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PERODIC TRAVELING WAVES IN FERMI–PASTA–ULAM TYPE SYSTEMS WITH NONLOCAL INTERACTION ON 2D-LATTICE

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The paper deals with the Fermi–Pasta–Ulam type systems that describe an infinite systems of nonlinearly coupled particles with nonlocal interaction on a two dimensional lattice. It is assumed that each particle interacts nonlinearly with several neighbors horizontally and vertically on both sides. The main result concerns the existence of traveling waves solutions with periodic relative displacement profiles. We obtain sufficient conditions for the existence of such solutions with the aid of critical point method and a suitable version of the Mountain Pass Theorem for functionals satisfying the Cerami condition instead of the Palais–Smale condition. We prove that under natural assumptions there exist monotone traveling waves.

1. Introduction. In the present paper we consider the Fermi–Pasta–Ulam type system that describes of an infinite system of nonlinearly coupled particles on a two dimensional lattice with nonlocal interaction, i.e. each particle interacts with l neighbors horizontally and vertically on both sides. The equations of motion of the system considered are of the form

$$\ddot{q}_{n,m}(t) = \sum_{j=1}^{l} \big(W'_{1j}(q_{n+j,m}(t) - q_{n,m}(t)) - W'_{1j}(q_{n,m}(t) - q_{n-j,m}(t)) + \\ + W'_{2j}(q_{n,m+j}(t) - q_{n,m}(t)) - W'_{2j}(q_{n,m}(t) - q_{n,m-j}(t)) \big),\ (n,m) \in \mathbb{Z}^2, \qquad (1)$$

where $q_{n,m}(t)$ is the coordinate of the (n,m)-th particle at time t, $W_{1j}, W_{2j} \in C^1(\mathbb{R};\mathbb{R})$ are the potentials of interaction ($j = 1, 2, \ldots, l$), in particular, W_{11}, W_{21} are the potentials of the horizontal and vertical interaction, respectively, of the (n,m)-th particle with nearest neighbors, W_{12}, W_{22} with second nearest neighbors, and so on. In the case $l = 1$ we obtain the Fermi–Pasta–Ulam type system on a two dimensional lattice with local interaction. Equations (1) form an infinite system of ordinary differential equations.

Notice that this system is a representative of a wide class of systems called lattice dynamical systems extensively studied in last decades. Such systems are of interest in view of numerous applications in physics [1, 11, 12, 13, 16].

Among the solutions of such systems, traveling waves deserve special attention. The existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam type systems with local interaction on 2D–lattice is studied in [6, 7, 8, 9]. While in papers [3, 4, 5, 10, 14]

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traveling waves for infinite systems of linearly and nonlinearly coupled oscillators on 2D–lattice are studied. A comprehensive presentation of existing results on traveling waves for 1D Fermi-Pasta-Ulam lattices with local interaction is given in [19].

The Fermi–Pasta–Ulam type systems with nonlocal interaction are not well-studied. The existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam type systems with nonlocal interaction on 1D–lattice is studied in [18] and [21]. G. Friesecke and K. Matthies [15] showed the existence of solitary traveling waves for a two-dimensional elastic lattice of particles interacting via harmonic springs between nearest and diagonal neighbors. While the present paper deal with traveling waves for more general lattice systems.

2. Statement of problem and main assumptions. A traveling wave solution of equation (1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct), \tag{2}$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$c^2u''(s) = \sum_{j=1}^{l} \big[W'_{1j}(u(s+j\cos\varphi) - u(s)) - W'_{1j}(u(s) - u(s-j\cos\varphi)) + \\ + W'_{2j}(u(s+j\sin\varphi) - u(s)) - W'_{2j}(u(s) - u(s-j\sin\varphi))\big],$$

or

$$c^2u''(s) = \sum_{j=1}^{l} \big[W'_{1j}(A_j^+u(s)) - W'_{1j}(A_j^-u(s)) + W'_{2j}(B_j^+u(s)) - W'_{2j}(B_j^-u(s))\big], \tag{3}$$

where

$$A_j^+u(s) := u(s+j\cos\varphi) - u(s),\ A_j^-u(s) := u(s) - u(s-j\cos\varphi),$$
$$B_j^+u(s) := u(s+j\sin\varphi) - u(s),\ B_j^-u(s) := u(s) - u(s-j\sin\varphi),$$
$$s = n\cos\varphi + m\sin\varphi - ct.$$

Note that the vector $\overrightarrow{l}(\cos\varphi, \sin\varphi)$ defines the direction of wave propagation. The constant $c \neq 0$ is called the speed of the wave. If $c < 0$, then the wave moves to the opposite direction corresponding $c > 0$.

In what follows, a solution of equation (3) is understood as a function $u(s)$ from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying equation (3) for all $s \in \mathbb{R}$.

We consider the case of periodic traveling waves. The profile function of such wave satisfies the following periodicity condition

$$u'(s+2k) = u'(s),\ s \in \mathbb{R}, \tag{4}$$

where $k > 0$ is a real number. Note that the profile of such wave is not necessarily periodic. But its relative displacement profiles

$$r_1^{\pm}(s) = \int_s^{s\pm\cos\varphi} u'(\tau)d\tau,\ r_2^{\pm}(s) = \int_s^{s\pm\sin\varphi} u'(\tau)d\tau$$

are periodic. Therefore, such waves are also called periodic (see [19]).

It is easy to see that if we have a solution $u(s)$ of equation (3) satisfying (4), then $u(s)$ + const is also a solution of this problem. Therefore, to obtain the main result, we impose an additional condition $u(0) = 0$.

In this paper we exclude trivial waves with linear profile functions $u(s) = as + b$. Notice that physically meaningful are waves with monotone, either nondecreasing or nonincreasing, profile functions. We note that from a physical point of view, increasing waves are *expansion waves*, while decreasing waves are *compression waves*.

We always assume that:

(i) $W_{ij}(r) = \frac{c_{ij}^2}{2}r^2 + f_{ij}(r)$, where $c_{ij} \in \mathbb{R}$, $f_{ij} \in C^1(\mathbb{R};\mathbb{R})$, moreover, $f_{ij}(0) = f'_{ij}(0) = 0$ and $f'_{ij}(r) = o(r)$ as $r \to 0$, $i \in \{1, 2\}$, $j \in \{1, 2, \dots, l\}$;

(ii) all functions $f_{ij}(r)$, $i \in \{1, 2\}$, $j \in \{1, 2, \dots, l\}$, are nonnegative and there exists l_0 such that
$$\lim_{r\to\pm\infty} \frac{f_{il_0}(r)}{r^2} = +\infty;$$
in addition, considering $2k$-periodic problem with integer period and $\varphi \equiv 0, \pi/2 \mod \pi$, we assume that $l_0 = 1$;

(iii) all functions $|r|^{-1}f'_{ij}(r)$, $i \in \{1, 2\}$, $j \in \{1, 2, \dots, l\}$, extended by 0 to $r = 0$, are nondecreasing and there exists l_0 such that $|r|^{-1}f'_{il_0}(r)$ are strictly increasing.

We note that, in the general case, not all of the functions $f_{ij}(r)$ are nonzero.

Remark 1. By assumption (iii), all functions
$$g_{ij}(r) := \frac{1}{2}rf'_{ij}(r) - f_{ij}(r),\ i \in \{1, 2\},\ j \in \{1, 2, \dots, l\},$$
are nondecreasing for $r > 0$ and nonincreasing for $r < 0$. Furthermore, $g_{il_0}(r) > 0$ for all $r \neq 0$.

An important role is played by quantity c_0 defined by the equality
$$c_0 = c_0(\varphi) := \left(\sum_{j=1}^{l}(c_{1j}^2\cos^2\varphi + c_{2j}^2\sin^2\varphi)j^2\right)^{1/2}.$$

3. Variational setting. We denote by E_k the Hilbert space
$$E_k = \left\{u \in H^1_{loc}(\mathbb{R}) \colon\ u'(s + 2k) = u'(s),\ u(0) = 0\right\}$$
with the scalar product
$$(u, v)_k = \int_{-k}^{k} u'(s)v'(s)ds$$
and corresponding norm $\|u\|_k = (u, u)^{\frac{1}{2}}$. The norm in the dual space E_k^* is denoted by $\|\cdot\|_{k,*}$. By the embedding theorem, $E_k \subset C([-k, k])$, where $C([-k, k])$ is the space of continuous functions on $[-k, k]$.

Note that the difference operators $A_j^{\pm}$ and $B_j^{\pm}$ are bounded linear operators on E_k, $A_j^{\pm}u(t) \to 0$ and $B_j^{\pm}u(t) \to 0$ as $|t| \to \infty$, moreover, these operators satisfy the inequalities (see [2], Lemma 6.1)

$$\begin{gathered}\|A_j^{\pm}u\|_{L^\infty(-k,k)} \le l_1(k)j^{1/2}\cdot\|u\|_k,\ \|A_j^{\pm}u\|_{L^2(-k,k)} \le |\cos\varphi| j\cdot\|u\|_k,\\ \|B_j^{\pm}u\|_{L^\infty(-k,k)} \le l_2(k)j^{1/2}\cdot\|u\|_k,\ \|B_j^{\pm}u\|_{L^2(-k,k)} \le |\sin\varphi| j\cdot\|u\|_k,\\ \|A_j^{\pm}u\|^2_{H^1(-k,k)} + \|B_j^{\pm}u\|^2_{H^1(-k,k)} \le (j^2+8)\cdot\|u\|_k^2,\end{gathered} \tag{5}$$

where

$$l_1(k) = \begin{cases} |\cos\varphi|\left(\left[\frac{1}{2k}\right]+1\right)^{1/2}, & 0<2k<1,\\ |\cos\varphi|, & 2k\ge 1,\end{cases} \qquad l_2(k) = \begin{cases} |\sin\varphi|\left(\left[\frac{1}{2k}\right]+1\right)^{1/2}, & 0<2k<1,\\ |\sin\varphi|, & 2k\ge 1,\end{cases}$$

and $\left[\frac{1}{2k}\right]$ is the integer part of $\frac{1}{2k}$.

It is easily seen that if $T := 2k \ge l$ is an integer and $\varphi \equiv 0 \mod \pi$, all operators $A_j^{\pm}$, $j \in \{1,2,\ldots,l\}$ have nontrivial kernels in the space E_k and $\ker A_1^{\pm} \subset \ker A_j^{\pm}$, $j \in \{2,3,\ldots,l\}$. Similarly if $T \ge l$ is an integer and $\varphi \equiv \pi/2 \mod \pi$, all operators $B_j^{\pm}$, $j \in \{1,2,\ldots,l\}$ have nontrivial kernels in the space E_k and $\ker B_1^{\pm} \subset \ker B_j^{\pm}$, $j \in \{2,3,\ldots,l\}$.

We denote by E_k^+ and $E_k^- = -E_k^+$ the cones of nondecreasing and nonincreasing functions in E_k, respectively. These cones are closed.

On the space E_k, we consider the functional

$$\begin{aligned} J_k(u) &= \int_{-k}^{k}\left[\frac{c^2}{2}(u'(s))^2 - \sum_{j=1}^{l}\left(W_{1j}(A_j^+u(s)) + W_{2j}(B_j^+u(s))\right)\right]ds = \\ &= \int_{-k}^{k}\left[\frac{c^2}{2}(u'(s))^2 - \sum_{j=1}^{l}\left(\frac{c_{1j}^2}{2}(A_j^+u(s))^2 + \frac{c_{2j}^2}{2}(B_j^+u(s))^2\right) - \right.\\ &\qquad\left. - \sum_{j=1}^{l}\left(f_{1j}(A_j^+u(s)) + f_{2j}(B_j^+u(s))\right)\right]ds. \end{aligned} \tag{6}$$

Remark 2. It is easily verified that, under the assumptions imposed, the functional J_k is well-defined C^1-functional on E_k, and its derivative is given by the formula

$$\begin{aligned} &\langle J_k'(u), h\rangle = \\ &= \int_{-k}^{k}\left[c^2u'(s)h'(s) - \sum_{j=1}^{l}\left(W_{1j}'(A_j^+u(s))A_j^+h(s) + W_{2j}'(B_j^+u(s))B_j^+h(s)\right)\right]ds = \\ &= \int_{-k}^{k}\left[c^2u'(s)h'(s) - \sum_{j=1}^{l}\left(c_{1j}^2A_j^+u(s)A_j^+h(s) + c_{2j}^2B_j^+u(s)B_j^+h(s)\right) - \right.\\ &\qquad\left. - \sum_{j=1}^{l}\left(f_{1j}'(A_j^+u(s))A_j^+h(s) + f_{2j}'(B_j^+u(s))B_j^+h(s)\right)\right]ds \end{aligned}$$

for $u, h \in E_k$. Moreover, any critical point of the functional J_k is a solution of equation (3) satisfying (4).

The formula for the derivative is obtained by direct calculation. We show that any critical point of the functional J_k is a solution of equation (3) satisfying (4). Indeed, let $g(s)$ be any C^∞-function satisfying (4). Then $h(s) = g(s) - g(0) \in E_k$. If u is a critical point of J_k, then

$$0 = \langle J_k'(u), h\rangle =$$

$$= \int_{-k}^{k} \Big[c^2 u'(s)h'(s) - \sum_{j=1}^{l} \big(W_{1j}'(A_j^+ u(s))A_j^+ h(s) + W_{2j}'(B_j^+ u(s))B_j^+ h(s)\big)\Big]ds =$$

$$= \int_{-k}^{k} \Big[c^2 u'(s)g'(s) - \sum_{j=1}^{l} \big(W_{1j}'(A_j^+ u(s))A_j^+ g(s) + W_{2j}'(B_j^+ u(s))B_j^+ g(s)\big)\Big]ds =$$

$$= \int_{-k}^{k} \Big[- c^2 u''(s)g(s) -$$

$$- \sum_{j=1}^{l} \big(\big\{W_{1j}'(A_j^- u(s)) - W_{1j}'(A_j^+ u(s))\big\}\, g(s) + \big\{W_{2j}'(B_j^- u(s)) - W_{2j}'(B_j^+ u(s))\big\}\, g(s)\big)\Big]ds =$$

$$= \int_{-k}^{k} \Big[- c^2 u''(s) +$$

$$+ \sum_{j=1}^{l} \big(W_{1j}'(A_j^+ u(s)) - W_{1j}'(A_j^- u(s)) + W_{2j}'(B_j^+ u(s)) - W_{2j}'(B_j^- u(s))\big)\Big] g(s)ds.$$

And this means that u is a weak solution of (3). Since $W_{ij}'(r)$ $(i \in \{1, 2\}, j \in \{1, 2, \dots, l\})$ are continuous, the right part of equation (3) is continuous. Therefore, $u''(s)$ is continuous, and hence, $u \in C^2(\mathbb{R}; \mathbb{R})$ is a solution of equation (3) in the classical sense.

Thus, to establish the existence of solutions to equation (3) satisfying (4), it is suffice to prove the existence of nontrivial critical points of the functional J_k. This requires a special form of the Mountain Pass Theorem.

Let $I\colon B \to \mathbb{R}$ be a C^1-functional on a Banach space B with the norm $\|\cdot\|$. We say that I satisfies the *Cerami condition*, if the following condition is satisfied.

(C) If $\{u_n\} \subset B$ is a Cerami sequence at a level b, i.e. $I(u_n) \to b$ and

$$(1 + \|u_n\|)\, \|I'(u_n)\|_* \to 0,\ n \to \infty,$$

then $\{u_n\}$ contains a convergent subsequence.

If there exist $e \in B$ and $\rho > 0$ such that $\|e\| > \rho$ and

$$\beta := \inf_{\|u\|=\rho} I(u) > I(0) = 0 \geq I(e),$$

then we say that the functional I possesses the *Mountain Pass Geometry.*

The following theorem of the Mountain-Pass-type can be found in [17, 18].

Theorem 1. *Suppose that a C^1-functional $I\colon B \to \mathbb{R}$ satisfies the Cerami condition and possesses the Mountain Pass Geometry. Let $P\colon B \to B$ be a continuous mapping such that*

$$I(Pu) \le I(u)$$

for all $u \in B$ and $P(e) = e$. Then there exists a critical point $u \in \overline{PB}$ (the closure of PB) of the functional I with the critical value

$$I(u) = d := \inf_{\gamma\in\Gamma_I} \max_{t\in[0,1]} I(\gamma(t)) \ge \beta,$$

where $\Gamma_I := \{\gamma \in C([0,1], B)\colon \gamma(0) = 0, \gamma(1) \ne 0, I(\gamma(1)) < 0)\}$.

Note that other forms of the Mountain Pass Theorem, in particular, with the Palais-Smale condition instead of the Cerami condition, can be found in [19, 20, 22].

4. Main result. The main result of this paper is the following theorem that establishes the existence of periodic waves with nondecreasing and nonincreasing profiles.

Theorem 2. *Assume (i)—(iii), $c > c_0$ and $\varphi \in [\pi n, \frac{\pi}{2} + \pi n]$ for any fixed $n \in \mathbb{Z}$. Then there exists $T_0 \ge l$ such that for every $T := 2k \ge T_0$ equation (3) has a nonconstant nondecreasing and nonincreasing solutions satisfying (4).*

Remark 3. Since we consider monotone waves, we may only suppose that the assumptions of Theorem 2 hold for $r \ge 0$ (respectively, for $r \le 0$), and obtain nondecreasing (respectively, nonincreasing) waves. On the other hand, proving the results we may assume, for instance, that $f_{ij}(r)$, $i \in \{1, 2\}$, $j \in \{1, 2, \dots, l\}$, are even functions.

Lemma 1. *Under the assumptions of Theorem 2 functional J_k possesses the Mountain Pass Geometry.*

Proof. Step 1. We first prove that there exist constants $\alpha > 0$ and $\rho > 0$ independent of k such that $J_k(v) \ge \alpha$ as $\|v\|_k = \rho$ and each nonzero critical point u of J_k satisfy $\|u\|_k \ge \rho$.

We note that (i) implies that all interaction potentials W_{ij} are subquadratic at 0. For convenience, we represent the functional J_k in the form

$$J_k(u) = \frac{1}{2} Q_k(u) - \Psi_k(u), \tag{7}$$

where

$$Q_k(u) = \int_{-k}^{k} \left[c^2(u'(s))^2 - \sum_{j=1}^{l} \left(c_{1j}^2 (A_j^+ u(s))^2 + c_{2j}^2 (B_j^+ u(s))^2 \right) \right] ds,$$

$$\Psi_k(u) = \int_{-k}^{k} \sum_{j=1}^{l} \left(f_{1j}(A_j^+ u(s)) + f_{2j}(B_j^+ u(s)) \right) ds.$$

Then, by (5), we see that given $\varepsilon > 0$, there exists $\rho > 0$ independent of $T := 2k \ge l$ and such that $\Psi_k(v) \le \varepsilon \|v\|_k^2$ for all $v \in E_k$ with $\|v\|_k \le \rho$. The same inequalities implies that

$$Q_k(v) \ge (c^2 - c_0^2) \|v\|_k^2.$$

Hence,

$$J_k(v) \ge \frac{c^2 - c_0^2}{2} \|v\|_k^2 - \varepsilon \|v\|_k^2, \ \|v\|_k \le \rho.$$

Similarly, changing ρ, we have that

$$\langle J_k'(v), v\rangle \ge (c^2 - c_0^2) \|v\|_k^2 - \varepsilon \|v\|_k^2, \ \|v\|_k \le \rho.$$

Since $\langle J_k'(v), v\rangle = 0$ for all critical points, after an appropriate choice of ε we obtain the required.

Step 2. Now we show that there exists an element $e \in E_k$ such that $\|e\|_k > \rho$ and $J_k(e) \le 0$.

Indeed, if $v \in E_k^+ \backslash \{0\}$, then $A_{l_0}^+ v > 0$, $B_{l_0}^+ v > 0$ on certain interval and, by (ii), $J_k(tv) \le 0$ for large enough $t > 0$. Hence, there exists $t_0 > 0$ such that $J_k(tv) \le 0$ for all $t > t_0$. Thus, we can fix $e = tv$ satisfying $\|e\|_k > \rho$ and $J_k(e) \le 0$, which proves the lemma. □

Remark 4. *The proof of Lemma 1 shows that $\langle J_k'(v), v\rangle > 0$, if $v \in E_k \setminus \{0\}$ and $\|v\|_k \le \rho$.*

Let

$$(Pu)(s) := \int_0^s |u'(t)| dt.$$

Remark 5. *It is easily verified that P is a continuous map from E_k into itself and PE_k consists of a nondecreasing functions.*

We note that $PE_k = E_k^+$ are closed. It is easy that, defining Mountain Pass value d_k for the functional J_k, we may assume that Γ_{J_k} consist of paths with values in E_k^+.

Denote by

$$N_k^+ = \{v \in E_k^+ \colon v \neq 0, \langle J_k'(v), v\rangle = 0\}$$

the partial Nehari manifold. Let $v \in E_k^+$, $T := 2k \ge l$. Since $A_j^+ v \neq 0$, $B_j^+ v \neq 0$ for all $j = 1, 2, \ldots, l$, then, by (ii) the function $\phi(t) = J_k(tv)$, $t > 0$ attains its maximum value, while assumption (iii) implies that

$$\phi'(t) = t(Q_k(v) - t^{-1}\Psi_k'(tv)v) = t^2 \langle J_k'(tv), tv\rangle$$

has the only positive zero t_0, and $t_0 v \in N_k^+$. Thus,

$$\inf_{v \in N_k^+} J_k(tv) = \inf_{v \in E_k^+ \backslash \{0\}} \max_{t>0} J_k(tv)$$

Denote by d_k^* their common value and we show that $d_k^* = d_k$. Indeed, if $v \in E_k^+$, then a part of the row $\{tv, t > 0\}$, after an appropriate rescaling, is a member of Γ_{J_k}. Hence, $d_k^* \ge d_k$. On the other hand, let $\gamma \in \Gamma_{J_k}$. By Remark 4, in a neighborhood of 0 there exists $t > 0$ such that $\langle J_k'(\gamma(t)), \gamma(t)\rangle > 0$. Since $J_k(\gamma(1)) < 0$, then, making use of Remark 1, we obtain that

$$\begin{aligned} \langle J_k'(\gamma(1)), \gamma(1)\rangle = Q_k(\gamma(1)) - \langle \Psi_k'(\gamma(1)), \gamma(1)\rangle \le \\ \le Q_k(\gamma(1)) - 2\Psi_k(\gamma(1)) = 2J_k(\gamma(1)) < 0. \end{aligned}$$

Hence, $\gamma(t_0) \in N_k^+$ for some $t_0 \in (0, 1)$. Since

$$d_k^* \le J_k(\gamma(t_0)) \le \max_{t \in [0,1]} J_k(\gamma(t)),$$

and $\gamma \in \Gamma_{J_k}$ is an arbitrary path, we have that $d_k^* \le d_k$. Thus, if $T := 2k \ge l$, then the mountain pass value d_k of J_k is given by the identities

$$d_k = \inf_{v \in N_k^+} J_k(tv) = \inf_{v \in E_k^+ \backslash \{0\}} \max_{t>0} J_k(tv). \tag{8}$$

Lemma 2. *Under the assumptions of Theorem 2 functional J_k satisfies the Cerami condition.*

Proof. Let $\{u_n\} \subset E_k$ be a Cerami sequence for the functional J_k at a level b, i.e. $J_k(u_n) \to b$ and $(1 + \|u_n\|_k) \|J_k'(u_n)\|_{k,*} \to 0$, $n \to \infty$.

Step 1. We prove that the sequence $\{u_n\}$ is bounded. Assuming the contrary and passing to a subsequence (with the same denotation), we have that $\|u_n\|_k \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_k}$. Then $\|v_n\|_k = 1$ and, passing to a subsequence again, we may assume that $v_n \to v_0$ weakly in E_k. By the inequalities (5) and the compactness of embedding $H^1(-k,k) \subset C(-k,k)$, we have that

$$A_j^+ v_n \to A_j^+ v_0,\ B_j^+ v_n \to B_j^+ v_0,\ j = 1, 2, \ldots, l,$$

uniformly on $[-k,k]$. Next, we consider two cases.

We first assume that $|A_j^+ v_0| + |B_j^+ v_0| \neq 0$ for some $j \in \{1, 2, \ldots, l\}$ and prove that this is impossible. If T is an integer and $\varphi \equiv 0, \pi/2 \mod \pi$, then $|A_1^+ v_0| + |B_1^+ v_0| \neq 0$, otherwise $|A_j^+ v_0| + |B_j^+ v_0| \neq 0$ for all $j = 1, 2, \ldots, l$. Anyway, $|A_{l_0}^+ v_0| + |B_{l_0}^+ v_0| \neq 0$, where l_0 is introduced in (ii). Therefore there exist an interval $I_0 \subset [-k,k]$ and $\varepsilon_0 > 0$ such that $|A_{l_0}^+ v_0| + |B_{l_0}^+ v_0| \geq \varepsilon_0$ on I_0 for all n large enough. Thus, $|A_{l_0}^+ u_n| + |B_{l_0}^+ u_n| \geq \varepsilon_0 \|u_n\|_k \to \infty$ on I_0. Since $J_k(u_n) \to b$, we get

$$\frac{1}{2} Q_k(u_n) - (b + o(1)) = \Psi_k(u_n).$$

Observing that $Q_k(u) \leq c^2 \|u\|_k^2$ and making use of assumption (ii), we obtain

$$\frac{c^2}{2} - \frac{b + o(1)}{\|u_n\|_k^2} \geq \int_\alpha^\beta \left[\frac{f_{1l_0}(A_{l_0}^+ u_n)}{|A_{l_0}^+ u_n|} |A_{l_0}^+ v_n|^2 + \frac{f_{2l_0}(B_{l_0}^+ u_n)}{|B_{l_0}^+ u_n|} |B_{l_0}^+ v_n|^2 \right] ds \to \infty.$$

We obtain a contradiction.

Now we rule out the case $A_j^+ v_0 = B_j^+ v_0 = 0$ for all $j = 1, 2, \ldots, l$. For this aim we choose $r_n \in [0,1]$ such that

$$J_k(r_n u_n) = \max_{r \in [0,1]} J_k(r u_n).$$

Let $w_n = p v_n$ with $p > 0$. Then for all n large enough, $0 < \frac{k}{\|u_n\|_k} < 1$ and, hence,

$$J_k(r_n u_n) \geq J_k(w_n) = \frac{(c^2 - c_0^2) p^2}{2} - \Psi_k(w_n).$$

We note that $A_j^+ v_n \to 0$ i $B_j^+ v_n \to 0$ uniformly on $[-k,k]$ for all $j = 1, 2, \ldots, l$, and, hence, $\Psi_k(w_n) \to 0$. Since $p > 0$ is an arbitrary number, we obtain that $J_k(r_n u_n) \to \infty$. Observe that $0 < r_n < 1$ for sufficiently large n because $J_k(0) = 0$ and $J_k(u_n) \to b$. Then r_n is an interior maximum point of the function $J_k(r u_n)$ on [0,1] and, therefore,

$$\langle J_k'(r_n u_n), r_n u_n \rangle = r_n (J_k(r u_n))'|_{r = r_n} = 0.$$

As consequence,

$$J_k(r_n u_n) = \frac{1}{2} \langle \Psi_k'(r_n u_n), r_n u_n \rangle - \Psi_k(r_n u_n).$$

By Remark 1, we have that

$$\frac{1}{2} \langle \Psi_k'(u_n), u_n \rangle - \Psi_k(u_n) \geq \langle \Psi_k'(r_n u_n), r_n u_n \rangle - \Psi_k(r_n u_n) = J_k(r_n u_n) \to \infty.$$

On the other hand,

$$\lim_{n\to\infty}\left(\frac{1}{2}\langle\Psi_k'(u_n),u_n\rangle-\Psi_k(u_n)\right)=\lim_{n\to\infty}\left(J_k(u_n)-\frac{1}{2}\langle J_k'(u_n),u_n\rangle\right)=b$$

and we get a contradiction again, which proves that the sequence $\{u_n\}$ is bounded in E_k.

Step 2. Since $\{u_n\}$ is bounded, then, up to a subsequence (with the same denotation), we have that $u_n\to u_0$ weakly in E_k, and hence, $A_j^+u_n\to A_j^+u_0$ and $B_j^+u_n\to B_j^+u_0$ weakly in $H^1(-k,k)$, and, by the compactness of Sobolev embedding, strongly in $C([-k,k])$, i.e. uniformly on $[-k,k]$ for all $j=1,2,\dots,l$. A straightforward calculation shows that

$$Q_k(u_n-u_0)=\langle J_k'(u_n)-J_k'(u_0),u_n-u_0\rangle+$$
$$+\sum_{j=1}^{l}\left[\int_{-k}^{k}\left(f_{1j}'(A_j^+u_n(s))-f_{1j}'(A_j^+u_0(s))\right)\left(A_j^+u_n(s)-A_j^+u_0(s)\right)ds+\right.$$
$$\left.+\int_{-k}^{k}\left(f_{2j}'(B_j^+u_n(s))-f_{2j}'(B_j^+u_0(s))\right)\left(B_j^+u_n(s)-B_j^+u_0(s)\right)ds\right].$$

Obviously that all the terms on the right hand part converge to 0 as $n\to\infty$ (first term converges to 0 by weak convergence and second term converges to 0 by strong convergence). Thus, $Q_k(u_n-u_0)\to 0$ as $n\to\infty$. Since the quadratic form Q_k is positive definite, $\|u_n-u\|_k\to 0$ as $n\to\infty$, and proof is complete. □

Proof of Theorem 2. Let the conditions of the theorem hold for $r\ge 0$. Lemmas 1 and 2 show that J_k satisfies almost all conditions of Theorem 1. It only remains to verify the inequality $J_k(Pu)\le J_k(u)$ for all $u\in E_k$.

Indeed, let $\varphi\in[2\pi n,\frac{\pi}{2}+2\pi n]$, $n\in\mathbb{Z}$. Then

$$A_j^+Pu(t)=\int_t^{t+j\cos\varphi}|u'(s)|ds\ge\left|\int_t^{t+j\cos\varphi}u'(s)ds\right|=|A_j^+u(t)|\ge A_j^+u(t),$$
$$B_j^+Pu(t)=\int_t^{t+j\sin\varphi}|u'(s)|ds\ge\left|\int_t^{t+j\sin\varphi}u'(s)ds\right|=|B_j^+u(t)|\ge B_j^+u(t).$$

Since $f_{ij}(r)$ are increasing for $r\ge 0$, we have that

$$J_k(Pu)=\int_{-k}^{k}\left[\frac{c^2}{2}((Pu)'(s))^2-\sum_{j=1}^{l}\left(\frac{c_{1j}^2}{2}(A_j^+Pu(s))^2+\frac{c_{2j}^2}{2}(B_j^+Pu(s))^2\right)-\right.$$
$$\left.-\sum_{j=1}^{l}\left(f_{1j}(A_j^+Pu(s))+f_{2j}(B_j^+Pu(s))\right)\right]ds=$$
$$=\int_{-k}^{k}\left[\frac{c^2}{2}(u'(s))^2-\sum_{j=1}^{l}\left(\frac{c_{1j}^2}{2}(A_j^+Pu(s))^2+\frac{c_{2j}^2}{2}(B_j^+Pu(s))^2\right)-\right.$$

$$-\sum_{j=1}^{l}\left(f_{1j}(A_j^+Pu(s))+f_{2j}(B_j^+Pu(s))\right)\Bigg]\,ds\le$$

$$\le\int_{-k}^{k}\Bigg[\frac{c^2}{2}(u'(s))^2-\sum_{j=1}^{l}\left(\frac{c_{1j}^2}{2}(A_j^+u(s))^2+\frac{c_{2j}^2}{2}(B_j^+u(s))^2\right)-$$

$$-\sum_{j=1}^{l}\left(f_{1j}(A_j^+u(s))+f_{2j}(B_j^+u(s))\right)\Bigg]\,ds=J_k(u).$$

Hence, by Theorem 1 there exists nontrivial critical point $u\in PE_k$ of the functional J_k such that $J_k(u)\ge\alpha$ with $\alpha>0$ from Lemma 1. By Remark 2, $u\in PE_k=E_k^+\subset E_k$ is a solution of problem (3), (4). Furthermore, by Remark 5, this solution is nondecreasing and nonconstant due to the definition of space E_k.

The case $r\le 0$ is similar (with P replaced by $-P$). In this case, nonincreasing solutions are obtained.

It is easy to see that for $\varphi\in[\pi+2\pi n,\frac{3\pi}{2}+2\pi n]$, $n\in\mathbb{Z}$, in the case $r\ge 0$ nonincreasing solutions are obtained, and in the case $r\le 0$ nondecreasing solutions are obtained. □

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