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ON RECOVERING THE SHAPE OF A QUANTUM TREE FROM THE SPECTRUM OF THE DIRICHLET BOUNDARY PROBLEM

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Spectral problems are considered generated by the Sturm-Liouville equation on equilateral trees with the Dirichlet boundary conditions at the pendant vertices and continuity and Kirchhoff's conditions at the interior vertices. It is proved that there are no co-spectral (i.e., having the same spectrum of such problem) among equilateral trees of ≤ 8 vertices. All co-spectral trees of 9 vertices are presented.

1. Introduction. It is usual in quantum graph theory to consider spectral problems generated by the Sturm-Liouville (Schrödinger) equations on equilateral metric graph domains with the Neumann or Dirichlet boundary conditions at the graphs pendant vertices and standard or in other words generalized Neumann (continuity and Kirchhoff's) conditions at its interior vertices. Here the problem of co-spectrality arises as well as in the classical graph theory.

It was shown in [2] that there exist co-spectral graphs (non-isomorphic graphs with the same spectrum of the Sturm-Liouville problem) in quantum graph theory. The example mentioned in [2] shows two non-isomorphic equilateral metric graphs of Fig. 1.

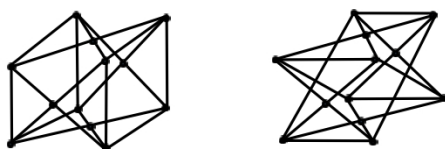


Figure 1. Nonisomorphic graphs with the same spectrum of the discrete Laplacians.

It should be noticed that in the case of graphs with non-commensurate edges the spectrum uniquely determines the shape of the graph [11].

The spectra of quantum graph problems on equilateral metric graphs are related to the normalized Laplacians of the corresponding combinatorial graphs in the following way: the eigenvalues of the normalized Laplacian are in one-to-one correspondence with the coefficients in the second term of the asymptotics of the eigenvalues of the Sturm-Liouville problem with standard conditions at the interior vertices and Neumann conditions at the pendant vertices

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of this graph (see [8] where the results of [3], [7], [10] and of [5] were used). This enables to obtain information on the form of a graph using the asymptotics of the eigenvalues.

Different versions of Ambarzumian's theorem for spectral problems on graphs were proved in [6], [12], [20], [18]. In [13], [4] a 'geometric' Ambarzumian's theorem was proved stating that if the spectrum of the Sturm-Liouville problem with the Neumann boundary conditions is such as in the case of the problem on a single interval with the zero potential then the graph is P_2 and the potential of the Sturm-Liouville equation is zero. In [8] a geometric Ambarzumian's theorem was proved for connected simple compact equilateral graphs of 5 or less vertices and for trees of 8 or less vertices. This theorem states that if the spectrum of the Sturm-Liouville problem with the Neumann boundary conditions at the pendant vertices and standard conditions at the interior vertices is such as the spectrum of this problem in case of zero potentials on the edges then this spectrum uniquely determines the shape of the graph and the zero potentials on the edges.

However, this result cannot be extended to the case of connected simple equilateral graphs of 6 vertices.

It is known [8] that the eigenvalues of the normalized Laplacian can be found from the asymptotics of eigenvalues of the Sturm-Liouville problem on a graph not only in the case of 'Ambarzumian's' asymptotics. Thus, putting aside Ambarzumian's theorem and the potentials admitting them to be arbitrary real L_2 functions, we put a question: can we find the shape of a simple connected equilateral graph using the asymptotics of the eigenvalues of the Sturm-Liouville spectral problem with the Neumann boundary conditions at the pendant vertices and standard conditions at the interior vertices? It follows from the results of [8] that if the number of vertices is ≤ 5 then the spectrum uniquely determines the shape of the (simple connected graph). In case of 6 vertices there is only one pair of co-spectral graphs (see [17], [9]). In case of trees the critical number of vertices is 9 ([17], [9]).

In present paper we are interested in what information on the shape of a tree can be obtained from the eigenvalue asymptotics in case of the Dirichlet conditions at the pendant vertices. It should be mentioned that in [1], [16] admitting Dirichlet conditions at some of the vertices a method for constructing families of co-spectral systems is proposed, using linear representations of finite groups. Finally, we mention paper [14] where the recovering of the shape of a graphs with an attached lead was considered and where the imaginary part of the Jost function is related with the characteristic function of a spectral problem with the Dirichlet condition at the vertex incident with the lead.

In Section 2 we formulate the spectral Sturm-Liouville problem on an equilateral tree with the standard conditions at the interior vertices and the Dirichlet conditions at the pendant vertices.

In Section 3 we give some auxiliary results. We show the relations between the spectrum of the Sturm-Liouville problem and the spectrum of the modified normalized Laplacian of the graphs interior sub-graph in the case of the Dirichlet conditions at the pendant vertices.

In Section 4 we show that if the number of vertices in a tree does not exceed 8 then the spectrum of the Dirichlet problem uniquely determines the shape of the tree.

In Section 5 we show all pairs or triples of non-isomorphic trees of 9 vertices having the same first and second terms in the asymptotics of the Dirichlet spectral problem.

2. Statement of the problems. Let G be a simple connected equilateral graph with $p \geq 3$ vertices, p_{pen} pendant vertices, g edges of the length l each. We denote by v_i the vertices, by $d(v_i)$ their degrees, by e_j the edges. We direct each peripheral (incident with the a pendant

vertex) edge away from its pendant vertex.

Orientation of the rest of the edges is arbitrary. Thus, for any interior vertex we consider the indegree $d^+(v_i)$ and its outdegree by $d^-(v_i) = d(v_i) - d^+(v_i)$. Denote by $W^-(v_i)$ the set of indices j_s ($s = 1, 2, \dots, d^-(v_i)$) of the edges outgoing from v_i and by $W^+(v_i)$ the set of indices k_s ($s = 1, 2, \dots, d^+(v_i)$) of the edges incoming into v_i .

Local coordinates for the edges identify each edge e_j with the interval $[0, l]$ so that the local coordinate increases in the direction of the edge. This means that each pendant vertex has the local coordinate 0. Each interior vertex has the local coordinate l on its incoming edge, while the local coordinate of the vertex is 0 on each outgoing edge. Functions y_j on the edges are subject to the system of g scalar Sturm-Liouville equations

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad (j = 1, 2, \dots, g) \quad (1)$$

where q_j are real-valued functions which belong to $L_2(0, l)$. For each interior vertex with outgoing edges e_j ($j \in W^-(v_i)$) and incoming edges e_k ($k \in W^+(v_i)$) the continuity conditions are

$$y_j(0) = y_k(l), \quad (2)$$

and Kirchhoff's condition is

$$\sum_{k \in W^+(v_i)} y_k'(l) = \sum_{j \in W^-(v_i)} y_j'(0). \quad (3)$$

We impose the Dirichlet boundary conditions

$$y_j(0) = 0 \quad (4)$$

at $r \leq p_{pen}$ of the pendant vertices and the Neumann conditions

$$y_j'(0) = 0 \quad (5)$$

at the rest $p_{pen} - r$ pendant vertices.

Let us denote by $s_j(\sqrt{\lambda}, x)$ the solution of the Sturm-Liouville equation (1) on the edge e_j which satisfies the conditions $s_j(\sqrt{\lambda}, 0) = s_j'(\sqrt{\lambda}, 0) - 1 = 0$ and by $c_j(\sqrt{\lambda}, x)$ the solution which satisfies the conditions $c_j(\sqrt{\lambda}, 0) - 1 = c_j'(\sqrt{\lambda}, 0) = 0$. Then the *characteristic function* $\Phi(\lambda)$, i.e. an entire function whose set of zeros coincides with the spectrum of the problem (1)-(5) can be expressed via $s_j(\sqrt{\lambda}, l)$, $s_j'(\sqrt{\lambda}, l)$, $c_j(\sqrt{\lambda}, l)$ and $c_j'(\sqrt{\lambda}, l)$. To do it we introduce the following system of vector-functions $\psi_j(\lambda, x) = \text{col}\{0, 0, \dots, s_j(\sqrt{\lambda}, x), \dots, 0\}$ and $\psi_{j+g}(\lambda, x) = \text{col}\{0, 0, \dots, c_j(\sqrt{\lambda}, x), \dots, 0\}$ for $j = 1, 2, \dots, g$. As in [19] we denote by L_j ($j = 1, 2, \dots, 2g$) the linear functionals generated by (1)-(5). Then $\Phi(\lambda) = \|L_j(\psi_k(\lambda, l))\|_{j,k}^{2g}$ is the characteristic matrix which represents the system of linear equations describing the continuity and Kirchhoff's conditions for the interior vertices. Then we call

$$\phi(\lambda) := \det(\Phi(\lambda))$$

the *characteristic function* of problem (1)-(5). The characteristic function is determined up to a constant multiple.

3. Auxiliary results. For a simple graph, the matrix $A = (A_{i,j})_{i,j=1}^p$ where $A_{i,i} = 0$ for all $i = 1, 2, \dots, p$ and for $i \neq j$: $A_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise} \end{cases}$ is called the *adjacency*

matrix. Let $D = \text{diag}\{d(v_1), d(v_2), \dots, d(v_p)\}$ be the degree matrix. Then $\tilde{A} = D^{-1/2}AD^{-1/2}$ is called the *weighted adjacency matrix* or *normalized Laplacian*.

Let G be a simple connected equilateral graph with $g \geq 2$ edges, p vertices, p_{pen} pendant vertices. Let r ($0 \leq r \leq p_{pen}$) be the number of pendant vertices with the Dirichlet conditions. The graph \hat{G} is obtained by removing the pendant vertices with Dirichlet boundary conditions and the edges incident with them in G . For convenience, we denote by $v_{r+1}, v_{r+2}, \dots, v_p$ the vertices of \hat{G} . Let \hat{A} be the adjacency matrix of \hat{G} , let $\hat{D}_G = \text{diag}\{d(v_{r+1}), d(v_{r+2}), \dots, d(v_p)\}$, where $d(v_i)$ is the degree of the vertex v_i in G . We consider the polynomial $P_{G,\hat{G}}$ defined by

$$P_{G,\hat{G}}(z) := \det(z\hat{D}_G - \hat{A}).$$

The following theorem was proved in [15] (Theorem 6.4.2).

Theorem 1. *Let G be a simple connected graph with at least two edges. Assume that all edges have the same length l and the same potentials symmetric with respect to the midpoints of the edges ($q(l-x) = q(x)$). Then the spectrum of problem (1)–(5) coincides with the set of zeros of the characteristic function $\phi_D(\lambda) = s^{g-p+r}(\sqrt{\lambda}, l)P_{G,\hat{G}}(c(\sqrt{\lambda}, l))$, where $s(\sqrt{\lambda}, x)$ and $c(\sqrt{\lambda}, x)$ are the solutions of the Sturm-Liouville equation which satisfies $s(\sqrt{\lambda}, 0) = s'(\sqrt{\lambda}, 0) - 1 = 0$ and $c(\sqrt{\lambda}, 0) - 1 = c'(\sqrt{\lambda}, 0) = 0$.*

Corollary 1. *Let T be an equilateral tree with at least 2 edges with the length of each edge l . Let $r = p_{pen}$, i.e. let the Dirichlet conditions be imposed at all the pendant vertices. Then $\phi_D(\lambda) = s(\sqrt{\lambda}, l)^{-1+p_{pen}}P_{T,\hat{T}}(c(\sqrt{\lambda}, l))$. If the potentials are zero on all edges then*

$$\tilde{\phi}_D(\lambda) = \left(\frac{\sin \sqrt{\lambda}l}{\sqrt{\lambda}}\right)^{-1+p_{pen}} P_{T,\hat{T}}(\cos \sqrt{\lambda}l).$$

Theorem 2. *Let T be an equilateral tree. The eigenvalues of problem (1)–(4) can be presented as the union of subsequences $\{\lambda_k\}_{k=1}^\infty = \bigcup_{i=1}^{2p-p_{pen}-1} \{\lambda_k^{(i)}\}_{k=1}^\infty$ with the following asymptotics*

$$\sqrt{\lambda_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{2\pi(k-1)}{l} + \frac{1}{l} \arccos \alpha_i + O\left(\frac{1}{k}\right) \quad \text{for } i = 1, 2, \dots, p - p_{pen}, \quad (6)$$

$$\sqrt{\lambda_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{2\pi k}{l} - \frac{1}{l} \arccos \alpha_i + O\left(\frac{1}{k}\right) \quad \text{for } i = p - p_{pen} + 1, \dots, 2(p - p_{pen}), \quad (7)$$

$$\sqrt{\lambda_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{\pi(k-1)}{l} + O\left(\frac{1}{k}\right) \quad \text{for } i = 2(p - p_{pen}) + 1, \dots, 2p - p_{pen} - 1. \quad (8)$$

Here $\alpha_1, \alpha_2, \dots, \alpha_{p-p_{pen}}$ are the zeros of the polynomial $P_{T,\hat{T}}(z)$.

Proof. From Corollary 1 we obtain the following asymptotics for the case of zero potentials:

$$\begin{aligned} \sqrt{\tilde{\lambda}_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{2\pi(k-1)}{l} + \frac{1}{l} \arccos \alpha_i & \quad \text{for } i = 1, 2, \dots, p - p_{pen}, \\ \sqrt{\tilde{\lambda}_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{2\pi k}{l} - \frac{1}{l} \arccos \alpha_i & \quad \text{for } i = p - p_{pen} + 1, \dots, 2(p - p_{pen}), \\ \sqrt{\tilde{\lambda}_k^{(i)}} \underset{k \rightarrow \infty}{=} \frac{\pi(k-1)}{l} & \quad \text{for } i = 2(p - p_{pen}) + 1, \dots, 2p - p_{pen} - 1. \end{aligned}$$

By Theorem 5.4 in [5] we conclude that $|\lambda_k^{(j)} - \tilde{\lambda}_k^{(j)}| \leq C < \infty$ where $\tilde{\lambda}_k^{(j)}$ are the eigenvalues of problem (1)–(4) on the same tree with $q_j \equiv 0$ for all j and therefore, presence of the $L_2(0, l)$ -potentials does not influence the first and the second terms of the asymptotics and (6)–(8) are true. \square

4. Inverse problem. It is clear from Theorem 3.3 that looking at the first two terms of the eigenvalue asymptotics we can't distinguish two trees only if the numbers of vertices are the same, the numbers of edges are the same and the sets $\{\alpha_k\}_{k=1}^{p-p_{pen}}$ corresponding to the two trees coincide. The latter means that the characteristic polynomial $P_{T,\hat{T}}(z)$ corresponding to one of the trees is equal to the characteristic polynomial of the other one multiplied by a non-zero constant. Let us consider all the trees of ≤ 8 vertices and check whether we can distinguish them using the first two terms of the eigenvalue asymptotics.

The only tree with $p = 3$ vertices is the graph P_3 , the path of 3 vertices. It has $p_{pen} = 2$ pendant vertices. The corresponding polynomial $P_{T,\hat{T}}$ we denote by $\phi_{3,2}(z)$ It is clear that $\phi_{3,2} = -2z$.

There are two non-isomorphic trees of $p = 4$ vertices. They are P_4 , the path of 4 vertices with $p_{pen} = 2$, and S_3 the star of 3 edges and $p_{pen} = 3$. The corresponding polynomials are $\phi_{4,2} = 4z^2 - 1$ and $\phi_{4,3} = -3z$. They have different sets of zeros.

There are three non-isomorphic trees with $p = 5$: P_5 , S_4 and the graph of Fig. 2. The corresponding polynomials are $\phi_{5,2}(z) = -8z^3 + 4z$, $\phi_{5,4}(z) = -4z$ and $\phi_{5,3}(z) = 6z^2 - 1$. Their sets of zeros are different.

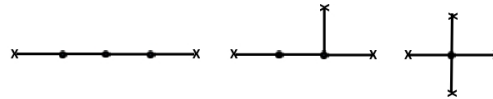


Figure 2. Trees with $p=5$.

There is only one tree P_6 with $p = 6$, $p_{pen} = 2$. The corresponding polynomial is $\phi_{6,2}(z) = 16z^4 - 12z^2 + 1$. There is only one tree S_5 with $p = 6$ and $p_{pen} = 5$. The corresponding polynomial is $\phi_{6,5}(z) = -5z$.

There are two pair of non-isomorphic trees with the same p and p_{pen} among the trees with $p = 6$. They are shown at Fig. 3 and Fig. 4.

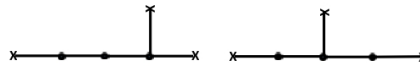


Figure 3. Trees with $p=6$. $p_{pen} = 3$

The corresponding polynomials $P_{T,\hat{T}}(z)$ are $\phi_{6,3}^1 = -12z^3 + 5z$, $\phi_{6,3}^2 = -12z^3 + 4z$.

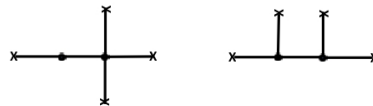


Figure 4. Trees with $p=6$. $p_{pen} = 4$

Here the corresponding polynomials $P_{T,\hat{T}}(z)$ are $\phi_{6,4}^1 = 8z^2 - 1$, $\phi_{6,4}^2 = 9z^2 - 1$. Since the sets of zeros of these polynomials are different we conclude that the spectrum of the Dirichlet problem uniquely determines the shape of the tree in case of $p \leq 6$.

Let $p = 7$. Then there is only one tree, namely P_7 , with $p_{pen} = 2$. The corresponding polynomial is $\phi_{7,2}(z) = -32z^5 + 32z^3 - 6z$. The only tree with $p = 7$ and $p_{pen} = 6$ is S_6 with the polynomial $\phi_{7,6} = -6z$.

Let $p = 7$ and $p_{pen} = 3$. There are three such non-isomorphic trees which are shown at Fig. 5.

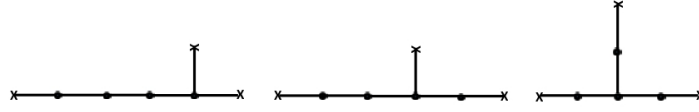


Figure 5. Trees with $p=7$. $p_{pen}=3$.

The corresponding polynomials are

$$\phi_{7,3}^1 = 24z^4 - 16z^2 + 1, \quad \phi_{7,3}^2 = 24z^4 - 14z^2 + 1, \quad \phi_{7,3}^3 = 24z^4 - 12z^2.$$

The sets of zeros of these polynomials are different and thus the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree in case of $p = 7$ and $p_{pen} = 3$.

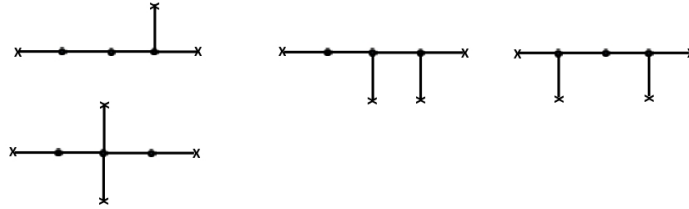


Figure 6. Trees with $p=7$. $p_{pen}=4$.

Let $p = 7$ and $p_{pen} = 4$. There are 4 such non-isomorphic trees which are shown at Fig. 6. The corresponding polynomials are

$$\phi_{7,4}^1 = -16z^3 + 6z, \quad \phi_{7,4}^2 = -18z^3 + 5z, \quad \phi_{7,4}^3 = -18z^3 + 6z, \quad \phi_{7,4}^4 = -16z^3 + 4z.$$

The sets of zeros of these polynomials are different. Therefore, the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree.

In case of $p = 7$ and $p_{pen} = 5$ we face a star and a double star graphs shown at Fig. 7 with the polynomials $\phi_{7,5}^1 = 10z^2 - 1$, $\phi_{7,5}^2 = 12z^2 - 1$.

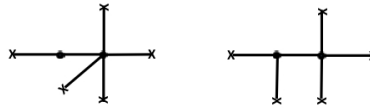


Figure 7. Trees with $p=6$. $p_{pen}=5$

These polynomials have different sets of zeros. Thus, we conclude the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree of 7 vertices.

Now let $p = 8$, $p_{pen} = 2$. There is only one such tree, namely P_8 . The corresponding polynomial is $\phi_{8,2}(z) = 64z^6 - 80z^4 + 24z^2 - 1$

Also there is only one tree S_7 with $p = 8$ and $p_{pen} = 7$ (the star of 7 edges). The corresponding polynomial is $\phi_{8,7} = -7z$.

Let $p = 8, p_{pen} = 3$. There are 4 such non-isomorphic trees shown at Fig. 8.

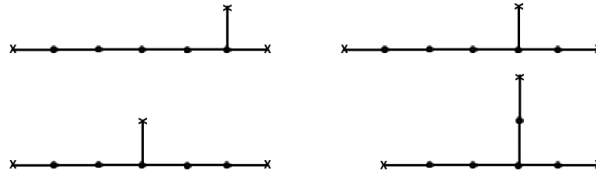


Figure 8. Trees with $p=8, p_{pen}=3$.

The corresponding polynomials are $\phi_{8,3}^1 = -48z^5 + 44z^3 - 7z, \phi_{8,3}^2 = -48z^5 + 40z^3 - 6z, \phi_{8,3}^3 = -48z^5 + 40z^3 - 7z, \phi_{8,3}^4 = -48z^5 + 36z^3 - 4z$. These polynomials have different sets of zeros. Thus, the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree in this case.

Let $p = 8, p_{pen} = 4$. There are 8 such non-isomorphic trees shown at Fig. 9.

The corresponding polynomials are $\phi_{8,4}^1 = 32z^4 - 20z^2 + 1, \phi_{8,4}^2 = 36z^4 - 19z^2 + 1, \phi_{8,4}^3 = 36z^4 - 18z^2 + 1, \phi_{8,4}^4 = 36z^4 - 21z^2 + 1, \phi_{8,4}^5 = 36z^4 - 16z^2 + 1, \phi_{8,4}^6 = 36z^4 - 12z^2, \phi_{8,4}^7 = 36z^4 - 16z^2, \phi_{8,4}^8 = 32z^4 - 12z^2 + 1$.

Since these polynomials have different sets of zeros, the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree with these data.

Let $p = 8, p_{pen} = 5$. There are 6 such non-isomorphic trees shown at Fig. 10.

The corresponding polynomials are $\phi_{8,5}^1 = -20z^3 + 7z, \phi_{8,5}^2 = -24z^3 + 6z, \phi_{8,5}^3 = -24z^3 + 7z, \phi_{8,5}^4 = -27z^3 + 6z, \phi_{8,5}^5 = -24z^3 + 5z, \phi_{8,5}^6 = -20z^3 + 4z$. All these polynomials have different sets of zeros. Thus, the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree with these data.

Let $p = 8, p_{pen} = 6$. There are 3 such trees (a star and two double stars) shown at Fig. 11. The corresponding polynomials $\phi_{8,6}^1 = 12z^2 - 1, \phi_{8,6}^2 = 15z^2 - 1, \phi_{8,6}^3 = 16z^2 - 1$. have different sets of zeros.

Thus, we have proved the following

Theorem 3. Let $\{\lambda_k\}_{k=1}^\infty$ be the spectrum of the Dirichlet spectral problem (1)–(4) on a simple connected graph. Let $\{\lambda_k\}_{k=1}^\infty = \bigcup_{i=1}^{2p-p_{pen}-1} \{\lambda_k^{(i)}\}_{k=1}^\infty$, where

$$\begin{aligned} \sqrt{\lambda_k^{(i)}} & \underset{k \rightarrow \infty}{=} \frac{2\pi(k-1)}{l} + \gamma_i + O\left(\frac{1}{k}\right) \quad \text{for } i = 1, 2, \dots, \tilde{p}, \\ \sqrt{\lambda_k^{(i)}} & \underset{k \rightarrow \infty}{=} \frac{2\pi k}{l} - \gamma_i + O\left(\frac{1}{k}\right) \quad \text{for } i = \tilde{p} + 1, \dots, 2\tilde{p}, \\ \sqrt{\lambda_k^{(i)}} & \underset{k \rightarrow \infty}{=} \frac{\pi(k-1)}{l} + O\left(\frac{1}{k}\right) \quad \text{for } i = 2\tilde{p} + 1, \dots, 2\tilde{p} + \tilde{p}_{pen} - 1, \end{aligned}$$

and $0 < \tilde{p} \leq 6, \tilde{p}_{pen} \geq 2$. Then these asymptotics uniquely determine the shape of the graph as a tree of $\tilde{p} + \tilde{p}_{pen}$ vertices, \tilde{p}_{pen} pendant vertices and $\cos(\gamma_1 l), \cos(\gamma_2 l), \dots, \cos(\gamma_{p_{pen}-1} l)$ are the zeros of one of the polynomial $\phi_{i,j}(z)$ or $\phi_{i,j}^s(x)$ described above.

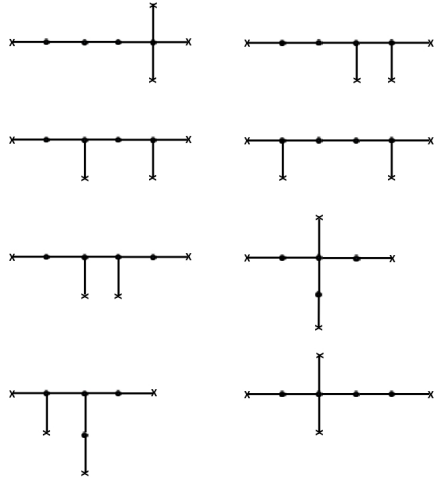


Figure 9. Trees with $p=8$. $p_{pen}=4$.

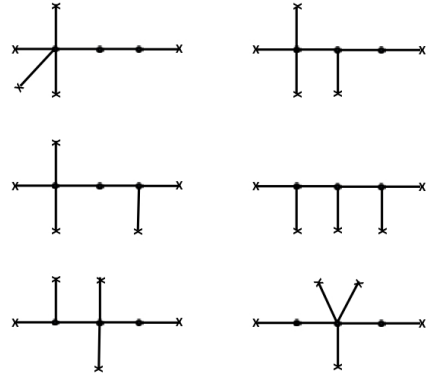


Figure 10. Trees with $p=8$. $p_{pen}=5$.

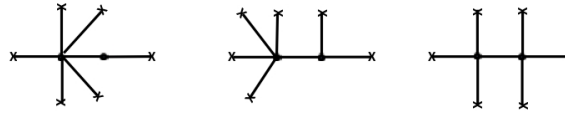


Figure 11. Trees with $p=8$. $p_{pen}=6$

5. Co-spectral non-isomorphic equilateral trees of 9 vertices. In this section we consider all trees of 9 edges and show that there are graphs with the same first and second terms of the asymptotics of eigenvalues of problem (1)–(4). These trees are co-spectral in case of zero potentials on all edges. There is only one tree (P_9) with $p = 9$ and $p_{pen} = 2$ and only one tree of $p = 9$ and $p_{pen} = 8$ (the star S_8). Let $p = 9$ and $p_{pen} = 3$.

There are 5 such non-isomorphic trees shown at Fig. 12. The corresponding polynomials are $\phi_{9,3}^1 = 96z^6 - 112z^4 + 30z^2 - 1$, $\phi_{9,3}^2 = 96z^6 - 104z^4 + 26z^2 - 1$, $\phi_{9,3}^3 = 96z^6 - 104z^4 + 28z^2 - 1$, $\phi_{9,3}^4 = 96z^6 - 96z^4 + 22z^2 - 1$, $\phi_{9,3}^5 = 96z^6 - 96z^4 + 20z^2$. All these polynomials have different sets of zeros. Thus, we conclude that there are no trees in Fig. 12 having the same second terms of eigenvalue asymptotics of the Dirichlet problem.

Let $p = 9$ and $p_{pen} = 4$. There are 14 such non-isomorphic trees shown at Fig. 13.

The corresponding polynomials are $\phi_{9,4}^1 = -64z^5 + 56z^3 - 8z$, $\phi_{9,4}^2 = -72z^5 + 56z^3 - 7z$, $\phi_{9,4}^3 = -72z^5 + 54z^3 - 8z$, $\phi_{9,4}^4 = -72z^5 + 60z^3 - 8z$, $\phi_{9,4}^5 = -72z^5 + 54z^3 - 7z$, $\phi_{9,4}^6 = -64z^5 + 48z^3 - 6z$, $\phi_{9,4}^7 = -72z^5 + 50z^3 - 7z$, $\phi_{9,4}^8 = -72z^5 + 48z^3 - 6z$, $\phi_{9,4}^9 = -64z^5 + 56z^3 + 8z^2 - 10z - 2$, $\phi_{9,4}^{10} = -72z^5 + 48z^3 - 4z$, $\phi_{9,4}^{11} = -72z^5 + 50z^3 - 5z$, $\phi_{9,4}^{12} = -64z^5 + 40z^3 - 4z$, $\phi_{9,4}^{13} = -72z^5 + 44z^3 - 4z$, $\phi_{9,4}^{14} = -64z^5 + 32z^3$.

Since the sets of zeros of these polynomials are different. Thus, we conclude that there are no trees in Fig. 13 having the same second terms of eigenvalue asymptotics of the Dirichlet problem.

Now let $p = 9$ and $p_{pen} = 5$. There are 14 such non-isomorphic trees shown at Fig. 14.

The corresponding polynomials are $\phi_{9,5}^1 = 40z^4 - 24z^2 + 1$, $\phi_{9,5}^2 = 48z^4 - 24z^2 + 1$,

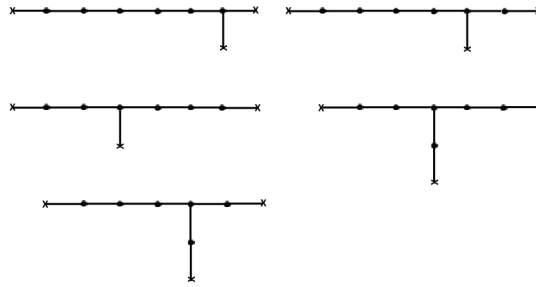


Figure 12. Trees with $p=9, p_{pen}=3$.

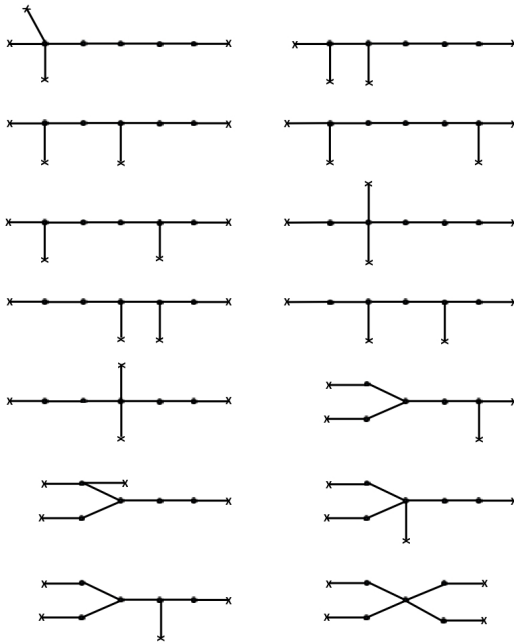


Figure 13. Trees with $p=9, p_{pen}=4$.

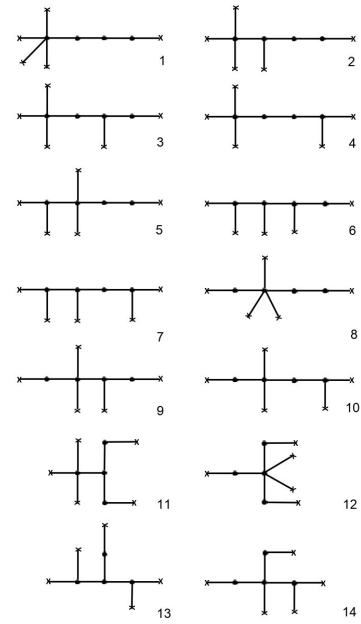


Figure 14. Trees with $p=9, p_{pen}=5$.

$$\begin{aligned} \phi_{9,5}^3 &= 48z^4 - 22z^2 + 1, \quad \phi_{9,5}^4 = 48z^4 - 26z^2 + 1, \quad \phi_{9,5}^5 = 48z^4 - 22z^2 + 1, \quad \phi_{9,5}^6 = 54z^4 - 21z^2 + 1, \\ \phi_{9,5}^7 &= 54z^4 - 24z^2 + 1, \quad \phi_{9,5}^8 = 40z^4 - 18z^2 + 1, \quad \phi_{9,5}^9 = 48z^4 - 18z^2 + 1, \quad \phi_{9,5}^{10} = 48z^4 - 20z^2 + 1, \\ \phi_{9,5}^{11} &= 48z^4 - 20z^2, \quad \phi_{9,5}^{12} = 40z^4 - 12z^2, \quad \phi_{9,5}^{13} = 54z^4 - 21z^2, \quad \phi_{9,5}^{14} = 32z^4 - 12z^2. \end{aligned}$$

We see that the graphs 3 and 5 have the same polynomials $\phi_{9,5}^3(z) \equiv \phi_{9,5}^5(z)$ and in case of zero potentials the trees are co-spectral.

Now $p = 9$ and $p_{pen} = 6$. There are 9 such non-isomorphic trees shown at Fig. 15.

Their polynomials are $\phi_{9,6}^1 = -24z^3 + 8z, \phi_{9,6}^2 = -30z^3 + 7z, \phi_{9,6}^3 = -30z^3 + 8z, \phi_{9,6}^4 = -32z^3 + 6z, \phi_{9,6}^5 = -36z^3 + 7z, \phi_{9,6}^6 = -32z^3 + 8z, \phi_{9,6}^7 = -36z^3 + 6z, \phi_{9,6}^8 = -24z^3 + 4z, \phi_{9,6}^9 = -30z^3 + 5z$. The polynomials $\phi_{9,6}^7(z), \phi_{9,6}^8(z)$ and $\phi_{9,6}^9$ have the same set of zeros and consequently in case of zero potentials on all edges the corresponding graphs are co-spectral.

The trees with $p = 9$ and $p_{pen} = 7$ are double stars shown at Fig.16.

Their polynomials are $P_{T,\hat{T}} = 14z^2 - 1, P_{T,\hat{T}} = 18^2 - 1, P_{T,\hat{T}} = 20z^2 - 1$. The sets of zeros of these polynomials are different.

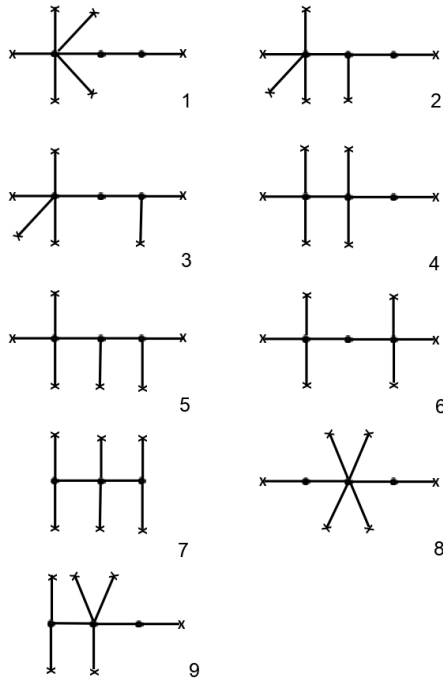


Figure 15. Trees with $p=9$. $p_{pen}=6$.

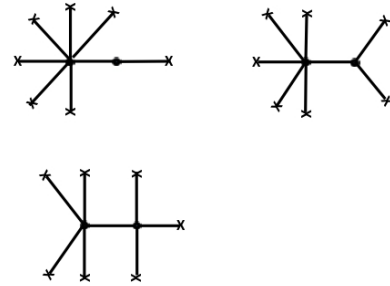


Figure 16. Trees with $p=9$. $p_{pen}=7$

Thus we arrive at

Theorem 4. *Among the trees of 9 vertices there is only one pair (trees 3 and 5 of Fig. 14) and one triple (trees 7, 8, 9 of Fig. 15) of graphs with the same spectrum of the Dirichlet problem with zero potentials on the edges.*

Remark 1. *In all cases of 9 vertex trees except of trees 3, 5 of Fig.14 and trees 7, 8, 9 of Fig. 15 the first and the second terms of the eigenvalue asymptotics uniquely determine the shape of the tree.*

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