\mu)(1-|\mu|^2)\rho - 3(5+|\mu|^2)(1-|\mu|^2)\rho^2 + 18\mu(1-|\mu|^2)(1-|\rho|^2)\psi\right]. \tag{17}$$

Taking modulus on both sides of (17), using $|\mu| = x \in [0,1]$, $|\rho| = y \in [0,1]$, $c_1 = c \in [0,2]$ and $|\psi| \leq 1$, we obtain

$$\left|H_{3,1}(f^{-1})\right| \leq \frac{\vartheta(c,x,y)}{8640}, \tag{18}$$

where $\vartheta\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\vartheta(c,x,y) = (4-c^2)^2\left[\frac{3}{4}c^2x^2 + \frac{3}{2}c^2x^3 + \frac{3}{4}c^2x^4 + (4-c^2)x^3 + \right.$$
$$\left. +18x(1-x^2) + 3cx(1+x)(1-x^2)y + 3(5-x)(1-x)(1-x^2)y^2\right]. \tag{19}$$

We can easily observe that $\vartheta(c,x,y)$ is an increasing function of y, since $3cx(1+x)\times\times(1-x^2)>0$ and $3(5-x)(1-x)(1-x^2)>0$ for $(c,x)\in[0,2]\times[0,1]$.

Hence,

$$\vartheta(c,x,y)\le\vartheta(c,x,1)=\left(4-c^2\right)^2\left[\frac{3c^2x^4}{4}+\frac{3c^2x^3}{2}+\left(4-c^2\right)x^3+\frac{3c^2x^2}{4}+\right.$$
$$\left.+3c(x+1)\left(1-x^2\right)x+3\left(1-x^2\right)\left(x^2+5\right)\right]=$$
$$=\left(4-c^2\right)^2\left[15+3cx+\left(-12+3c+\frac{3c^2}{4}\right)x^2+\left(4-3c+\frac{c^2}{2}\right)x^3+\left(-3-3c+\frac{3c^2}{4}\right)x^4\right]\le$$
$$\le\left(4-c^2\right)^2\left[15+3cx+\left(-12+3c+\frac{3c^2}{4}\right)x^2+\left(4-3c+\frac{c^2}{2}\right)x^2\right]=$$
$$=\left(4-c^2\right)^2\left[15+3cx+\left(-8+\frac{5c^2}{4}\right)x^2\right]:=\Psi(c,x),\ (c,x)\in[0,2]\times[0,1]. \tag{20}$$

For $c=0$ and $c=2$, we obtain

$$\Psi(0,x)=16(15-8x^2)\le 240 \text{ and } \Psi(2,x)=0, \quad \text{for } x\in[0,1]. \tag{21}$$

For $x=0$ and $x=1$, we have

$$\Psi(c,0)=15(4-c^2)^2\le 240 \text{ and } \Psi(c,1)=(4-c^2)^2\left(7+3c+\frac{5c^2}{4}\right)\le 125,\ \text{for } c\in[0,2]. \tag{22}$$

Now, it remains to show that $\Psi(c,x)\le 240$ on $(c,x)\in(0,2)\times(0,1)$. We have $\partial\Psi/\partial x=0$ if and only if

$$x=\frac{-6c^5+48c^3-96c}{5c^6-72c^4+336c^2-512}:=x_0\in(0,1),\quad \frac{\partial^2\Psi}{\partial x^2}(c,x_0)=2\left(4-c^2\right)^2\left(-8+\frac{5c^2}{4}\right)<0.$$

Therefore $\Psi(c,x)$ attains maximum at (c,x_0). Hence

$$\Psi(c,x)\le\Psi(c,x_0)=\frac{6(-4+c^2)^2(-80+11c^2)}{-32+5c^2}<240. \tag{23}$$

In view of equations (20) – (23) we obtain

$$\max\left\{\vartheta(c,x,y):c\in[0,2],\ x\in[0,1],\ y\in[0,1]\right\}=240. \tag{24}$$

From expression (18) and (24), we get $|H_{3,1}(f^{-1})|\le\frac{1}{36}$.

For $p_0\in\mathcal{P}$, we obtain $t_2=t_3=t_5=0,\ t_4=1/6$, which implies the result. □

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