УДК 517.53

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GLOBAL ESTIMATES FOR SUMS OF ABSOLUTELY CONVERGENT DIRICHLET SERIES IN A HALF-PLANE

P. V. Filevych, O. B. Hrybel. *Global estimates for sums of absolutely convergent Dirichlet series in a half-plane*, Mat. Stud. **59** (2023), 60–67.

Let $(\lambda_n)_{n=0}^{+\infty}$ be a nonnegative sequence increasing to $+\infty$, $F(s) = \sum_{n=0}^{+\infty} a_n e^{s\lambda_n}$ be an absolutely convergent Dirichlet series in the half-plane $\{s \in \mathbb{C}\colon \operatorname{Re} s < 0\}$, and let, for every $\sigma < 0$, $\mathfrak{M}(\sigma, F) = \sum_{n=0}^{+\infty} |a_n| e^{\sigma\lambda_n}$.

Suppose that $\Phi\colon (-\infty, 0) \to \overline{\mathbb{R}}$ is a function, and let $\widetilde{\Phi}(x)$ be the Young-conjugate function of $\Phi(\sigma)$, i.e. $\widetilde{\Phi}(x) = \sup\{x\sigma - \alpha(\sigma)\colon \sigma < 0\}$ for all $x \in \mathbb{R}$. The following two statements are proved:

(i) There exist constants $\theta \in (0, 1)$ and $C \in \mathbb{R}$ such that $\ln \mathfrak{M}(\sigma, F) \le \Phi(\theta\sigma) + C$ for all $\sigma < 0$ if and only if there exist constants $\delta \in (0, 1)$ and $c \in \mathbb{R}$ such that

$$\ln \sum_{m=0}^{n} |a_m| \le -\widetilde{\Phi}(\lambda_n/\delta) + c$$

for all integers $n \ge 0$ (Theorem 2);

(ii) For every $\theta \in (0, 1)$ there exists a real constant $C = C(\delta)$ such that $\ln \mathfrak{M}(\sigma, F) \le \Phi(\theta\sigma) + C$ for all $\sigma < 0$ if and only if for every $\delta \in (0, 1)$ there exists a real constant $c = c(\delta)$ such that

$$\ln \sum_{m=0}^{n} |a_m| \le -\widetilde{\Phi}(\lambda_n/\delta) + c$$

for all integers $n \ge 0$ (Theorem 3). ;

(iii) Let Φ be a continuous positive increasing function on $\mathbb{R}$ such that $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$ and F be a entire Dirichlet series. For every $q > 1$ there exists a constant $C = C(q) \in \mathbb{R}$ such that $\ln \mathfrak{M}(\sigma, F) \le \Phi(q\sigma) + C,\ \sigma \in \mathbb{R}$, holds if and only if for every $\delta \in (0, 1)$ there exist constants $c = c(\delta) \in \mathbb{R}$ and $n_0 = n_0(\delta) \in \mathbb{N}_0$ such that

$$\ln \sum_{m=n}^{+\infty} |a_m| \le -\widetilde{\Phi}(\delta\lambda_n) + c, \quad n \ge n_0$$

(Theorem 5). These results are analogous to some results previously obtained by M. M. Sheremeta for entire Dirichlet series.

1. Introduction. We denote by $\mathbb{N}_0$ the set of all nonnegative integers, and let Λ be the class of all nonnegative sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ increasing to $+\infty$.

Suppose that $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ is a sequence from the class Λ. Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{+\infty} a_n e^{s\lambda_n} \tag{1}$$

and denote by $\sigma_a(F)$ the abscissa of absolute convergence of this series. Put

$$\beta(F) = \varliminf_{n\to+\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

2010 *Mathematics Subject Classification:* 30B50, 30D15, 30D20.
Keywords: Dirichlet series; abscissa of absolute convergence; maximal term; Young-conjugate function.
doi:10.30970/ms.59.1.60-67

In the case when $\sigma_a(F) > -\infty$, we also put $\mathfrak{M}(\sigma, F) = \sum_{n=0}^{+\infty} |a_n|e^{\sigma\lambda_n}$ for all $\sigma < \sigma_a(F)$.

It is easy to see that if $\sigma < \beta(F)$, then $|a_n|e^{\sigma\lambda_n} \to 0$ as $n \to +\infty$. Therefore, for all $\sigma < \beta(F)$, we can define the maximal term $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}$ of series (1). Note also that in the case when $\sigma > \beta(F)$ we have

$$\varlimsup_{n\to+\infty} |a_n|e^{\sigma\lambda_n} = +\infty.$$

Assume that series (1) is absolutely convergent at the point $s = 0$. For each $n \in \mathbb{N}_0$ we put $S_n = \sum_{m=n}^{+\infty} |a_m|$ and consider the Dirichlet series

$$F_1(s) = \sum_{n=0}^{+\infty} S_n e^{s\lambda_n}.$$

Then, as well-known (see, for example, [1, Theorem I.2.8]), $\sigma_a(F) = \beta(F_1)$.

If series (1) is absolutely divergent at the point $s = 0$, then for each $n \in \mathbb{N}_0$ we set $T_n = \sum_{m=0}^{n} |a_m|$ and consider the Dirichlet series

$$F_2(s) = \sum_{n=0}^{+\infty} T_n e^{s\lambda_n}.$$

Then, as well-known (see, for example, [1, Theorem I.2.8]), $\sigma_a(F) = \min\{0, \beta(F_2)\}$.

In what follows we assume that, for a given Dirichlet series F of the form (1), the sequence $(S_n)_{n\in\mathbb{N}_0}$ and the series F_1 (in the case when the series F is absolutely convergent at the point $s = 0$), and also the sequence $(T_n)_{n\in\mathbb{N}_0}$ and the series F_2 are always defined by F as above.

Let $A \in (-\infty, +\infty]$, and let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ. Denote by $\mathcal{D}_A(\lambda)$ the class of all Dirichlet series of the form (1) such that $\sigma_a(F) \ge A$, and let $\mathcal{D}_A^*(\lambda)$ be the class of all Dirichlet series of the form (1), for which $\beta(F) \ge A$. Put

$$\mathcal{D}_A = \bigcup_{\lambda\in\Lambda} \mathcal{D}_A(\lambda), \quad \mathcal{D}_A^* = \bigcup_{\lambda\in\Lambda} \mathcal{D}_A^*(\lambda).$$

By X we denote the class of all functions $\alpha\colon \mathbb{R} \to \overline{\mathbb{R}}$. For a function $\alpha \in X$ let $\widetilde{\alpha}(x)$ be the Young-conjugate function to $\alpha(\sigma)$, i.e.

$$\widetilde{\alpha}(x) = \sup\{x\sigma - \alpha(\sigma) \colon \sigma \in \mathbb{R}\}, \quad x \in \mathbb{R}.$$

Note (see, for example, [2]) that $\widetilde{\alpha}(x)$ is a convex function, that is, for arbitrary $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 < x_2 < x_3$, we have

$$\widetilde{\alpha}(x_2)(x_3 - x_1) \le \widetilde{\alpha}(x_1)(x_3 - x_2) + \widetilde{\alpha}(x_3)(x_2 - x_1).$$

By Ω' we denote the class of all continuously differentiable, positive functions Φ on $\mathbb{R}$ such that $\Phi'(\sigma)$ is an increasing, positive function on $\mathbb{R}$. It is clear that $\Omega' \subset X$.

If $\Phi \in \Omega'$, $\varphi(x)$ is the inverse function of $\Phi'(\sigma)$, and

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in \mathbb{R},$$

then it is easy to verify that $\widetilde{\Phi}(x) = x\Psi(\varphi(x))$ for all $x \in \mathbb{R}$.

M. M. Sheremeta [3] proved that for every entire Dirichlet series F (i.e., for every Dirichlet series $F\in\mathcal{D}_{+\infty}$) of the form (1) the following inequalities

$$\mu(\sigma,F_1)\le\mathfrak{M}(\sigma,F)\le\frac{\sigma+\varepsilon}{\varepsilon}\mu(\sigma+\varepsilon,F_1)\quad(\sigma\ge0,\ \varepsilon>0)\tag{2}$$

are true, and applied these inequalities to establish conditions under which some global estimates for sums of entire Dirichlet series hold.

Theorem A ([3])**.** *Let $\Phi\in\Omega'$ and $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*. Then there exist positive constants C_1 and C_2 such that*

$$\ln\mathfrak{M}(\sigma,F)\le C_1\Phi(\sigma+C_2),\quad \sigma\in\mathbb{R},\tag{3}$$

if and only if there exist positive constants c_1 and c_2 such that

$$\ln\sum_{\lambda_n\ge x}|a_n|\le -c_1\widetilde{\Phi}(x/c_1)+c_2x,\quad x\ge0.\tag{4}$$

Theorem B ([3])**.** *Let $\Phi\in\Omega'$ and $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*. Then for every $q>1$ there exists a positive constant $C=C(q)$ such that*

$$\ln\mathfrak{M}(\sigma,F)\le\Phi(q\sigma)+C,\quad \sigma\in\mathbb{R},\tag{5}$$

if and only if for every $\delta\in(0,1)$ there exists a positive constant $c=c(\delta)$ such that

$$\ln\sum_{\lambda_n\ge x}|a_n|\le -\widetilde{\Phi}(\delta x)+c,\quad x\ge0.$$

Note that inequalities (2) can also be effectively applied to study other properties of entire Dirichlet series (see [4–6]).

In connection with inequalities (2), we also note that different estimates for the sum of a Dirichlet series by the maximal term of the same series have been obtained in many works (see, for example, [2, 7–10] and the bibliography therein). It is well known that such estimates can be correct for Dirichlet series only under additional conditions on their exponents. Inequalities (2) qualitatively differ from estimates of this kind precisely in that, for a given entire Dirichlet series, they do not depend on the behavior of the sequence of its exponents.

In this paper, we obtain analogues of inequalities (2) and Theorems A and B for absolutely convergent Dirichlet series in the half-plane $\{s\in\mathbb{C}\colon\ \operatorname{Re}s<0\}$, that is, for series from the class $\mathcal{D}_0$. In addition, we show that Theorem A contains an inaccuracy that is easy to fix, and we will give slightly more general versions of Theorems A and B.

2. Main results. To establish global estimates for sums of absolutely convergent Dirichlet series in the half-plane $\{s\in\mathbb{C}\colon\ \operatorname{Re}s<0\}$, we use the following theorem.

Theorem 1. *Let $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ, and let F be an arbitrary Dirichlet series of the form* (1)*. Then:*

(i) $F\in\mathcal{D}_0$ if and only if $F_2\in\mathcal{D}_0^*$;

(ii) if $F\in\mathcal{D}_0$, for every $\sigma<0$ and $\delta\in(0,1)$ we have

$$\mu(\sigma,F_2)\le\mathfrak{M}(\sigma,F)\le\frac{\mu(\delta\sigma,F_2)}{1-\delta}e^{\lambda_0(1-\delta)\sigma}.\tag{6}$$

For any function $\alpha \in X$ we put

$$D_\alpha = \{\sigma \in \mathbb{R} \colon \alpha(\sigma) < +\infty\}$$

and suppose that $A \in (-\infty, +\infty]$. By X_A we denote the class of all functions $\alpha \in X$ for which $D_\alpha \subset (-\infty, A)$.

Theorem 2. *Let $\Phi \in X_0$, and let $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1). *Then there exist constants $\theta \in (0,1)$ and $C \in \mathbb{R}$ such that*

$$\ln \mathfrak{M}(\sigma, F) \le \Phi(\theta\sigma) + C, \quad \sigma < 0, \tag{7}$$

if and only if there exist constants $\eta \in (0,1)$ and $c \in \mathbb{R}$ such that

$$\ln T_n \le -\widetilde{\Phi}(\lambda_n/\eta) + c, \quad n \in \mathbb{N}_0. \tag{8}$$

Theorem 3. *Let $\Phi \in X_0$, and let $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1). *Then for every $\theta \in (0,1)$ there exists a real constant $C = C(\delta)$ such that we have* (7) *if and only if for every $\eta \in (0,1)$ there exists a real constant $c = c(\eta)$ such that* (8) *holds.*

Returning to Theorem A, we note that for every function $\Phi \in \Omega'$ such that $\Phi(-\infty) = 0$, this theorem is not correct in its sufficient part. To confirm this fact, we put $F(s) = a_0 e^{\lambda_0 s}$, where $a_0 > 1$ and $\lambda_0 > 0$. Then $\mathfrak{M}(\sigma, F) = a_0 e^{\lambda_0 \sigma}$ for all $\sigma \in \mathbb{R}$, and, as it is easy to see, for any function $\Phi \in \Omega'$ there exist positive constants C_1 and C_2 such that (3) holds. On the other hand, if $\Phi(-\infty) = 0$, then directly from the definition of the function $\widetilde{\Phi}(x)$ it follows that $\widetilde{\Phi}(0) = 0$. Since the function $\widetilde{\Phi}(x)$ is continuous on $\mathbb{R}$, for arbitrary positive constants c_1 and c_2 there exists $x_0 > 0$ such that $-\widetilde{\Phi}(c_1 x)/c_1 + c_2 x < \ln a_0$ for all $x \in [0, x_0)$. On the other hand, for all $x \in [0, \lambda_0]$ we have

$$\ln \textstyle\sum_{\lambda_n \ge x} |a_n| = \ln a_0.$$

Therefore, (4) does not hold.

By Ω we denote the class of all continuous, positive, increasing functions Φ on $\mathbb{R}$ such that $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$. Then $\Omega' \subset \Omega$. The following theorems are correct.

Theorem 4. *Let $\Phi \in \Omega$ and $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1) *such that $a_n\lambda_n \ne 0$ for some $n \in \mathbb{N}_0$. Put $k = \min\{n \in \mathbb{N}_0 \colon a_n\lambda_n \ne 0\}$. Then there exist constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that* (3) *holds if and only if simultaneously there exists a constant $\delta > 0$ such that*

$$\varlimsup_{\sigma \to -\infty} \left(\Phi^{-1}\left(\delta \ln \sum_{n \le k} |a_n| e^{\sigma\lambda_n}\right) - \sigma\right) < +\infty \tag{9}$$

and there exist constants $c_1 > 0$, $c_2 \in \mathbb{R}$ and $n_0 \in \mathbb{N}_0$ such that

$$\ln S_n \le -c_1 \widetilde{\Phi}(\lambda_n/c_1) + c_2\lambda_n, \quad n \ge n_0. \tag{10}$$

Theorem 5. *Let $\Phi \in \Omega$ and $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1). *Then for every $q > 1$ there exists a constant $C = C(q) \in \mathbb{R}$ such that* (5) *holds if and only if for every $\delta \in (0,1)$ there exist constants $c = c(\delta) \in \mathbb{R}$ and $n_0 = n_0(\delta) \in \mathbb{N}_0$ such that*

$$\ln S_n \le -\widetilde{\Phi}(\delta\lambda_n) + c, \quad n \ge n_0. \tag{11}$$

3. Auxiliary results. The following statement is trivial (see, for example, [2]).

Lemma 1. *Let $\alpha, \beta \in X$. Then the following conditions are equivalent:*

(i) $\widetilde{\alpha}(\sigma) \le \beta(\sigma)$ *for all* $\sigma \in \mathbb{R}$*;*

(ii) $\alpha(x) \ge \widetilde{\beta}(x)$ *for all* $x \in \mathbb{R}$*.*

Lemma 2. *Let $A \in (-\infty, +\infty]$, $\beta \in X_A$, and let $F \in \mathcal{D}_A^*$ be a Dirichlet series of the form* (1)*. Then the following conditions are equivalent:*

(i) $\ln \mu(\sigma, F) \le \beta(\sigma)$ *for all* $\sigma < A$*;*

(ii) $\ln |a_n| \le -\widetilde{\beta}(\lambda_n)$ *for all* $n \in \mathbb{N}_0$*.*

Proof. Consider the function $\alpha \in X$ such that $\alpha(\lambda_n) = -\ln |a_n|$ for all $n \in \mathbb{N}_0$, and $\alpha(x) = +\infty$ for every $x \in \mathbb{R}$ such that $x \ne \lambda_n$ for each $n \in \mathbb{N}_0$. Then $\ln \mu(\sigma, F) = \widetilde{\alpha}(\sigma)$ for all $\sigma < A$. Furthermore, if $A < +\infty$, then $\beta(\sigma) = +\infty$ for all $\sigma \ge A$. Therefore, the first condition of Lemma 2 coincides with the first condition of Lemma 1. It remains to note that the second condition of Lemma 2 coincides with the second condition of Lemma 1. □

Lemma 3. *Let $A \in (-\infty, +\infty]$, $\Phi \in X_A$, a and b be positive constants and c and d be an arbitrary real constants. Then for the function $\beta(\sigma) = a\Phi(b\sigma + c) + d$, $\sigma \in \mathbb{R}$, we have $\beta \in X_{Ab+c}$ and*

$$\widetilde{\beta}(x) = a\widetilde{\Phi}\left(\frac{x}{ab}\right) - \frac{cx}{b} - d, \quad x \in \mathbb{R}.$$

Proof. The fact that $\beta \in X_{Ab+c}$ is obvious. In addition, taking $y = b\sigma + c$, we get

$$\widetilde{\beta}(x) = \sup_{\sigma \in \mathbb{R}}(x\sigma - \beta(\sigma)) = \sup_{\sigma \in \mathbb{R}}(x\sigma - a\Phi(b\sigma + c) - d) = \sup_{y \in \mathbb{R}}\left(\frac{x}{b}(y - c) - a\Phi(y) - d\right) =$$
$$= a \sup_{y \in \mathbb{R}}\left(\frac{x}{ab}y - \Phi(y)\right) - \frac{cx}{b} - d = a\widetilde{\Phi}\left(\frac{x}{ab}\right) - \frac{cx}{b} - d$$

for all $x \in \mathbb{R}$. □

Using Lemma 2, it is easy to prove the following lemma (see [2]).

Lemma 4. *Let $\beta \in \Omega$, and let $F \in \mathcal{D}_{+\infty}^*$ be a Dirichlet series of the form* (1)*. Then the following conditions are equivalent:*

(i) there exists $\sigma_0 \in \mathbb{R}$ *such that* $\ln \mu(\sigma, F) \le \beta(\sigma)$ *for all* $\sigma \ge \sigma_0$*;*

(ii) there exists $n_0 \in \mathbb{N}_0$ *such that* $\ln |a_n| \le -\widetilde{\beta}(\lambda_n)$ *for all integers* $n \ge n_0$*.*

4. Proofs of the theorems.

Proof of Theorem 1. Let $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ be a sequence from the class Λ, and let F be an arbitrary Dirichlet series of the form (1).

(i) It was already noted above that $\sigma_a(F) \ge \min\{0, \beta(F_2)\}$. In addition, in the case when $\beta(F_2) < 0$ we have $\sigma_a(F) = \beta(F_2)$. Therefore, $\sigma_a(F) \ge 0$ if and only if $\beta(F_2) \ge 0$, i.e. $F \in \mathcal{D}_0$ if and only if $F_2 \in \mathcal{D}_0^*$.

(ii) For any $\sigma < 0$ and $n \in \mathbb{N}_0$ we have $\mathfrak{M}(\sigma, F) \ge \sum_{m=0}^{n} |a_m| e^{\sigma\lambda_m} \ge e^{\sigma\lambda_n} \sum_{m=0}^{n} |a_m| = T_n e^{\sigma\lambda_n}$, and therefore $\mathfrak{M}(\sigma, F) \ge \mu(\sigma, F_2)$.

Let $\sigma < 0$, $\delta \in (0,1)$, and $\varepsilon = -\sigma(1-\delta)$. Noting that $\varepsilon > 0$ and $\sigma + \varepsilon < 0$, we get

$$\mathfrak{M}(\sigma, F) = \sum_{n=0}^{+\infty} |a_n| e^{\sigma\lambda_n} = T_0 e^{\sigma\lambda_0} + \sum_{n=1}^{+\infty} (T_n - T_{n-1}) e^{\sigma\lambda_n} = \sum_{n=0}^{+\infty} T_n (e^{\sigma\lambda_n} - e^{\sigma\lambda_{n+1}}) =$$
$$= -\sigma \sum_{n=0}^{+\infty} T_n \int_{\lambda_n}^{\lambda_{n+1}} e^{\sigma x} dx = -\sigma \sum_{n=0}^{+\infty} T_n e^{(\sigma+\varepsilon)\lambda_n} \int_{\lambda_n}^{\lambda_{n+1}} \frac{1}{e^{(\sigma+\varepsilon)\lambda_n - \sigma x}}\, dx \le$$
$$\le -\sigma \sum_{n=0}^{+\infty} T_n e^{(\sigma+\varepsilon)\lambda_n} \int_{\lambda_n}^{\lambda_{n+1}} \frac{1}{e^{(\sigma+\varepsilon)x - \sigma x}}\, dx \le -\sigma\mu(\sigma+\varepsilon, F_2) \sum_{n=0}^{+\infty} \int_{\lambda_n}^{\lambda_{n+1}} \frac{dx}{e^{\varepsilon x}} =$$
$$= -\sigma\mu(\sigma+\varepsilon, F_2) \int_{\lambda_0}^{+\infty} \frac{dx}{e^{\varepsilon x}} = -\sigma\mu(\sigma+\varepsilon, F_2) \left(-\frac{1}{\varepsilon e^{\varepsilon x}} \Big|_{\lambda_0}^{+\infty} \right) =$$
$$= -\sigma\mu(\sigma+\varepsilon, F_2) \frac{1}{\varepsilon e^{\varepsilon\lambda_0}} = \frac{\mu(\delta\sigma, F_2)}{1-\delta} e^{\lambda_0(1-\delta)\sigma}.$$

□

Proof of Theorem 2. Let $\Phi \in X_0$, and let $F \in \mathcal{D}_0$ be a Dirichlet series of the form (1).

Necessity. Suppose that (7) holds for some constants $\theta \in (0,1)$ and $C \in \mathbb{R}$. Taking $\beta(\sigma) = \Phi(\theta\sigma) + C$ for all $\sigma \in \mathbb{R}$ and using Theorem 1 and (7), we have

$$\ln \mu(\sigma, F_2) \le \ln \mathfrak{M}(\sigma, F) \le \Phi(\theta\sigma) + C = \beta(\sigma), \quad \sigma < 0.$$

Hence, by Lemmas 2 and 3, for all $n \in \mathbb{N}_0$ we obtain

$$\ln T_n \le -\widetilde{\beta}(\lambda_n) = -\widetilde{\Phi}(\lambda_n/\theta) + C,$$

that is, (8) holds with $\eta = \theta$ and $c = C$.

Sufficiency. Suppose that (8) holds for some constants $\eta \in (0,1)$ and $c \in \mathbb{R}$. Let $\beta(\sigma) = \Phi(\eta\sigma) + c$ for all $\sigma \in \mathbb{R}$. According to Lemma 3, we can write (8) in the form

$$\ln T_n \le -\widetilde{\beta}(\lambda_n), \quad n \in \mathbb{N}_0.$$

Therefore, fixing an arbitrary $\theta \in (0, \eta)$ and setting $\delta = \theta/\eta$ and $C = c - \ln(1-\delta)$, by Theorem 1 and Lemma 2 for all $\sigma < 0$ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\delta\sigma, F_2) - \ln(1-\delta) \le \beta(\delta\sigma) - \ln(1-\delta) =$$
$$= \Phi(\eta\delta\sigma) + c - \ln(1-\delta) = \Phi(\theta\sigma) + C,$$

that is, (7) holds. □

Proof of Theorem 3. Let $\Phi \in X_0$, and let $F \in \mathcal{D}_0$ be a Dirichlet series of the form (1).

Necessity. Let $\eta \in (0,1)$ be an arbitrary constant. If there exists a constant $C = C(\eta)$ such that $\ln \mathfrak{M}(\sigma, F) \le \Phi(\eta\sigma) + C$ for all $\sigma < 0$, then, reasoning as in the proof of Theorem 2, we see that (8) holds with $c = c(\eta) = C(\eta)$.

Sufficiency. Let $\theta \in (0,1)$ be an arbitrary constant. We fix some $\delta \in (0,1)$ such that the inequality $\delta\theta < 1$ holds, and set $\eta = \delta\theta$. If there exists a constant $c = c(\eta)$ such that (8) holds, then, reasoning again as in the proof of Theorem 2, we see that (7) holds with $C = C(\theta) = c(\eta)$. □

Proof of Theorem 4. Let $\Phi \in \Omega$, and let $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form (1), which satisfies the hypotheses of Theorem 4.

Let $\sigma \in \mathbb{R}$. Put $P(\sigma) = \sum_{n\le k} |a_n| e^{\sigma\lambda_n}$ and note that $P(\sigma) = a + |a_k| e^{\sigma\lambda_k}$, where $a = 0$ if $k = 0$ and $a = |a_0|$ if $k \ge 1$. Then

$$\mathfrak{M}(\sigma, F) - a \sim |a_k| e^{\sigma\lambda_k}$$

as $\sigma \to -\infty$. This implies that there exists a number $\sigma_1 \in \mathbb{R}$ such that

$$\mathfrak{M}(\sigma, F) \le P(\sigma + 1), \quad \sigma \le \sigma_1. \tag{12}$$

Necessity. Suppose that there exist constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that (3) holds. Putting $\delta = 1/C_1$, by (3) we have

$$\delta \ln P(\sigma) \le \delta \ln \mathfrak{M}(\sigma, F) \le \Phi(\sigma + C_2)$$

for each $\sigma \in \mathbb{R}$. This implies (9). Next, let $\beta(\sigma) = C_1\Phi(\sigma + C_2)$ for all $\sigma \in \mathbb{R}$. Then $\beta \in \Omega$ and, by Lemma 3, $\widetilde{\beta}(x) = C_1\widetilde{\Phi}(x/C_1) - C_2 x$ for every $x \in \mathbb{R}$. Using (2) and (3), for all $\sigma \ge 0$ we have $\ln \mu(\sigma, F_1) \le \beta(\sigma)$. Therefore, by Lemma 4 there exists $n_0 \in \mathbb{N}_0$ such that $\ln S_n \le -\widetilde{\beta}(\lambda_n)$ for all integers $n \ge n_0$, i.e., we obtain (10) with $c_1 = C_1$ and $c_2 = C_2$.

Sufficiency. Now suppose that there exist constants $\delta > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, and $n_0 \in \mathbb{N}_0$ such that (9) and (10) hold. Taking $c_4 = 1/\delta$ and using (9) and (12), for some constants $c_5, \sigma_2 \in \mathbb{R}$ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln P(\sigma + 1) \le c_4\Phi(\sigma + c_5), \quad \sigma \le \sigma_2. \tag{13}$$

Let $\beta(\sigma) = c_1\Phi(\sigma + c_2)$ for all $\sigma \in \mathbb{R}$. Then $\beta \in \Omega$ and by (10) and by Lemma 3 we obtain $\ln S_n \le -\widetilde{\beta}(\lambda_n)$ for all integers $n \ge n_0$. Using (2) with $\varepsilon = 1$ and Lemma 4, for some $\sigma_3 > \sigma_2$ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\sigma + 1, F_1) + \ln(\sigma + 1) \le 2\beta(\sigma + 1) = 2c_1\Phi(\sigma + 1 + c_2), \quad \sigma \ge \sigma_3. \tag{14}$$

Next, we choose $\sigma_4 > \sigma_3$ such that the inequality $\Phi(\sigma_4) \ge \ln \mathfrak{M}(\sigma_3, F)$ holds, and we set $c_6 = \sigma_4 - \sigma_2$. Then

$$\forall \sigma \in [\sigma_2, \sigma_3]: \quad \ln \mathfrak{M}(\sigma, F) \le \ln \mathfrak{M}(\sigma_3, F) \le \Phi(\sigma_4) = \Phi(\sigma_2 + c_6) \le \Phi(\sigma + c_6). \tag{15}$$

Taking $C_1 = \max\{c_4, 2c_1, 1\}$, $C_2 = \max\{c_5, c_2 + 1, c_6\}$, from (13), (14) and (15) we see that (3) holds. □

Proof of Theorem 5. Let $\Phi \in \Omega$, and let $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form (1).

Necessity. Suppose that for each $q > 1$ there exists a constant $C = C(q) \in \mathbb{R}$ such that (5) holds. Let $\delta \in (0, 1)$ be an arbitrary fixed number, $q = 1/\delta$, and $C = C(q)$. Put $\beta(\sigma) = \Phi(q\sigma) + C$ for all $\sigma \in \mathbb{R}$. Note that by Lemma 3 we have $\widetilde{\beta}(x) = \widetilde{\Phi}(\delta x) - C$ for all $x \in \mathbb{R}$. Since, according to (2) and (5), the inequality $\ln \mu(\sigma, F_1) \le \beta(\sigma)$ holds for all $\sigma \ge 0$, by Lemma 4 there exists $n_0 \in \mathbb{N}_0$ such that for all integers $n \ge n_0$ we have

$$\ln S_n \le -\widetilde{\beta}(\lambda_n) = -\widetilde{\Phi}(\delta\lambda_n) + C,$$

i.e., (11) holds with $c = C$.

Sufficiency. Suppose that for each $\delta \in (0, 1)$ there exist constants $c = c(\delta) \in \mathbb{R}$ and $n_0 = n_0(\delta) \in \mathbb{N}_0$ such that (11) holds. Let $q > 1$ be an arbitrary fixed number, $p = (q + 1)/2$,

$\delta = 1/p$, $c = c(\delta)$, and $n_0 = n_0(\delta)$. Put $\beta(\sigma) = \Phi(p\sigma) + c$ for all $\sigma \in \mathbb{R}$. According to (11) and Lemma 3, we have

$$\ln S_n \le -\widetilde{\Phi}(\delta\lambda_n) + c = -\widetilde{\beta}(\lambda_n)$$

for all integers $n \ge n_0$. Then, by Lemma 4, there exists $\sigma_0 > 0$ such that

$$\ln \mu(\sigma, F) \le \beta(\sigma) = \Phi(p\sigma) + c$$

for all $\sigma \ge \sigma_0$. Therefore, using (2) with $\varepsilon = (\delta q - 1)\sigma$, for every $\sigma \ge p\sigma_0/q$ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\delta q\sigma, F_1) + \ln \frac{2q}{q-1} \le \Phi(q\sigma) + c + \ln \frac{2q}{q-1}.$$

So, taking

$$C = \max\left\{c + \ln \frac{2q}{q-1}, \ln \mathfrak{M}\left(\frac{p\sigma_0}{q}, F\right)\right\},$$

we see that (5) holds. □

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Received 29.09.2022
Revised 24.03.2023