Introduction. Let $\Lambda = (\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$,
\[
f(z) = \sum_{k=0}^{\infty} f_k z^k
\]  
be an entire transcendental function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Suppose that the series
\[
A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)
\]  
in the system $f(\lambda_n z)$ converges regularly in $C$, i.e. for all $r \in [0, +\infty)$
\[
\mathfrak{M}(r, A) := \sum_{n=1}^{\infty} |a_n| M_f(r \lambda_n) < +\infty.
\]  
Many authors studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We mention here only the monographs of A. F. Leont’ev [1] and B. V. Vynnytskyi [2], and there in references.

Since series (2) converges regularly in $C$, the function $A$ is entire. To study its growth, generalized orders are used. For this purpose, as in [3] by $L$ we denote the class of continuous non-negative on $(-\infty, +\infty)$ functions $\alpha$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow$.

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spaces of series in systems of functions

$+\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. $\alpha$ is a slowly increasing function. Clearly, $L_{si} \subseteq L^0$. For $\alpha \in L$ and $\beta \in L$

quantity

$$g_{\alpha, \beta}[f] = \lim_{r \to +\infty} \frac{\alpha(\ln M_f(r))}{\beta(r)}$$

is called ([3]) generalized $(\alpha, \beta)$-order of the entire function $f$. In terms of the generalized $(\alpha, \beta)$-orders in the paper [4–5, 14] the relationship between the growth of functions $M_f(r)$, $\mathfrak{M}(r, A)$ and $M_f^{-1}(\mathfrak{M}(r, A))$ was studied.

The study of various spaces of analytic functions represented by power series and Dirichlet series has been studied by many authors (we only point out here [6–10]). For the Laplace-Stieltjes integrals, the Banach and Fréchet spaces were studied in [11–12].

The present paper is devoted to the study of Banach and Fréchet spaces for entire functions. At first we remark that if

$$\ln n = o(\Gamma_f(\lambda_n)) \text{ as } n \to \infty \text{ then series (2) converges regularly in } \mathbb{C} \text{ if and only if}$$

$$\lim_{n \to \infty} \frac{1}{\lambda_n} M_f^{-1} \left( \frac{1}{|a_n|} \right) = +\infty. \quad (4)$$

Indeed, if series (2) regularly converges in $\mathbb{C}$ then $|a_n|M_f(r\lambda_n) \to 0$ as $n \to \infty$, i.e. for all $r \in [0, +\infty)$ and $n \geq n_0(r)$ one has $|a_n|M_f(r\lambda_n) \leq 1$, whence $\frac{1}{\lambda_n} M_f^{-1} \left( \frac{1}{|a_n|} \right) \geq r$ for all $n \geq n_0(r)$. In view of the arbitrariness of $r$ we get (4).

On the other hand, if $r \in [1, +\infty)$ is an arbitrary number and (4) holds then for every $K > r$ and all $n \geq n_0 = n_0(K)$ we have $\frac{1}{\lambda_n} M_f^{-1} \left( \frac{1}{|a_n|} \right) \geq K$, i.e. $|a_n|M_f(K\lambda_n) \leq 1$. Therefore,

$$\sum_{n=n_0}^{\infty} |a_n|M_f(r\lambda_n) \leq \sum_{n=n_0}^{\infty} |a_n|M_f(K\lambda_n) \leq \sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f(K\lambda_n)} =$$

$$= \sum_{n=n_0}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{K\lambda_n} \frac{d\ln M_f(t)}{d\ln t} dt \right\} \leq \sum_{n=n_0}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{K\lambda_n} \Gamma_f(t) dt \right\} \leq$$

$$\leq \sum_{n=n_0}^{\infty} \exp \left\{ -\Gamma_f(r\lambda_n) \ln (K/r) \right\} \leq \sum_{n=n_0}^{\infty} \exp \left\{ -\Gamma_f(\lambda_n) \ln (K/r) \right\} < +\infty,$$

because $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$, what was required to prove.

Let $h$ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and such that

$$|a_n|M_f(\lambda_nh(\lambda_n)) \to 0, \quad n \to +\infty. \quad (5)$$
Let us show that if \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then the series

\[
B_h(z) = \sum_{n=1}^{\infty} \frac{f(z\lambda_n)}{M_f(\lambda_n h(\lambda_n))}
\]  

converges regularly in \( \mathbb{C} \).

Indeed, for every \( r \geq 1 \) as above we have

\[
\sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f(\lambda_n h(\lambda_n))} = \sum_{n=1}^{\infty} \exp \left\{- \frac{\lambda_n h(\lambda_n)}{\Gamma_f(t) \ln t} \right\} \leq \sum_{n=1}^{\infty} \exp \left\{-\Gamma_f(\lambda_n) \ln \frac{h(\lambda_n)}{r} \right\} < +\infty,
\]

because \( h(\lambda_n) \to +\infty \) and \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \).

By \( S_h(f, \Lambda) \) we denote the class of series (2) such that (5) holds. On \( S_h(f, \Lambda) \) for \( A_j(z) = \sum_{n=1}^{\infty} \lambda_{n,j} f(\lambda_n z) \) \( (j = 1, 2) \) we define operations

\[
(A_1 + A_2)(z) = \sum_{n=1}^{\infty} (a_{n,1} + a_{n,2}) f(\lambda_n z), \quad (\lambda A)(z) = \sum_{n=1}^{\infty} \lambda a_n f(\lambda_n z).
\]

Put \( \|A\|_h = \max\{\|a_n\|_{M_f(\lambda_n h(\lambda_n))} : n \geq 1\} \). Under these operations \( S_h(f, \Lambda) \) becomes a normalized linear space.

**Theorem 1.** If \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then \( (S_h(f, \Lambda), \| \cdot \|_h) \) is a non-uniformly convex Banach space which is separable also.

**Proof.** Let \( (A_p) \) be a Cauchy sequence in \( S_h(f, \Lambda) \), \( A_p(z) = \sum_{n=1}^{\infty} a_{n,p} f(\lambda_n z) \). Then in view of (5) \( |a_{n,p}|_{M_f(\lambda_n h(\lambda_n))} \to 0 \) as \( x \to +\infty \) for each \( p \), and for a given \( \varepsilon > 0 \) there exists \( j_0 = j_0(\varepsilon) \in \mathbb{N} \) such that \( \|A_p - A_q\|_h < \varepsilon \) for all \( p \geq j_0 \) and \( q \geq j_0 \), i.e.

\[
\max\{\|a_{n,p} - a_{n,q}\|_{M_f(\lambda_n h(\lambda_n))} : n \geq 1\} < \varepsilon.
\]

Thus, \( |a_{n,p} M_f(\lambda_n h(\lambda_n)) - a_{n,q} M_f(\lambda_n h(\lambda_n))| < \varepsilon \) for all \( n \geq 1 \), \( p \geq j_0 \) and \( q \geq j_0 \). This shows that \( (|a_{n,p}|_{M_f(\lambda_n h(\lambda_n))})_{p=1}^{\infty} \) is a Cauchy sequence, so it converges to \( |a_{n,0}|_{M_f(\lambda_n h(\lambda_n))} \) (say) as \( p \to \infty \). Since

\[
|a_{n,0}|_{M_f(\lambda_n h(\lambda_n))} \leq |a_{n,0} M_f(\lambda_n h(\lambda_n)) - a_{n,p} M_f(\lambda_n h(\lambda_n))| + |a_{n,p}|_{M_f(\lambda_n h(\lambda_n))} \to 0
\]

as \( p \to \infty \), the series \( A_0(z) = \sum_{n=1}^{\infty} a_{n,0} f(\lambda_n z) \) belongs to \( S_h(f, \Lambda) \). Also we have

\[
\|A_p - A_0\|_h = \max\{|a_{n,p} M_f(\lambda_n h(\lambda_n)) - a_{n,0} M_f(\lambda_n h(\lambda_n))| : n \geq 1\} \to 0, \quad p \to \infty,
\]

i.e. the space \( (S_h(f, \Lambda), \| \cdot \|_h) \) is complete and, thus, a Banach space.

Further consider \( B_{1,h} \) and \( B_{2,h} \) defined as follows:

\[
B_{1,h}(z) = \frac{f(z\lambda_{n_0})}{M_f(\lambda_{n_0} h(\lambda_{n_0}))}, \quad B_{2,h}(z) = \frac{f(z\lambda_n)}{M_f(\lambda_n h(\lambda_n))} + \frac{f(z\lambda_k)}{M_f(\lambda_k h(\lambda_k))},
\]

where \( n_0 \) and \( k \) are positive fixed integers. Obviously, \( B_{j,h} \in S_h(f, \Lambda) \), but \( \|B_{1,h}\|_h = 1, \|B_{2,h}\| = 1, \|B_{1,h} + B_{2,h}\| = 2 \) and \( \|B_{2,h} - B_{1,h}\| = 1 \neq 0 \), i.e. the space \( (S_h(f, \Lambda), \| \cdot \|_h) \) is non-uniformly convex (see, for example, [7, p. 183]).
It is still to be shown that \((S_h(f, \Lambda), \| \cdot \|_h)\) is separable. For this, at first, consider the set of all function \(A_m \in (S_h(f, \Lambda), \| \cdot \|_h)\), which has the representation \(A_m(z) = \sum_{n=1}^{m} b_n f(\lambda_n z)\), where \(b_n = c_n + id_n\) and \(c_n, d_n\) are rational numbers for every \(n\). This set is readily seen to be a countable one. It is also everywhere dense in \((S_h(f, \Lambda), \| \cdot \|_h)\).

Since \(A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \in S_h(f, \Lambda), \| \cdot \|_h)\), i.e. \(|a_n| M_f(\lambda_n h(\lambda_n)) \to 0\) as \(n \to +\infty\), for every \(\varepsilon > 0\) and \(n \geq n_0 = n_0(\varepsilon)\), we have \(|a_n| M_f(\lambda_n h(\lambda_n)) < \varepsilon/2\), i.e. \(\max \{ |a_n| M_f(\lambda_n h(\lambda_n)) : n \geq n_0 \} < \varepsilon/2\).

Now let \(G \in (S_h(f, \Lambda), \| \cdot \|_h)\) be defined as \(G(s) = \sum_{n=1}^{\infty} b_n f(\lambda_n z)\), where \(b_n\) are given as \(b_n = 0\) for \(n \geq n_0\) and \(|a_n - b_n| M_f(\lambda_n h(\lambda_n)) < \varepsilon/2\) for \(1 \leq n \leq n_0 - 1\). Then

\[
\|A - G\|_h \leq \max \{ |a_n - b_n| M_f(\lambda_n h(\lambda_n)) : n \leq n_0 - 1 \} + \max \{ |a_n| M_f(\lambda_n h(\lambda_n)) : n \geq n_0 \} < \varepsilon.
\]

Therefore, \((S_h(f, \Lambda), \| \cdot \|_h)\) is separable, and Theorem 1 is proved.

The following statement concerns uniform convergence of \((A_m)\).

**Theorem 2.** Let \(\ln n = o(\Gamma_f(\lambda_n))\) as \(n \to \infty\). In order that \((A_m) \subset S_h(f, \Lambda)\) converges to \(A \in S_h(f, \Lambda)\) by \(\| \cdot \|_h\), it is necessary and sufficient that \(A_m(z)\) converges uniformly to \(A(z)\) over each compact subset of \(\mathbb{C}\).

**Proof.** If \(A_m(z) = \sum_{n=1}^{\infty} a_{n,m} f(\lambda_n z)\) and \(\|A_m - A\|_h < \varepsilon\) for every \(\varepsilon > 0\) and all \(m \geq m_0(\varepsilon)\) then

\[
\max \{ |a_{n,m} - a_n| M_f(\lambda_n h(\lambda_n)) : n \geq 1 \} < \varepsilon
\]

and, thus, \(|a_{n,m} - a_n| M_f(\lambda_n h(\lambda_n)) < \varepsilon\) for every \(\varepsilon > 0\), all \(m \geq m_0(\varepsilon)\) and all \(n \geq 1\). Therefore, if \(m \geq m_0(\varepsilon)\) and \(r \leq r_0 < +\infty\) then, as above,

\[
|A_m(z) - A(z)| = \sum_{n=1}^{\infty} |a_{n,m} - a_n| f(\lambda_n z) \leq \sum_{n=1}^{\infty} |a_{n,m} - a_n| M_f(r \lambda_n) \leq \sum_{n=1}^{\infty} |a_{n,m} - a_n| M_f(\lambda_n h(\lambda_n)) \frac{M_f(r_0 \lambda_n)}{M_f(\lambda_n h(\lambda_n))} \leq \varepsilon \sum_{n=1}^{\infty} \exp \left\{ -\Gamma_f(\lambda_n) \ln \frac{h(\lambda_n)}{r_0} \right\} = K_0 \varepsilon.
\]

From hence it follows that \(A_m(z)\) converges uniformly to \(A(z)\) on \(\{z : |z| \leq r_0\}\).

Conversely, let \(A_m(z)\) converges uniformly to \(A(z)\) on each \(\{z : |z| \leq r\}\). Then \(|a_{n,m} - a_n| M_f(r \lambda_n) < \varepsilon\) for every \(\varepsilon > 0\), all \(n \geq 1\) and all \(m \geq m_0 = m_0(\varepsilon, r)\), whence

\[
|a_{n,m} - a_n| M_f(\lambda_n h(\lambda_n)) \leq \varepsilon \frac{M_f(\lambda_n h(\lambda_n))}{M_f(rh(\lambda_n))}.
\]

Choosing \(r = h(\lambda_n)\) from hence we get \(|a_{n,m} - a_n| M_f(\lambda_n h(\lambda_n)) \leq \varepsilon\) for every \(\varepsilon > 0\), all \(n \geq 1\) and all \(m \geq m_0 = m_0(\varepsilon, n)\), i.e. \(\|A_m - A\|_h \to 0\) as \(m \to \infty\).
Corollary 1. Let $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$, $A_m(z) = \sum_{n=1}^{m} a_n f(\lambda_n z)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$. Then $A_m(z) \to A(z)$ as $m \to \infty$ for all $z$ if and only if $|a_n| M_f(\lambda_n h(\lambda_n)) \to 0$ as $n \to \infty$, i.e. $A \in S_h(f, \Lambda)$.

Indeed, if $A \in S_h(f, \Lambda)$ then $|a_n| M_f(\lambda_n h(\lambda_n)) \to 0$ as $n \to \infty$ and $\|A_m - A\|_h = \max\{|a_n| M_f(\lambda_n h(\lambda_n)) : n \geq m\} \to 0$ as $m \to \infty$, i.e. $A_m \to A$ by $\| \cdot \|_h$ and, therefore, by Theorem 2 $A_m(z) \to A(z)$ as $m \to \infty$ for all $z$.

Conversely, if $A \notin S_h(f, \Lambda)$ then $|a_n| M_f(\lambda_n h(\lambda_n)) \geq \eta > 0$ for some sequence $(n_j) \uparrow \infty$. Therefore, if $m \leq p < q < \infty$ and $A_{p,q}(z) = \sum_{n=p}^{q} a_n f(\lambda_n z)$ then

$$\|A_{p,q}\|_h = \max\{|a_n| M_f(\lambda_n h(\lambda_n)) : p \leq n \leq q\} \geq \eta$$

provided $p \leq n_j \leq q$. Hence it follows that $(A_m)$ is not even a Cauchy sequence.

Now, for $(S_h(f, \Lambda), \| \cdot \|_h)$ by $S_h^*(f, \Lambda)$ we denote the dual space, i.e. $S_h^*(f, \Lambda)$ is the family of all continuous linear functionals on $(S_h(f, \Lambda), \| \cdot \|_h)$. Let $L(A) = \sum_{n=1}^{\infty} a_n g_n$, where real numbers are such that

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = K < +\infty. \tag{7}$$

Theorem 3. Let $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$. Then every bounded linear functional defined on $(S_h(f, \Lambda), \| \cdot \|_h)$ is of the form

$$L(A) = \sum_{n=1}^{\infty} a_n g_n, \quad A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z), \tag{8}$$

where $g_n$ is real-valued sequence satisfying (7).

Proof. In view of (7) we have

$$\sum_{n=1}^{\infty} |a_n g_n| = \sum_{n=1}^{\infty} |a_n| M_f(\lambda_n h(\lambda_n)) \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} \leq \max\{|a_n| M_f(\lambda_n h(\lambda_n)) : n \geq 1\} \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = K \|A\|_h < +\infty,$$

i.e. $L$ is well-defined functional on $(S_h(f, \Lambda), \| \cdot \|_h)$. Moreover,

$$|L(A)| \leq \|A\|_h \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))},$$

whence

$$\|L\|_h \leq \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))}. \tag{9}$$

Conversely, we first remark that if $A \in (S_h(f, \Lambda), \| \cdot \|_h)$ and $A_m(z) = \sum_{n=1}^{m} a_n f(\lambda_n z)$ then

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} < +\infty.$$
Proof. For $p \in \mathbb{N}$ and let $a_n = \frac{\text{sign}(g_n)}{M_f(\lambda_n, h(\lambda_n))}$ for $1 \leq n \leq p$ and $a_n = 0$ for $n > p$. If we define $A(z) = \sum_{n=1}^{\infty} a_n f(z \lambda_n)$ then obviously $A \in \mathcal{S}_h(f, A)$ and $\|A\|_h = 1$. Hence

$$|L(A)| = \left| \sum_{n=1}^{p} \frac{\text{sign}(g_n)}{M_f(\lambda_n, h(\lambda_n))} L(f(z \lambda_n)) \right| = \sum_{n=1}^{p} \frac{|g_n|}{M_f(\lambda_n, h(\lambda_n))}$$

and $|L(A)| \leq \|A\|_h \|L\|_h = \|L\|_h$, so that $\sum_{n=1}^{p} \frac{|g_n|}{M_f(\lambda_n, h(\lambda_n))} \leq \|L\|_h$ and

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n, h(\lambda_n))} = \sup_p \sum_{n=1}^{p} \frac{|g_n|}{M_f(\lambda_n, h(\lambda_n))} \leq \sup_p \|L\|_h = \|L\|_h. \quad (10)$$

Inequalities (9) and (10) together show that

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n, h(\lambda_n))} = \|L\|_h$$

and this completes the proof of the Theorem 3. \(\square\)

3. Growth of $\mathbb{M}(r, A)$. Let $\mu(r, A) = \max\{ |a_n| M_f(r \lambda_n) : n \geq 1 \}$ be the maximal term and $\nu(r, A) = \max\{ n \geq 1 : |a_n| M_f(r \lambda_n) = \mu(r, A) \}$ be the central index of series (3).

Lemma 1. The functions $\ln \mu(r, A)$, $\lambda_{\nu(r, A)}$ and $\nu(r, A)$ are non-decreasing and

$$\ln \mu(r, A) - \ln \mu(r_0, A) = \int_{r_0}^{r} \frac{\Gamma_f(t \lambda_{\nu(t, A)})}{t} dt, \quad 0 \leq r_0 \leq r < +\infty. \quad (11)$$

Proof. For $h > 0$ we have

$$\mu(r + h, A) = |a_{\nu(r+h, A)}| M_f((r + h) \lambda_{\nu(r+h, A)}) = |a_{\nu(r+h, A)}| M_f((r \lambda_{\nu(r, A)}) \frac{M_f((r + h) \lambda_{\nu(r+h, A)})}{M_f(r \lambda_{\nu(r, A)})} \leq \mu(r, A) \exp\{ \ln M_f((r + h) \lambda_{\nu(r+h, A)}) - \ln M_f(r \lambda_{\nu(r, A)}) \} = \mu(r, A) \exp\left\{ \int_{r \lambda_{\nu(r, A)}}^{(r+h) \lambda_{\nu(r+h, A)}} \Gamma_f(t) dt \ln t \right\} \leq$$

$$= \mu(r, A) \exp\left\{ \int_{r \lambda_{\nu(r, A)}}^{(r+h) \lambda_{\nu(r+h, A)}} \Gamma_f(t) dt \ln t \right\} \leq$$
\[ \mu(r, A) \exp \left\{ \Gamma_f((r + h)\lambda_{\nu(r+h,A)}) \ln (1 + h/r) \right\}, \]
i.e.
\[ \ln \mu(r + h, A) - \ln \mu(r, A) \leq \Gamma_f((r + h)\lambda_{\nu(r+h,A)}) \ln (1 + h/r). \]  
(12)

Similarly,
\[ \mu(r, A) = |a_{\nu(r,A)}| M_f((r + h)\lambda_{\nu(r,A)}) \frac{M_f(r\lambda_{\nu(r,A)})}{M_f((r + h)\lambda_{\nu(r,A)})} \leq \]
\[ \mu(r + h, A) \exp \left\{ - \int_{r\lambda_{\nu(r,A)}}^{(r+h)\lambda_{\nu(r,A)}} \Gamma_f(t)d\ln t \right\} \leq \mu(r + h, A) \exp \left\{ -\Gamma_f(r\lambda_{\nu(r,A)}) \ln (1 + h/r) \right\}, \]
i.e.
\[ \ln \mu(r + h, A) - \ln \mu(r, A) \geq \Gamma_f(r\lambda_{\nu(r,A)}) \ln (1 + h/r). \]  
(13)

From (12) and (13) we obtain
\[ \Gamma_f(r\lambda_{\nu(r,A)}) \frac{\ln (1 + h/r)}{h} \leq \frac{\ln \mu(r + h, A) - \ln \mu(r, A)}{h} \leq \frac{\Gamma_f((r + h)\lambda_{\nu(r+h,A)}) \ln (1 + h/r)}{h}. \]
Hence it follows that the functions \( \ln \mu(r, A), \lambda_{\nu(r,A)} \) and \( \nu(r, A) \) are non-decreasing. Our reasoning is also correct if \( h < 0 \). Therefore, if \((r_1, r_2)\) is an interval of constancy of the function \( \nu(r, A) \) then at \( h \to 0 \) we obtain
\[ \frac{d\ln \mu(r, A)}{dr} = \frac{\Gamma_f(r\lambda_{\nu(r,A)})}{r}, \quad r \in (r_1, r_2). \]

Since the function \( \Gamma_f(r\lambda_{\nu(r,A)}) \) has a finite number of discontinuities on each finite interval, we obtain the equality (11).

We need also the following lemma.

**Lemma 2.** If \( \ln n = O(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then for some \( q > 1 \) and all \( r \geq 1 \)
\[ \mu(r, A) \leq \mathcal{M}(r, A) \leq K\mu(qr, A), \quad K = \text{const} > 0, \]  
(14)
and if \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then for every \( \varepsilon > 0 \) and all \( r \geq 1 \)
\[ \mu(r, A) \leq \mathcal{M}(r, A) \leq K(\varepsilon)\mu((1 + \varepsilon)r, A), \quad K(\varepsilon) > 0. \]  
(15)

**Proof.** From (3) for \( q > 1 \) and \( r \geq 1 \) as above have
\[ \mu(r, A) \leq \mathcal{M}(r, A) \leq \sum_{n=1}^{\infty} |a_n| M_f(qr\lambda_n) \frac{M_f(r\lambda_n)}{M_f(qr\lambda_n)} \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{qr\lambda_n} \Gamma_f(t)d\ln t \right\} \leq \]
\[ \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \left\{ -\Gamma_f(r\lambda_n) \ln q \right\} \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \left\{ -\Gamma_f(\lambda_n) \ln q \right\}. \]
If \( \ln n = O(\Gamma_f(\lambda_n)) \) as \( n \to \infty \), that is \( \ln n \leq c \Gamma_f(\lambda_n) \) for some \( c > 0 \) and all \( n \geq 1 \), then for \( q = e^{c+1} \) we obtain

\[
\mathfrak{M}(r, A) \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \left\{ -\frac{c+1}{c} \ln n \right\} = K \mu(qr, A),
\]
i.e. (14) holds. If \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) that is \( \ln n \leq \frac{\ln(1+\varepsilon)}{2} \Gamma_f(\lambda_n) \) for every \( \varepsilon > 0 \) and all \( n \geq n_0(\varepsilon) \), then for \( q = 1 + \varepsilon \) we get

\[
\sum_{n=n_0(\varepsilon)}^{\infty} \exp \left\{ -\Gamma_f(\lambda_n) \ln q \right\} \leq \sum_{n=n_0(\varepsilon)}^{\infty} \exp \left\{ -2 \ln n \right\},
\]
whence (15) follows. Lemma 2 is proved.

Let \( \alpha \in L, \beta \in L \) and

\[
\varrho_{\alpha, \beta}[A] = \lim_{r \to +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(r)}
\]
be the generalized \((\alpha, \beta)\)-order of an entire function \( A \). Lemma 2 implies the following statement.

**Proposition 1.** Let \( \alpha(\ln x) \in L_{si} \). If either \( \ln n = O(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) and \( \beta \in L_{si} \) or \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) and \( \beta \in L^0 \) then \( \varrho_{\alpha, \beta}[A] = \lim_{r \to +\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \).

**Proof.** If \( \ln n = O(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) and \( \beta \in L_{si} \) then (14) implies

\[
\lim_{r \to +\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \leq \lim_{r \to +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(r)} \leq \lim_{r \to +\infty} \frac{\alpha(\ln \mu(qr, A) + \ln K)}{\beta(qr)} \lim_{r \to +\infty} \frac{\beta(qr)}{\beta(r)} = \lim_{r \to +\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)}. \]

If \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then similarly from (15) we obtain

\[
\lim_{r \to +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(r)} \leq \lim_{r \to +\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \lim_{r \to +\infty} \frac{\beta((1+\varepsilon)r)}{\beta(r)}. \]

It is known [8] that if \( \beta \in L^0 \) then \( \lim_{r \to +\infty} \frac{\beta((1+\varepsilon)r)}{\beta(r)} \searrow 1 \) as \( \varepsilon \searrow 0 \).

Using Lemma 1 and Proposition 1 we prove the following theorem.

**Theorem 4.** Let \( \alpha(e^x) \in L^0, \beta(x) \in L^0 \), \( \frac{\ln r}{\ln \alpha^{-1}(c\beta(r))} \to 0 \) as \( r \to +\infty \) for each \( c \in (0, +\infty) \) and \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \). Suppose that \( \ln M_f(r) = O(\Gamma_f(r)) \) and \( \Gamma_f(r) = O(r) \) as \( r \to +\infty \). Then

\[
\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[A] = \lim_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta} \left( 1 \frac{M_f^{-1}(\lambda_n)}{|a_n|} \right). \] (16)
Proof. Suppose that $\varrho_{\alpha, \beta}[A] < +\infty$. Then by Lemma 3 $\ln \mu(r, A) \leq \alpha^{-1}(q_{\beta}(r))$ for every $\varrho > \varrho_{\alpha, \beta}[A]$ and all $r \geq r_0$, i.e.

$$\ln |a_n| \leq \alpha^{-1}(q_{\beta}(r)) - \ln M_f(r\lambda_n)$$

for all $n \geq 1$ and $r \geq r_0$.

Choosing $r = \beta^{-1}(\alpha(\lambda_n)/\varrho))$ we get

$$e^{\lambda_n}/|a_n| \geq M_f(\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho)))$$

for all $n \geq n_0$, i.e.

$$M_f^{-1}(e^{\lambda_n}/|a_n|)) \geq \lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho)), \quad n \geq n_0.$$  

If $\ln M_f(r) = O(\Gamma_f(r))$ as $r \to +\infty$, that is $\frac{d\ln M_f(r)}{d\ln r} \geq c > 0$ for all $r$, then

$$\frac{d\ln M_f^{-1}(e^x)}{d\ln x} \leq \frac{1}{c} < +\infty$$

for all $x \geq 0$.

Hence it follows that the function $\gamma(x) = M_f^{-1}(e^x)$ belongs to $L^0$ and, thus,

$$M_f^{-1}(e^{(1+o(1))x}) = (1 + o(1))(M_f^{-1}(e^x)), \quad x \to \infty.$$  \hspace{1cm} (17)

Therefore, $\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho)) \leq (1 + o(1))M_f^{-1}(1/|a_n|)$ as $n \to \infty$ and

$$\beta \left(\frac{1 + o(1)}{\lambda_n}M_f^{-1}(1/|a_n|)\right) \geq \frac{\alpha(\lambda_n)}{\varrho}, \quad n \to \infty,$$

whence in view of the condition $\beta(x) \in L^0$ and the arbitrariness of $\varrho$ we obtain the inequality $\varpi_{\alpha, \beta}[A] \leq \varrho_{\alpha, \beta}[A]$, which is obvious if $\varrho[A] = +\infty$.

Now, to prove the equality $\varpi_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[A]$, suppose by the contrary that $\varpi_{\alpha, \beta}[A] < \varrho_{\alpha, \beta}[A]$ and choose $\varpi_{\alpha, \beta}[A] < \varpi < \varrho_{\alpha, \beta}[A]$. Then

$$\frac{1}{\lambda_n} \leq M_f(\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varpi)))$$

for $n \geq n_0(\varpi)$.

Therefore, for $r \geq r_0(\varpi)$

$$\mu(r, A) = |a_{\nu(r)}| M_f(r\lambda_{\nu(r)}) \leq \frac{M_f(r\lambda_{\nu(r)})}{M_f(\lambda_{\nu(r)}\beta^{-1}(\alpha(\lambda_{\nu(r)}/\varpi)))}$$

and, since $\mu(r, A) \to +\infty$ as $r \to +\infty$, we obtain $r \leq \beta^{-1}(\alpha(\lambda_{\nu(r)}/\varpi))$, i.e.

$$\lambda_{\nu(r), A} \leq \alpha^{-1}(\varpi_{\beta}(r))$$

for all $r \geq r_0 = r_0(\varpi)$.

Since $\ln r / \ln \alpha^{-1}(\varpi_{\beta}(r)) \to 0$ as $r \to +\infty$ and $\alpha(e^x) \in L^0$, we have

$$\alpha(ra^{-1}(\varpi_{\beta}(r))) = \alpha(\exp{\ln a^{-1}(\varpi_{\beta}(r))}) = (1 + o(1))(1 + o(1))\alpha(1) \leq q_{\beta}(r), \quad r \geq r_0(\varpi).$$

Therefore, $\Gamma_f(r\lambda_{\nu(r), A}) \leq \Gamma(\alpha^{-1}(q_{\beta}(r)))$ for $r \geq r_1$ and by Lemma 1

$$\ln \mu(r, A) - \ln \mu(r_1, A) \leq \int_{r_1}^{r} \frac{\Gamma_f(\alpha^{-1}(q_{\beta}(t)))}{t} dt \leq \Gamma_f(\alpha^{-1}(q_{\beta}(r))) \ln \frac{r}{r_1} \leq C\alpha^{-1}(q_{\beta}(r)) \ln \frac{r}{r_1},$$

because $\Gamma_f(r) \leq Cr$ for all $r$. Since $\alpha(e^x) \in L^0$ and
for every $\alpha > \alpha_0$, as above we get
\[
\alpha(C^{-1}(\beta_1)) \ln r = (1 + o(1)) \beta_1
\]
as $r \to +\infty$, and, \(g_{\alpha,\beta}[A] \leq q\), which is a contradiction to the condition \(q < g_{\alpha,\beta}[A]\).

**Remark 1.** The conditions $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta(x) \in L^0$ in Theorem 4 can be replaced by conditions $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta(x) \in L_n$.

Since the functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ satisfy the assumptions of Theorem 4, the following statement is correct.

**Corollary 2.** Let a function $f$ and a sequence $(\lambda_n)$ satisfy the conditions of Theorem 4. Then
\[
\lim_{r \to +\infty} \ln \mathfrak{M}(r, A) = \lim_{n \to \infty} \frac{\ln n}{M_f^{-1}(1/|a_n|)}.
\]

The functions $\alpha(x) = \beta(x) = \ln^+ x$ do not satisfy the conditions of Theorem 4. In this case we put
\[
g(A) = \frac{\ln n}{M_f^{-1}(1/|a_n|)}
\]
and prove the following theorem.

**Theorem 5.** Let $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$. Suppose that $\ln M_f(r) = O(\Gamma_f(r))$, $r = o(\ln M_f(r))$ as $r \to +\infty$ and $\lim_{n \to \infty} \ln M_f(r) \leq 1$. Then
\[
g(A) = \mathcal{K}[A] + 1, \quad \mathcal{K}[A] := \lim_{n \to \infty} \frac{\ln \lambda_n}{\ln \left(\frac{1}{\lambda_n M_f^{-1}(1/|a_n|)}\right)}.
\]

**Proof.** Let $1 \leq g[A] < +\infty$. Since Lemma 1 implies
\[
\lim_{r \to +\infty} \ln \mu(r, A) = g[A], \quad \text{for every } g > g[A] \text{ and all } r \geq r_0(g)
\]
we have $\ln \mu(r, A) \leq r^g$ for $r \geq r_0(g)$, i.e.
\[
\ln |a_n| + \ln M_f(r \lambda_n) \leq r^g \text{ for all } n \geq 1 \text{ and } r \geq r_0(g)
\]
Choose $r = r_n = \lambda_n^{1/(g - 1)}$. Then $r_n \geq r_0(g)$ for $n \geq n_0(g)$ and, therefore,
\[
\ln |a_n| \leq \lambda_n^{g/(g - 1)} - \ln M_f(\lambda_n^{g/(g - 1)}) \quad n \geq n_0(g)
\]
The condition $r = o(\ln M_f(r))$ as $r \to +\infty$ implies
\[
\ln |a_n| \leq -(1 + o(1)) \ln M_f(\lambda_n^{g/(g - 1)}) \quad n \to \infty,
\]
i.e. in view of (17)
\[
\lambda_n^{g/(g - 1)} \leq M_f^{-1}(\exp\{(1 + o(1)) \ln (1/|a_n|)\}) = (1 + o(1))M_f^{-1}(1/|a_n|), \quad n \to \infty,
\]
whence \( \mathcal{N}[A] \leq \varrho - 1 \). In view of the arbitrariness of \( \varrho \) we get the inequality \( \mathcal{N}[A] + 1 \leq \varrho[A] \), which is obvious if \( \varrho[A] = +\infty \).

Now, to prove the equality \( \mathcal{N}[A] + 1 = \varrho[A] \), suppose that \( \mathcal{N}[A] < \varrho[A] - 1 \). Then for every \( \mathcal{N} \in (\mathcal{N}[A], \varrho[A] - 1) \) we have

\[
|a_n| \leq \frac{1}{M_f(\lambda_n^{1+1/\mathcal{N}})}
\]

for all \( n \geq n_0(\mathcal{N}) \). Therefore, as in the proof of Theorem 4 we obtain \( \lambda_{\nu(r,A)} \leq r^\mathcal{N} \) for \( r \geq r_0 \) and, thus, in view of (11)

\[
\ln \mu(r, A) - \ln \mu(r_0, A) \leq \int_{r_0}^{r} \frac{\Gamma_f(t^{1+\mathcal{N}})}{t} dt = \frac{1}{1 + \mathcal{N}} \int_{r_0}^{r^{1+\mathcal{N}}} \Gamma_f(t) d\ln t =
\]

\[
= \frac{1}{1 + \mathcal{N}} \int_{r_0^{1+\mathcal{N}}} d\ln M_f(t) d\ln t = \frac{1}{1 + \mathcal{N}} (\ln M_f(r^{1+\mathcal{N}}) - \ln M_f(r_0^{1+\mathcal{N}})),
\]

whence

\[
\varrho[A] = \lim_{r \to +\infty} \frac{\ln \mu(r, A)}{\ln r} \leq \lim_{r \to +\infty} \frac{\ln M_f(r^{1+\mathcal{N}})}{\ln r} = (1 + \mathcal{N})\varrho[f] \leq 1 + \mathcal{N},
\]

because \( \varrho[f] \leq 1 \), which contradicts to the condition \( \mathcal{N} < \varrho[A] - 1 \). Theorem 5 is proved. \( \square \)

4. Fréchet spaces (see more details in [13]). For fixed \( \varrho < +\infty \) by \( \mathcal{S}_{\varrho} \) we denote the class of function (2), such that \( \varrho_{\alpha, \beta}[A] \leq \varrho \). Then (18) implies

\[
|a_n| \leq \frac{1}{M_f(\lambda_n^{\beta-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right))}, \quad n \to \infty.
\]

Using an idea of the article [8], for \( q \in \mathbb{N} \) we define

\[
\|A\|_{q\varrho} = \sum_{n=1}^{\infty} |a_n| M_f(\lambda_n^{\beta-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right))
\]

If \( \beta \in L^{0} \) then

\[
\frac{\beta^{-1}((1 + c)x)}{\beta^{-1}(x)} \geq Q(c) > 1
\]

for every \( c > 0 \) and all \( x \geq x_0 \) and, as above, we have

\[
\frac{M_f(\lambda_n^{\beta-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right))}{M_f(\lambda_n^{\beta-1}\left(\frac{\alpha(\lambda_n)}{\varrho}\right))} \leq \exp\left\{-\frac{\alpha(\lambda_n)}{\varrho + 1/q} \right\} \leq \exp\left\{-\frac{\alpha(\lambda_n)}{\varrho + 1/q} \ln Q(1/(q\varrho))\right\} < 1.
\]

Therefore, if \( \ln n = o(\Gamma_f(\lambda_n)) \) as \( n \to \infty \) then in view of (19) \( \|A\|_{q\varrho} \) exists for each \( q \in \mathbb{N} \) and it easily to check that \( \|A\|_{q\varrho} \) is a norm on \( \mathcal{S}_{\varrho} \).
Clearly, \( \|A\|_{\varrho,q} \leq \|A\|_{\varrho,q+1} \). Therefore [6], the family \( \|A\|_{\varrho,q} : q \in \mathbb{N} \) induces on \( \mathcal{S}_\varrho \) the unique topology such that \( \mathcal{S}_\varrho \) becomes a local convex vector space and this topology is given by the metric \( d \), where

\[
d(A_1, A_2) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|A_1 - A_2\|_{\varrho,q}}{1 + \|A_1 - A_2\|_{\varrho,q}}.
\]

(20)

The space with the metric \( d \) we denote by \( \mathcal{S}_{\varrho,d} \).

**Theorem 6.** If the functions \( \alpha, \beta, f \) and the sequence \( (\lambda_n) \) satisfy the hypotheses of Theorem 4 then \( \mathcal{S}_{\varrho,d} \) is a Fréchet space.

**Proof.** It is sufficient to show that \( \mathcal{S}_{\varrho,d} \) is complete. Let therefore \( (A_j) \) be a \( d \)-Cauchy sequence in \( \mathcal{S}_{\varrho,d} \) and so far for a given \( \varepsilon > 0 \) there corresponds an \( m = m(\varepsilon) \) such that

\[
\|A_j - A_k\|_{\varrho,q} < \varepsilon
\]

for all \( j, k \geq m \) and \( q \in \mathbb{N} \). Consequently for these \( j, k \) and \( q \) we have

\[
\sum_{n=1}^{\infty} |a_n^{(j)} - a_n^{(k)}| M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) < \varepsilon,
\]

(21)
i.e. \( |a_n^{(j)} - a_n^{(k)}| < \varepsilon \) and \( (a_n^{(j)})_{j \geq 1} \) is a Cauchy sequence. Therefore, \( a_n^{(j)} \to a_n \) as \( j \to \infty \). Letting \( k \to \infty \) in (21) one has for \( j \geq j_0 \)

\[
\sum_{n=1}^{\infty} |a_n^{(j)} - a_n| M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) < \varepsilon,
\]

(22)
and consequently taking \( j = j_0 \) in (22) we get for a fixed \( q \)

\[
\sum_{n=1}^{\infty} |a_n^{(j_0)} - a_n| M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) < \varepsilon,
\]

whence in view of (19) with \( a_n^{(j_0)} \) instead of \( a_n \) we obtain

\[
|a_n| M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) \leq |a_n^{(j_0)}| M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) + \varepsilon \leq M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) / \left( M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + o(1)} \right) \right) + 2\varepsilon \right),
\]

i.e.

\[
\lim_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} M_f^{-1} \left( \frac{1}{|a_n|} \right) \right)} \leq \lim_{n \to \infty} \frac{\alpha(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} M_f^{-1} \left( \frac{M_f \left( \lambda_n^{\beta-1} \left( \frac{\alpha(\lambda_n)}{\varrho + 1/q} \right) \right) / (2\varepsilon) \right) \right) \right) \right) = \varrho + 1/q,
\]

because \( M_f^{-1} \) is a slowly increasing function.

By Theorem 4 in view of the arbitrariness of \( q \) we get \( \varrho_{\alpha,\beta} \|A\| \leq \varrho \). Thus, using (21) again we see that \( \|A_j - A\|_{\varrho,q} < \varepsilon \) for \( j \geq j_0 \) and the result is proved.

\( \square \)

For \( \mathcal{S}_{\varrho,d} \) by \( \mathcal{S}_{\varrho,d}^* \) we denote the dual space. The following analog of Theorem 3 is true.
Theorem 7. If the functions $\alpha$, $\beta$, $f$ and the sequence $(\lambda_n)$ satisfy the conditions of Theorem 4 then continuous linear functional $L$ on $\mathcal{S}_{\varrho,d}$ is of form (8) if and only if for all $n \in \mathbb{N}$ and $q \in \mathbb{N}$

$$|g_n| \leq KM_f \left( \lambda_n \beta^{-1} \left( \frac{\alpha(\lambda_n)}{q + 1/q} \right) \right), \quad K = \text{const} > 0.$$  \hspace{1cm} (23)

Proof. Let $L \in \mathcal{S}_{\varrho,d}^\ast$. This clearly means if $A_m \to A$ in $\mathcal{S}_{\varrho,d}$ then $L(A_m) \to L(A)$.

Now let $a_n$ satisfy (19) and $A_m(s) = \sum_{n=1}^{m} a_n f(z \lambda_n)$. Then we claim that $A_m \to A$ in the norm $\|\cdot\|_{\varrho,q}$ for every $q \in \mathbb{N}$.

So let $q$ be fixed integer. Choose $\varepsilon \in (0, 1/q)$. Then in view of (19) we can determine an integer $m = m(\varepsilon)$ such that

$$|a_n| \leq \frac{1}{M_f \left( \lambda_n \beta^{-1} \left( \frac{\alpha(\lambda_n)}{q + \varepsilon} \right) \right)}, \quad n \geq m + 1,$$

and it follows as above that

$$\|A_m - A\|_{\varrho,q} = \left\| \sum_{n=m+1}^{\infty} a_n f(z \lambda_n) \right\|_{\varrho,q} \leq \sum_{n=m+1}^{\infty} \frac{M_f \left( \lambda_n \beta^{-1} \left( \frac{\alpha(\lambda_n)}{q + 1/q} \right) \right)}{M_f \left( \lambda_n \beta^{-1} \left( \frac{\alpha(\lambda_n)}{q + \varepsilon} \right) \right)} \to 0, \quad m \to \infty,$$

and this ascertains our claim. Combining this with the continuity of $L$ we have

$$\lim_{m \to \infty} L(A_m) = L(A)$$

in the topology given by $d$.

Note that $L(A_m) = \sum_{n=1}^{m} d_n g_n$, where $g_n = L(f(z \lambda_n))$ for each $n$. Since $L$ is continuous on $(\mathcal{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})$, there exists a $K > 0$ such that

$$|g_n| = |L(f(z \lambda_n))| \leq K\|f(z \lambda_n)\|_{\varrho,q}$$

for each $q \in \mathbb{N}$ and so, using the definition of the norm $\|f(z \lambda_n)\|_{\varrho,q}$, we get (23).

To prove the other part, let now $g_n$ satisfy (23). Then

$$|L(A)| \leq K \sum_{n=1}^{\infty} |a_n| \exp \left\{ \lambda_n \beta^{-1} \left( \frac{\alpha(\lambda_n)}{q + 1/q} \right) \right\}, \quad q \in \mathbb{N},$$

and so $|L(A)| \leq K\|A\|_{\varrho,q}$ for all $q \in \mathbb{N}$. Therefore, $L \in (\mathcal{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})^\ast$ for all $q \in \mathbb{N}$. Since $\|A\|_{\varrho,q} \leq \|A\|_{\varrho,q+1}$, from (19) it follows that $\mathcal{S}_{\varrho,d} = \bigcup_{q \geq 1} (\mathcal{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})^\ast$. Thus, $L \in \mathcal{S}_{\varrho,d}^\ast$. $\square$
REFERENCES


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