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SPACES OF SERIES IN SYSTEMS OF FUNCTIONS

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The Banach and Fréchet spaces of series $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ regularly converging in $\mathbb{C}$, where f is an entire transcendental function and (λ_n) is a sequence of positive numbers increasing to $+\infty$, are studied.

Let $M_f(r) = \max\{|f(z)|: |z| = r\}$, $\Gamma_f(r) = \frac{d\ln M_f(r)}{d\ln r}$, h be positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $\mathbf{S}_h(f, \Lambda)$ be a class of the function A such that $|a_n|M_f(\lambda_n h(\lambda_n))$ $\to 0$ as $n \to +\infty$. Define $\|A\|_h = \max\{|a_n|M_f(\lambda_n h(\lambda_n)) : n \geq 1\}$. It is proved that if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then $(\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$ is a non-uniformly convex Banach space which is also separable.

In terms of generalized orders, the relationship between the growth of $\mathfrak{M}(r, A) = \sum_{n=1}^{\infty} |a_n|M_f(r\lambda_n)$, the maximal term $\mu(r, A) = \max\{|a_n|M_f(r\lambda_n): n \geq 1\}$ and the central index $\nu(r, A) = \max\{n \geq 1: |a_n|M_f(r\lambda_n) = \mu(r, A)\}$ and the decrease of the coefficients a_n. The results obtained are used to construct Fréchet spaces of series in systems of functions.

1. Introduction. Let $\Lambda = (\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$,

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire transcendental function and $M_f(r) = \max\{|f(z)|: |z| = r\}$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

in the system $f(\lambda_n z)$ converges regularly in $\mathbb{C}$, i.e. for all $r \in [0, +\infty)$

$$\mathfrak{M}(r, A) := \sum_{n=1}^{\infty} |a_n|M_f(r\lambda_n) < +\infty. \tag{3}$$

Many authors studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We mention here only the monographs of A. F. Leont'ev [1] and B. V. Vynnytskyi [2], and there in references.

Since series (2) converges regularly in $\mathbb{C}$, the function A is entire. To study its growth, generalized orders are used. For this purpose, as in [3] by L we denote the class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow$

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$+\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$. For $\alpha \in L$ and $\beta \in L$ quantity

$$\varrho_{\alpha,\beta}[f] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(r)}$$

is called ([3]) generalized (α,β)-order of the entire function f. In terms of the generalized (α,β)-orders in the paper [4–5,14] the relationship between the growth of functions $M_f(r)$, $\mathfrak{M}(r, A)$ and $M_f^{-1}(\mathfrak{M}(r, A))$ was studied.

The study of various spaces of analytic functions represented by power series and Dirichlet series has been studied by many authors (we only point out here [6–10]). For the Laplace-Stieltjes integrals, the Banach and Fréchet spaces were studied in [11–12].

The present paper is devoted to the study of Banach and Fréchet spaces for entire functions represented by series (2) regularly converging in $\mathbb{C}$. Note that the function $\ln M_f(r)$ is logarithmically convex and, therefore,

$$\Gamma_f(r) := \frac{d\ln M_f(r)}{d\ln r} \nearrow +\infty, \quad r \to +\infty,$$

(at points where the derivative does not exist, $\frac{d\ln M_f(r)}{d\ln r}$ means right-hand side derivative). The function $\Gamma_f(r)$ will play an important role in our research.

2. Banach spaces of series in the system of functions. At first we remark that if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then series (2) converges regularly in $\mathbb{C}$ if and only if

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) = +\infty. \tag{4}$$

Indeed, if series (2) regularly converges in $\mathbb{C}$ then $|a_n|M_f(r\lambda_n) \to 0$ as $n \to \infty$, i.e. for all $r \in [0, +\infty$ and $n \ge n_0(r)$ one has $|a_n|M_f(r\lambda_n) \le 1$, whence $\frac{1}{\lambda_n}M_f^{-1}(\frac{1}{|a_n|}) \ge r$ for all $n \ge n_0(r)$. In view of the arbitrariness of r we get (4).

On the other hand, if $r \in [1, +\infty)$ is an arbitrary number and (4) holds then for every $K > r$ and all $n \ge n_0 = n_0(K)$ we have $\frac{1}{\lambda_n}M_f^{-1}(\frac{1}{|a_n|}) \ge K$, i.e. $|a_n|M_f(K\lambda_n) \le 1$. Therefore,

$$\sum_{n=n_0}^{\infty} |a_n|M_f(r\lambda_n) = \sum_{n=n_0}^{\infty} |a_n|M_f(K\lambda_n)\frac{M_f(r\lambda_n)}{M_f(K\lambda_n)} \le \sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f(K\lambda_n)} =$$

$$= \sum_{n=n_0}^{\infty} \exp\left\{-\int_{r\lambda_n}^{K\lambda_n} \frac{d\ln M_f(t)}{d\ln t} d\ln t\right\} = \sum_{n=n_0}^{\infty} \exp\left\{-\int_{r\lambda_n}^{K\lambda_n} \Gamma_f(t) d\ln t\right\} \le$$

$$\le \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(r\lambda_n)\ln(K/r)\} \le \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(\lambda_n)\ln(K/r)\} < +\infty,$$

because $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$, what was required to prove.

Let h be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and such that

$$|a_n|M_f(\lambda_n h(\lambda_n)) \to 0, \quad n \to +\infty. \tag{5}$$

Let us show that if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then the series

$$B_h(z) = \sum_{n=1}^{\infty} \frac{f(z\lambda_n)}{M_f(\lambda_n h(\lambda_n))} \tag{6}$$

converges regularly in $\mathbb{C}$.

Indeed, for every $r \geq 1$ as above we have

$$\sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f(\lambda_n h(\lambda_n))} = \sum_{n=1}^{\infty} \exp\left\{ - \int\limits_{r\lambda_n}^{\lambda_n h(\lambda_n)} \Gamma_f(t) d\ln t \right\} \leq \sum_{n=1}^{\infty} \exp\left\{ -\Gamma_f(\lambda_n) \ln \frac{h(\lambda_n)}{r} \right\} < +\infty,$$

because $h(\lambda_n) \to +\infty$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$.

By $\mathbf{S}_h(f,\Lambda)$ we denote the class of series (2) such that (5) holds. On $\mathbf{S}_h(f,\Lambda)$ for $A_j(z) = \sum_{n=1}^{\infty} \lambda a_{n,j} f(\lambda_n z)$ $(j = 1,2)$ we define operations

$$(A_1 + A_2)(z) = \sum_{n=1}^{\infty} (a_{n,1} + a_{n,2}) f(\lambda_n z), \quad (\lambda A)(z) = \sum_{n=1}^{\infty} \lambda a_n f(\lambda_n z).$$

Put $\|A\|_h = \max\{|a_n| M_f(\lambda_n h(\lambda_n))\colon n \geq 1\}$. Under these operations $\mathbf{S}_h(f,\Lambda)$ becomes a normalized linear space.

Theorem 1. *If* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$ *then* $(\mathbf{S}_h(f,\Lambda), \|\cdot\|_h)$ *is a non-uniformly convex Banach space which is separable also.*

Proof. Let (A_p) be a Cauchy sequence in $\mathbf{S}_h(f,\Lambda)$, $A_p(z) = \sum_{n=1}^{\infty} a_{n,p} f(\lambda_n z)$. Then in view of (5) $|a_{n,p}| M_f(\lambda_n h(\lambda_n)) \to 0$ as $x \to +\infty$ for each p, and for a given $\varepsilon > 0$ there exists $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that $\|A_p - A_q\|_h < \varepsilon$ for all $p \geq j_0$ and $q \geq j_0$, i.e.

$$\max\{|a_{n,p} - a_{n,q}| M_f(\lambda_n h(\lambda_n))\colon n \geq 1\} < \varepsilon.$$

Thus, $|a_{n,p} M_f(\lambda_n h(\lambda_n)) - a_{n,q} M_f(\lambda_n h(\lambda_n))| < \varepsilon$ for all $n \geq 1$, $p \geq j_0$ and $q \geq j_0$. This shows that $(|a_{n,p}| M_f(\lambda_n h(\lambda_n)))_{p=1}^{\infty}$ is a Cauchy sequence, so it converges to $|a_{n,0}| M_f(\lambda_n h(\lambda_n))$ (say) as $p \to \infty$. Since

$$|a_{n,0}| M_f(\lambda_n h(\lambda_n)) \leq |a_{n,0} M_f(\lambda_n h(\lambda_n)) - a_{n,p} M_f(\lambda_n h(\lambda_n))| + |a_{n,p}| M_f(\lambda_n h(\lambda_n)) \to 0$$

as $p \to \infty$, the series $A_0(z) = \sum\limits_{n=1}^{\infty} a_{n,0} f(\lambda_n z)$ belongs to $\mathbf{S}_h(f,\Lambda)$. Also we have

$$\|A_p - A_0\|_h = \max\{|a_{n,p} M_f(\lambda_n h(\lambda_n)) - a_{n,0} M_f(\lambda_n h(\lambda_n))|\colon n \geq 1\} \to 0, \quad p \to \infty,$$

i.e. the space $(\mathbf{S}_h(f,\Lambda), \|\cdot\|_h)$ is complete and, thus, a Banach space.

Further consider $B_{1,h}$ and $B_{2,h}$ defined as follows:

$$B_{1,h}(z) = \frac{f(z\lambda_{n_0})}{M_f(\lambda_{n_0} h(\lambda_{n_0}))}, \quad B_{2,h}(z) = \frac{f(z\lambda_{n_0})}{M_f(\lambda_{n_0} h(\lambda_{n_0}))} + \frac{f(z\lambda_k)}{M_f(\lambda_k h(\lambda_k))},$$

where n_0 and k are positive fixed integers. Obviously, $B_{j,h} \in \mathbf{S}_h(f,\Lambda)$, but $\|B_{1,h}\|_h = 1$, $\|B_{2,h}\| = 1$, $\|B_{1,h} + B_{2,h}\| = 2$ and $\|B_{2,h} - B_{1,h}\| = 1 \not\to 0$, i.e. the space $(\mathbf{S}_h(f,\Lambda), \|\cdot\|_h)$ is non-uniformly convex (see, for example, [7, p. 183]).

It is still to be shown that $(\mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$ is separable. For this, at first, consider the set of all function $A_m \in (\mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$, which has the representation $A_m(z)=\sum\limits_{n=1}^{m} b_n f(\lambda_n z)$, where $b_n=c_n+id_n$ and c_n, d_n are rational numbers for every n. This set is readily seen to be a countable one. It is also everywhere dense in $(\mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$.

Since $A(z)=\sum\limits_{n=1}^{\infty} a_n f(\lambda_n z)\in \mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$, i.e. $|a_n|M_f(\lambda_n h(\lambda_n))\to 0$ as $n\to+\infty$, for every $\varepsilon>0$ and $n\ge n_0=n_0(\varepsilon)$ we have $|a_n|M_f(\lambda_n h(\lambda_n))<\varepsilon/2$, i.e. $\max\{|a_n|M_f(\lambda_n h(\lambda_n))\colon n\ge n_0\}<\varepsilon/2$.

Now let $G\in(\mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$ be defined as $G(s)=\sum\limits_{n=1}^{\infty} b_n f(\lambda_n z)$, where b_n are given as $b_n=0$ for $n\ge n_0$ and $|a_n-b_n|M_f(\lambda_n h(\lambda_n))<\varepsilon/2$ for $1\le n\le n_0-1$. Then

$$\|A-G\|_h\le\max\{|a_n-b_n|M_f(\lambda_n h(\lambda_n))\colon n\le n_0-1\}+\max\{|a_n|M_f(\lambda_n h(\lambda_n))\colon n\ge n_0\}<\varepsilon.$$

Therefore, $(\mathbf{S}_h(f,\Lambda),\|\cdot\|_h)$ is separable, and Theorem 1 is proved. □

The following statement concerns uniform convergence of (A_m).

Theorem 2. *Let* $\ln n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$. *In order that* $(A_m)\subset\mathbf{S}_h(f,\Lambda)$ *converges to* $A\in\mathbf{S}_h(f,\Lambda)$ *by* $\|\cdot\|_h$ *it is necessary and sufficient that* $A_m(z)$ *converges uniformly to* $A(z)$ *over each compact subset of* $\mathbb{C}$.

Proof. If $A_m(z)=\sum\limits_{n=1}^{\infty} a_{n,m} f(\lambda_n z)$ and $\|A_m-A\|_h<\varepsilon$ for every $\varepsilon>0$ and all $m\ge m_0(\varepsilon)$ then

$$\max\{|a_{n,m}-a_n|M_f(\lambda_n h(\lambda_n))\colon n\ge 1\}<\varepsilon$$

and, thus, $|a_{n,m}-a_n|M_f(\lambda_n h(\lambda_n))<\varepsilon$ for every $\varepsilon>0$, all $m\ge m_0(\varepsilon)$ and all $n\ge 1$. Therefore, if $m\ge m_0(\varepsilon)$ and $r\le r_0<+\infty$ then, as above,

$$|A_m(z)-A(z)|=\left|\sum_{n=1}^{\infty}(a_{n,m}-a_n)f(\lambda_n z)\right|\le\sum_{n=1}^{\infty}|a_{n_m}-a_n|M_f(r\lambda_n)\le$$
$$\le\sum_{n=1}^{\infty}|a_{n,m}-a_n|M_f(\lambda_n h(\lambda_n))\frac{M_f(r_0\lambda_n)}{M_f(\lambda_n h(\lambda_n))}\le\varepsilon\sum_{n=1}^{\infty}\frac{M_f(r_0\lambda_n)}{M_f(\lambda_n h(\lambda_n))}\le$$
$$\le\varepsilon\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(\lambda_n)\ln\frac{h(\lambda_n)}{r_0}\right\}=K_0\varepsilon.$$

From hence it follows that $A_m(z)$ converges uniformly to $A(z)$ on $\{z\colon |z|\le r_0\}$.

Conversely, let $A_m(z)$ converges uniformly to $A(z)$ on each $\{z\colon |z|\le r\}$. Then $|a_{n,m}-a_n|M_f(r\lambda_n)<\varepsilon$ for every $\varepsilon>0$, all $n\ge 1$ and all $m\ge m_0=m_0(\varepsilon,r)$, whence

$$|a_{n,m}-a_n|M_f(\lambda_n h(\lambda_n))\le\varepsilon\frac{M_f(\lambda_n h(\lambda_n))}{M_f(rh(\lambda_n))}.$$

Choosing $r=h(\lambda_n)$ from hence we get $|a_{n,m}-a_n|M_f(\lambda_n h(\lambda_n))\le\varepsilon$ for every $\varepsilon>0$, all $n\ge 1$ and all $m\ge m_0=m_0(\varepsilon,n)$, i.e. $\|A_m-A\|_h\to 0$ as $m\to\infty$. □

Corollary 1. *Let* $A(z) = \sum\limits_{n=1}^{\infty} a_n f(\lambda_n z)$, $A_m(z) = \sum\limits_{n=1}^{m} a_n f(\lambda_n z)$ *and* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$. *Then* $A_m(z) \to A(z)$ *as* $m \to \infty$ *for all* z *if and only if* $|a_n| M_f(\lambda_n h(\lambda_n)) \to 0$ *as* $n \to \infty$, *i.e.* $A \in \mathbf{S}_h(f, \Lambda)$.

Indeed, if $A \in \mathbf{S}_h(f, \Lambda)$ then $|a_n| M_f(\lambda_n h(\lambda_n)) \to 0$ as $n \to \infty$ and $\|A_m - A\|_h = \max\{|a_n| M_f(\lambda_n h(\lambda_n))\colon n \ge m\} \to 0$ as $m \to \infty$, i.e. $A_m \to A$ by $\|\cdot\|_h$ and, therefore, by Theorem 2 $A_m(z) \to A(z)$ as $m \to \infty$ for all z.

Conversely, if $A \notin \mathbf{S}_h(f, \Lambda)$ then $|a_{n_j}| M_f(\lambda_{n_j} h(\lambda_{n_j})) \ge \eta > 0$ for some sequence $(n_j) \uparrow \infty$. Therefore, if $m \le p < q < \infty$ and $A_{p,q}(z) = \sum\limits_{n=p}^{q} a_n f(\lambda_n z)$ then

$$\|A_{p,q}\|_h = \max\{|a_n| M_f(\lambda_n h(\lambda_n))\colon p \le n \le q\} \ge \eta$$

provided $p \le n_j \le q$. Hence it follows that (A_m) is not even a Cauchy sequence.

Now, for $(\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$ by $\mathbf{S}_h^*(f, \Lambda)$ we denote the dual space, i.e. $\mathbf{S}_h^*(f, \Lambda)$ is the family of all continuous linear functionals on $(\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$. Let $L(A) = \sum\limits_{n=1}^{\infty} a_n g_n$, where real numbers are such that

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = K < +\infty. \tag{7}$$

Theorem 3. *Let* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$. *Then every bounded linear functional defined on* $(\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$ *is of the form*

$$L(A) = \sum_{n=1}^{\infty} a_n g_n, \quad A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z), \tag{8}$$

where g_n *is real-valued sequence satisfying* (7).

Proof. In view of (7) we have

$$\sum_{n=1}^{\infty} |a_n g_n| = \sum_{n=1}^{\infty} |a_n| M_f(\lambda_n h(\lambda_n)) \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} \le$$
$$\le \max\{|a_n| M_f(\lambda_n h(\lambda_n))\colon n \ge 1\} \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = K\|A\|_h < +\infty,$$

i.e. L is well-defined functional on $(\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$. Moreover,

$$|L(A)| \le \|A\|_h \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))},$$

whence

$$\|L\|_h \le \sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))}. \tag{9}$$

Conversely, we first remark that if $A \in (\mathbf{S}_h(f, \Lambda), \|\cdot\|_h)$ and $A_m(z) = \sum\limits_{n=1}^{m} a_n f(\lambda_n z)$ then

$$\|A_m - A\|_h = \max_{n>m} |a_n| M_f(\lambda_n h(\lambda_n)) \to 0$$

as $m \to \infty$ and by Corollary 1 from Theorem 2 $A_m(z))$ converges uniformly to $A(z)$ over each compact subset of $\mathbb{C}$. Therefore, if $L \in \mathbf{S}_h^*(f, \Lambda)$ and be define $L(f(z\lambda_n)) = g_n$ for each n then

$$L(A) = L\biggl(\lim_{m\to\infty} \sum_{n=1}^{m} a_n f(z\lambda_n)\biggr) = \lim_{m\to\infty} \sum_{n=1}^{m} a_n L(f(z\lambda_n)) = \sum_{n=1}^{\infty} a_n g_n.$$

Now we show that $\sum\limits_{n=1}^{\infty} \dfrac{|g_n|}{M_f(\lambda_n h(\lambda_n))} \le \|L\|_h$ so that $\sum\limits_{n=1}^{\infty} \dfrac{|g_n|}{M_f(\lambda_n h(\lambda_n))} < +\infty$. We take $p \in \mathbb{N}$ and let $a_n = \dfrac{\operatorname{sign}(g_n)}{M_f(\lambda_n h(\lambda_n))}$ for $1 \le n \le p$ and $a_n = 0$ for $n > p$. If we define $A(z) = \sum\limits_{n=1}^{\infty} a_n f(z\lambda_n)$ then obviously $A \in \mathbf{S}_h(f, \Lambda)$ and $\|A\|_h = 1$. Hence

$$|L(A)| = \left| \sum_{n=1}^{p} \frac{\operatorname{sign}(g_n)}{M_f(\lambda_n h(\lambda_n))} L(f(z\lambda_n) \right| = \sum_{n=1}^{p} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))}$$

and $|L(A)| \le \|A\|_h \|L\|_h = \|L\|_h$, so that $\sum\limits_{n=1}^{p} \dfrac{|g_n|}{M_f(\lambda_n h(\lambda_n))} \le \|L\|_h$ and

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = \sup_p \sum_{n=1}^{p} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} \le \sup_p \|L\|_h = \|L\|_h. \tag{10}$$

Inequalities (9) and (10) together show that

$$\sum_{n=1}^{\infty} \frac{|g_n|}{M_f(\lambda_n h(\lambda_n))} = \|L\|_h$$

,

and this completes the proof of the Theorem 3. □

3. Growth of $\mathfrak{M}(r, A)$. Let $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) \colon n \ge 1\}$ be the maximal term and $\nu(r, A) = \max\{n \ge 1 \colon |a_n| M_f(r\lambda_n) = \mu(r, A)\}$ be the central index of series (3).

Lemma 1. *The functions* $\ln\, \mu(r, A)$, $\lambda_{\nu(r,A)}$ *and* $\nu(r, A)$ *are non-decreasing and*

$$\ln\, \mu(r, A) - \ln\, \mu(r_0, A) = \int_{r_0}^{r} \frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t} dt, \quad 0 \le r_0 \le r < +\infty. \tag{11}$$

Proof. For $h > 0$ we have

$$\begin{aligned}
\mu(r + h, A) &= |a_{\nu(r+h,A)}| M_f((r + h)\lambda_{\nu(r+h,A)}) = \\
&= |a_{\nu(r+h,A)}| M_f(r\lambda_{\nu(r+h,A)}) \frac{M_f((r + h)\lambda_{\nu(r+h,A)})}{M_f(r\lambda_{\nu(r+h,A)})} \le \\
&\le \mu(r, A) \exp\{\ln\, M_f((r + h)\lambda_{\nu(r+h,A)}) - \ln\, M_f(r\lambda_{\nu(r+h,A)})\} = \\
&= \mu(r, A) \exp\left\{ \int_{r\lambda_{\nu(r+h,A)}}^{(r+h)\lambda_{\nu(r+h,A)}} \Gamma_f(t) d\ln\, t \right\} \le
\end{aligned}$$

$$\le \mu(r,A)\exp\left\{\Gamma_f((r+h)\lambda_{\nu(r+h,A)})\ln(1+h/r)\right\},$$

i.e.

$$\ln\,\mu(r+h,A)-\ln\,\mu(r,A)\le \Gamma_f((r+h)\lambda_{\nu(r+,A)})\ln(1+h/r). \tag{12}$$

Similarly,

$$\mu(r,A)=|a_{\nu(r,A)}|M_f((r+h)\lambda_{\nu(r,A)})\frac{M_f(r\lambda_{\nu(r,A)})}{M_f((r+h)\lambda_{\nu(r,A)})}\le$$

$$=\mu(r+h,A)\exp\left\{-\int\limits_{r\lambda_{\nu(r,A)}}^{(r+h)\lambda_{\nu(r,A)}}\Gamma_f(t)d\ln\,t\right\}\le \mu(r+h,A)\exp\left\{-\Gamma_f(r\lambda_{\nu(r,A)})\ln(1+h/r)\right\},$$

i.e.

$$\ln\,\mu(r+h,A)-\ln\,\mu(r,A)\ge \Gamma_f(r\lambda_{\nu(r,A)})\ln(1+h/r). \tag{13}$$

From (12) and (13) we obtain

$$\Gamma_f(r\lambda_{\nu(r,A)})\frac{\ln(1+h/r)}{h}\le \frac{\ln\,\mu(r+h,A)-\ln\,\mu(r,A)}{h}\le \Gamma_f((r+h)\lambda_{\nu(r+h,A)})\frac{\ln(1+h/r)}{h}.$$

Hence it follows that the functions $\ln\,\mu(r,A)$, $\lambda_{\nu(r,A)}$ and $\nu(r,A)$ are non-decreasing. Our reasoning is also correct if $h<0$. Therefore, if $(r_1,\,r_2)$ is an interval of constancy of the function $\nu(r,A)$ then at $h\to 0$ we obtain

$$\frac{d\ln\,\mu(r,A)}{dr}=\frac{\Gamma_f(r\lambda_{\nu(r,A)})}{r},\quad r\in(r_1,\,r_2).$$

Since the function $\Gamma_f(r\lambda_{\nu(r,A)})$ has a finite number of discontinuities on each finite interval, we obtain the equality (11). □

We need also the following lemma.

Lemma 2. *If* $\ln\,n=O(\Gamma_f(\lambda_n))$ *as* $n\to\infty$ *then for some* $q>1$ *and all* $r\ge 1$

$$\mu(r,A)\le \mathfrak{M}(r,A)\le K\mu(qr,A),\quad K=const>0, \tag{14}$$

and if $\ln\,n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$ *then for every* $\varepsilon>0$ *and all* $r\ge 1$

$$\mu(r,A)\le \mathfrak{M}(r,A)\le K(\varepsilon)\mu((1+\varepsilon)r,A),\quad K(\varepsilon)>0. \tag{15}$$

Proof. From (3) for $q>1$ and $r\ge 1$ as above have

$$\mu(r,A)\le \mathfrak{M}(r,A)\le \sum_{n=1}^{\infty}|a_n|M_f(qr\lambda_n)\frac{M_f(r\lambda_n)}{M_f(qr\lambda_n)}\le \mu(qr,A)\sum_{n=1}^{\infty}\exp\left\{-\int\limits_{r\lambda_n}^{qr\lambda_n}\Gamma_f(t)d\ln\,t\right\}\le$$

$$\le \mu(qr,A)\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(r\lambda_n)\ln\,q\right\}\le \mu(qr,A)\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(\lambda_n)\ln\,q\right\}.$$

If $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$, that is $\ln n \le c\Gamma_f(\lambda_n))$ for some $c > 0$ and all $n \ge 1$, then for $q = e^{c+1}$ we obtain

$$\mathfrak{M}(r, A) \le \mu(qr, A) \sum_{n=1}^{\infty} \exp\left\{-\frac{c+1}{c} \ln n\right\} = K\mu(qr, A),$$

i.e. (14) holds. If $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ that is $\ln n \le \frac{\ln(1+\varepsilon)}{2}\Gamma_f(\lambda_n))$ for every $\varepsilon > 0$ and all $n \ge n_0(\varepsilon)$, then for $q = 1 + \varepsilon$ we get

$$\sum_{n=n_0(\varepsilon)}^{\infty} \exp\{-\Gamma_f(\lambda_n) \ln q\} \le \sum_{n=n_0(\varepsilon)}^{\infty} \exp\{-2\ln n\},$$

whence (15) follows. Lemma 2 is proved. □

Let $\alpha \in L$, $\beta \in L$ and

$$\varrho_{\alpha,\beta}[A] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(r)}$$

be the generalized (α, β)-order of an entire function A. Lemma 2 implies the following statement.

Proposition 1. *Let $\alpha(\ln x) \in L_{si}$. If either $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L_{si}$ or $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L^0$ then $\varrho_{\alpha,\beta}[A] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)}$.*

Proof. If $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L_{si}$ then (14) implies

$$\varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mathfrak{M}(r, A)}{\beta(r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(qr, A) + \ln K)}{\beta(qr)} \varlimsup_{r\to+\infty} \frac{\beta(qr)}{\beta(r)} =$$
$$= \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \varlimsup_{r\to+\infty} \frac{\beta(qr)}{\beta(r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)}.$$

If $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then similarly from (15) we obtain

$$\varlimsup_{r\to+\infty} \frac{\alpha(\ln \mathfrak{M}(r, A)}{\beta(r)} \le \varlimsup_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \varlimsup_{r\to+\infty} \frac{\beta((1+\varepsilon)r)}{\beta(r)}.$$

It is known [8] that if $\beta \in L^0$ then $\varlimsup_{r\to+\infty} \frac{\beta((1+\varepsilon)r)}{\beta(r)} \searrow 1$ as $\varepsilon \searrow 0$. □

Using Lemma 1 and Proposition 1 we prove the following theorem.

Theorem 4. *Let $\alpha(e^x) \in L^0$, $\beta(x) \in L^0$, $\frac{\ln r}{\ln \alpha^{-1}(c\beta(r))} \to 0$ as $r \to +\infty$ for each $c \in (0, +\infty)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$. Suppose that $\ln M_f(r) = O(\Gamma_f(r))$ and $\Gamma_f(r) = O(r)$ as $r \to +\infty$. Then*

$$\varrho_{\alpha,\beta}[A] = \varkappa_{\alpha,\beta}[A], \quad \varkappa_{\alpha,\beta}[A] := \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right)\right)}. \tag{16}$$

Proof. Suppose that $\varrho_{\alpha,\beta}[A] < +\infty$. Then by Lemma 3 $\ln\, \mu(r, A)) \le \alpha^{-1}(\varrho\beta(r))$ for every $\varrho > \varrho_{\alpha,\beta}[A]$ and all $r \ge r_0$, i.e.

$$\ln\, |a_n| \le \alpha^{-1}(\varrho\beta(r)) - \ln\, M_f(r\lambda_n) \text{ for all } n \ge 1 \text{ and } r \ge r_0.$$

Choosing $r = \beta^{-1}(\alpha(\lambda_n)/\varrho))$ we get

$$e^{\lambda_n}/|a_n| \ge M_f(\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho)))$$

for all $n \ge n_0$, i.e.

$$M_f^{-1}(e^{\lambda_n}/|a_n|)\}) \ge \lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho))), \quad n \ge n_0.$$

If $\ln\, M_f(r) = O(\Gamma_f(r))$ as $r \to +\infty$, that is $\dfrac{d\ln\, \ln\, M_f(r)}{d\ln\, r} \ge c > 0$ for all r, then

$$\frac{d\ln\, M_f^{-1}(e^x)}{d\ln\, x} \le \frac{1}{c} < +\infty \text{ for all } x \ge 0.$$

Hence it follows that the function $\gamma(x) = M_f^{-1}(e^x)$ belongs to L^0 and, thus,

$$M_f^{-1}(e^{(1+o(1))x}) = (1 + o(1))(M_f^{-1}(e^x)), \quad x \to \infty. \tag{17}$$

Therefore, $\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varrho))) \le (1 + o(1))M_f^{-1}(1/|a_n|)$ as $n \to \infty$ and

$$\beta\left(\frac{1 + o(1)}{\lambda_n} M_f^{-1}\left(1/|a_n|\right)\right) \ge \frac{\alpha(\lambda_n)}{\varrho}, \quad n \to \infty,$$

whence in view of the condition $\beta(x) \in L^0$ and the arbitrariness of ϱ we obtain the inequality $\varkappa_{\alpha,\beta}[A] \le \varrho_{\alpha,\beta}[A]$, which is obvious if $\varrho[A] = +\infty$.

Now, to prove the equality $\varkappa_{\alpha,\beta}[A] = \varrho_{\alpha,\beta}[A]$, suppose by the contrary that $\varkappa_{\alpha,\beta}[A] < < \varrho_{\alpha,\beta}[A]$ and choose $\varkappa_{\alpha,\beta}[A] < \varkappa < q < \varrho_{\alpha,\beta}[A]$. Then

$$|a_n| \le \frac{1}{M_f(\lambda_n\beta^{-1}(\alpha(\lambda_n)/\varkappa)))} \text{ for } n \ge n_0(\varkappa).$$

Therefore, for $r \ge r_0(\varkappa)$

$$\mu(r, A) = |a_{\nu(r)}|M_f(r\lambda_{\nu(r)}) \le \frac{M_f(r\lambda_{\nu(r)})}{M_f(\lambda_{\nu(r)}\beta^{-1}(\alpha(\lambda_{\nu(r)})/\varkappa)))}$$

and, since $\mu(r, A) \to +\infty$ as $r \to +\infty$, we obtain $r \ge \beta^{-1}(\alpha(\lambda_{\nu(r,A)})/\varkappa))$, i.e.

$$\lambda_{\nu(r,A)} \le \alpha^{-1}(\varkappa\beta(r)) \text{ for all } r \ge r_0 = r_0(\varkappa).$$

Since $\dfrac{\ln\, r}{\ln\, \alpha^{-1}(\varkappa\beta(r))} \to 0$ as $r \to +\infty$ and $\alpha(e^x) \in L^0$, we have

$$\alpha(r\alpha^{-1}(\varkappa\beta(r))) = \alpha(\exp\{\ln\, \alpha^{-1}(\varkappa\beta(r)) + \ln\, r\}) = \alpha(\exp\{(1 + o(1))\ln\, \alpha^{-1}(\varkappa\beta(r))\}) =$$
$$= (1 + o(1))\varkappa\beta(r) \le q\beta(r), \quad r \ge r_0(q).$$

Therefore, $\Gamma_f(r\lambda_{\nu(r,A)}) \le \Gamma(\alpha^{-1}(q\beta(r))$ for $r \ge r_1$ and by Lemma 1

$$\ln\, \mu(r, A) - \ln\, \mu(r_1, A) \le \int\limits_{r_1}^{r} \frac{\Gamma_f(\alpha^{-1}(q\beta(t))}{t}dt \le \Gamma_f(\alpha^{-1}(q\beta(r))\ln\, \frac{r}{r_1} \le C\alpha^{-1}(q\beta(r))\ln\, \frac{r}{r_1},$$

because $\Gamma_f(r) \le Cr$ for all r. Since $\alpha(e^x) \in L^0$ and

$$\frac{\ln \ln r}{\ln \alpha^{-1}(q\beta(r))} \to 0$$

as $r \to +\infty$, as above we get

$$\alpha(C\alpha^{-1}(q\beta(r)) \ln (r/r_1)) = (1 + o(1))q\beta(r)$$

as $r \to +\infty$, and, $\varrho_{\alpha,\beta}[A] \le q$, which is a contradiction to the condition $q < \varrho_{\alpha,\beta}[A]$. □

Remark 1. The conditions $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta(x) \in L^0$ in Theorem 4 can be replaced by conditions $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta(x) \in L_{si}$.

Since the functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ satisfy the assumptions of Theorem 4, the following statement is correct.

Corollary 2. *Let a function f and a sequence (λ_n) satisfy the conditions of Theorem 4. Then*

$$\varlimsup_{r\to+\infty} \frac{\ln \ln \mathfrak{M}(r, A))}{r} = \varlimsup_{n\to\infty} \frac{\lambda_n \ln \lambda_n}{M_f^{-1}(1/|a_n|)}.$$

The functions $\alpha(x) = \beta(x) = \ln^+ x$ do not satisfy the conditions of Theorem 4. In this case we put

$$\varrho[A] = \varlimsup_{r\to+\infty} \frac{\ln \ln \mathfrak{M}(r, A))}{\ln r}$$

and prove the following theorem.

Theorem 5. *Let $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$. Suppose that $\ln M_f(r) = O(\Gamma_f(r))$, $r = o(\ln M_f(r)$ as $r \to +\infty$ and $\varlimsup_{n\to\infty} \dfrac{\ln \ln M_f(r)}{\ln r} \le 1$. Then*

$$\varrho[A] = \varkappa[A] + 1, \quad \varkappa[A] := \varlimsup_{n\to\infty} \frac{\ln \lambda_n}{\ln \left(\dfrac{1}{\lambda_n} M_f^{-1}\left(\dfrac{1}{|a_n|}\right)\right)}. \tag{18}$$

Proof. Let $1 \le \varrho[A] < +\infty$. Since Lemma 1 implies

$$\varlimsup_{r\to+\infty} \frac{\ln \ln \mu(r, A)}{\ln r} = \varrho[A],$$

for every $\varrho > \varrho[A]$ and all $r \ge r_0(\varrho)$ we have $\ln \mu(r, A) \le r^\varrho$ for $r \ge r_0(\varrho)$, i.e.

$$\ln |a_n| + \ln M_f(r\lambda_n) \le r^\varrho \text{ for all } n \ge 1 \text{ and } r \ge r_0(\varrho)$$

Choose $r = r_n = \lambda_n^{1/(\varrho-1)}$. Then $r_n \ge r_0(\varrho)$ for $n \ge n_0(\varrho)$ and, therefore,

$$\ln |a_n| \le \lambda_n^{\varrho/(\varrho-1)} - \ln M_f(\lambda_n^{\varrho/(\varrho-1)}) \quad n \ge n_0(\varrho)$$

The condition $r = o(\ln M_f(r)$ as $r \to +\infty$ implies

$$\ln |a_n| \le -(1 + o(1)) \ln M_f(\lambda_n^{\varrho/(\varrho-1)}) \text{ as } n \to \infty,$$

i.e. in view of (17)

$$\lambda_n^{\varrho/(\varrho-1)} \le M_f^{-1}(\exp\{(1 + o(1)) \ln (1/|a_n|)\}) = (1 + o(1))M_f^{-1}(1/|a_n|), \quad n \to \infty,$$

whence $\varkappa[A] \le \varrho - 1$. In view of the arbitrariness of ϱ we get the inequality $\varkappa[A] + 1 \le \varrho[A]$, which is obvious if $\varrho[A] = +\infty$.

Now, to prove the equality $\varkappa[A] + 1 = \varrho[A]$, suppose that $\varkappa[A] < \varrho[A] - 1$. Then for every $\varkappa \in (\varkappa[A], \varrho[A] - 1)$ we have

$$|a_n| \le 1/M_f(\lambda_n^{1+1/\varkappa})$$

for all $n \ge n_0(\varkappa)$. Therefore, as in the proof of Theorem 4 we obtain $\lambda_{\nu(r,A)} \le r^{\varkappa}$ for $r \ge r_0$ and, thus, in view of (11)

$$\ln \mu(r, A) - \ln \mu(r_0, A) \le \int\limits_{r_0}^{r} \frac{\Gamma_f(t^{1+\varkappa})}{t} dt = \frac{1}{1+\varkappa} \int\limits_{r_0^{1+\varkappa}}^{r^{1+\varkappa}} \Gamma_f(t) d\ln t =$$

$$= \frac{1}{1+\varkappa} \int\limits_{r_0^{1+\varkappa}}^{r^{1+\varkappa}} \frac{d\ln M_f(t)}{d\ln t} d\ln t = \frac{1}{1+\varkappa}(\ln M_f(r^{1+\varkappa}) - \ln M_f(r_0^{1+\varkappa})),$$

whence

$$\varrho[A] = \varlimsup_{r\to+\infty} \frac{\ln\ln \mu(r, A)}{\ln r} \le \varlimsup_{r\to+\infty} \frac{\ln\ln M_f(r^{1+\varkappa})}{\ln r} = (1+\varkappa)\varrho[f] \le 1+\varkappa,$$

because $\varrho[f] \le 1$, which contradicts to the condition $\varkappa < \varrho[A] - 1$. Theorem 5 is proved. □

4. Fréchet spaces (see more details in [13]). For fixed $\varrho < +\infty$ by $\overline{S}_\varrho$ we denote the class of function (2), such that $\varrho_{\alpha,\beta}[A] \le \varrho$. Then (18) implies

$$|a_n| \le \frac{1}{M_f\left(\lambda_n \beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho + o(1)}\right)\right)}, \quad n \to \infty. \tag{19}$$

Using an idea of the article [8], for $q \in \mathbb{N}$ we define

$$\|A\|_{\varrho;q} = \sum_{n=1}^{\infty} |a_n| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right).$$

If $\beta \in L^0$ then

$$\frac{\beta^{-1}((1+c)x)}{\beta^{-1}(x)} \ge Q(c) > 1$$

for every $c > 0$ and all $x \ge x_0$ and, as above, we have

$$\frac{M_f\left(\lambda_n \beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right)}{M_f\left(\lambda_n \beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho}\right)\right)} \le \exp\left\{-\Gamma_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right) \ln \frac{\beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho}\right)}{\beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho + 1/q}\right)}\right\} \le$$

$$\le \exp\{-\Gamma_f(\lambda_n) \ln Q(1/(q\varrho))\} < 1.$$

Therefore, if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then in view of (19) $\|A\|_{\varrho;q}$ exists for each $q \in \mathbb{N}$ and it easily to check that $\|A\|_{\varrho;q}$ is a norm on $\overline{S}_\varrho$.

Clearly, $\|A\|_{\varrho;q} \le \|A\|_{\varrho;q+1}$. Therefore [6], the family $\|A\|_{\varrho;q}\colon q \in \mathbb{N}$ induces on $\overline{S}_\varrho$ the unique topology such that $\overline{S}_\varrho$ becomes a local convex vector space and this topology is given by the metric d, where

$$d(A_1,\, A_2) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|A_1 - A_2\|_{\varrho;q}}{1 + \|A_1 - A_2\|_{\varrho;q}}. \tag{20}$$

The space with the metric d we denote by $\overline{S}_{\varrho,d}$.

Theorem 6. *If the functions α, β, f and the sequence (λ_n) satisfy the hypotheses of Theorem 4 then $\overline{S}_{\varrho,d}$ is a Fréchet space.*

Proof. It is sufficient to show that $\overline{S}_{\varrho,d}$ is complete. Let therefore (A_j) be a d-Cauchy sequence in $\overline{S}_{\varrho,d}$ and so far for a given $\varepsilon > 0$ there corresponds an $m = m(\varepsilon)$ such that

$$\|A_j - A_k\|_{\varrho;q} < \varepsilon$$

for all j, $k \ge m$ and $q \in \mathbb{N}$. Consequently for these j, k and q we have

$$\sum_{n=1}^{\infty} |a_n^{(j)} - a_n^{(k)}| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q)}\right)\right) < \varepsilon, \tag{21}$$

i.e. $|a_n^{(j)} - a_n^{(k)}| < \varepsilon$ and $(a_n^{(j)})_{j\ge 1})$ is a Cauchy sequence. Therefore, $a_n^{(j)} \to a_n$ as $j \to \infty$. Letting $k \to \infty$ in (21) one has for $j \ge j_0$

$$\sum_{n=1}^{\infty} |a_n^{(j)} - a_n| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q)}\right)\right) < \varepsilon, \tag{22}$$

and consequently taking $j = j_0$ in (22) we get for a fixed q

$$\sum_{n=1}^{\infty} |a_n^{(j_0)} - a_n| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q)}\right)\right) < \varepsilon,$$

whence in view of (19) with $a_n^{(j_0)}$ instead of a_n we obtain

$$\begin{aligned} |a_n| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right) &\le |a_n^{(j_0)}| M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right) + \varepsilon \le \\ &\le M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right) \Big/ M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + o(1)}\right)\right) + \varepsilon \le 2\varepsilon, \end{aligned}$$

i.e.

$$\varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right)\right)} \le \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n} M_f^{-1}\left(M_f\left(\lambda_n \beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho + 1/q}\right)\right) \big/ (2\varepsilon)\right)\right)} = \varrho + 1/q,$$

because M_f^{-1} is a slowly increasing function.

By Theorem 4 in view of the arbitrariness of q we get $\varrho_{\alpha,\beta}[A] \le \varrho$. Thus, using (21) again we see that $\|A_j - A\|_{\varrho;q} < \varepsilon$ for $j \ge j_0$ and the result is proved. □

For $\overline{S}_{\varrho,d}$ by $\overline{S}^*_{\varrho,d}$ we denote the dual space. The following analog of Theorem 3 is true.

Theorem 7. *If the functions α, β, f and the sequence (λ_n) satisfy the conditions of Theorem 4 then continuous linear functional L on $\overline{S}_{\varrho,d}$ is of form* (8) *if and only if for all $n \in \mathbb{N}$ and $q \in \mathbb{N}$*

$$|g_n| \le KM_f\left(\lambda_n\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho+1/q)}\right)\right), \quad K = const > 0. \tag{23}$$

Proof. Let $L \in \overline{S}^*_{\varrho,d}$. This clearly means if $A_m \to A$ in $\overline{S}_{\varrho,d}$ then $L(A_m) \to L(A)$.

Now let a_n satisfy (19) and $A_m(s) = \sum\limits_{n=1}^{m} a_n f(z\lambda_n)$. Then we claim that $A_m \to A$ in $\overline{S}_{\varrho,d}$ (observe that $A_m \in \overline{S}_{\varrho,d}$). To ascertain this, it is sufficient to prove that $A_m \to A$ in the norm $\|\cdot\|_{\varrho,q}$ for every $q \in \mathbb{N}$.

So let q be fixed integer. Choose $\varepsilon \in (0,\, 1/q)$. Then in view of (19) we can determine an integer $m = m(\varepsilon)$ such that

$$|a_n| \le \frac{1}{M_f\left(\lambda_n\beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho+\varepsilon}\right)\right)}, \quad n \ge m+1,$$

and it follows as above that

$$\|A_m - A\|_{\varrho,q} = \|\sum_{n=m+1}^{\infty} a_n f(z\lambda_n)\|_{\varrho,q} \le \sum_{n=m+1}^{\infty} \frac{M_f\left(\lambda_n\beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho+1/q}\right)\right)}{M_f\left(\lambda_n\beta^{-1}\left(\dfrac{\alpha(\lambda_n)}{\varrho+\varepsilon}\right)\right)} \to 0, \quad m \to \infty,$$

and this ascertains our claim. Combining this with the continuity of L we have

$$\lim_{m\to\infty} L(A_m) = L(A)$$

in the topology given by d.

Note that $L(A_m) = \sum\limits_{n=1}^{m} d_n g_n$, where $g_n = L(f(z\lambda_n))$ for each n. Since L is continuous on $(\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})$, there exists a $K > 0$ such that

$$|g_n| = |L(f(z\lambda_n))| \le K\|f(z\lambda_n))\|_{\varrho,q} \text{ for each } q \in \mathbb{N}$$

and so, using the definition of the norm $\|f(z\lambda_n))\|_{\varrho,q}$, we get (23).

To prove the other part, let now g_n satisfy (23). Then

$$|L(A)| \le K\sum_{n=1}^{\infty} |a_n| \exp\left\{\lambda_n\beta^{-1}\left(\frac{\alpha(\lambda_n)}{\varrho+1/q}\right)\right\}, \quad q \in \mathbb{N},$$

and so $|L(A)| \le K\|A\|_{\varrho,q}$ for all $q \in \mathbb{N}$. Therefore, $L \in (\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})^*$ for all $q \in \mathbb{N}$. Since $\|A\|_{\varrho;q} \le \|A\|_{\varrho;q+1}$, from (19) it follows that $\overline{S}^*_{\varrho,d} = \bigcup\limits_{q\ge1}(\overline{S}_{\varrho,d}, \|\cdot\|_{\varrho,q})^*$. Thus, $L \in \overline{S}^*_{\varrho,d}$. □

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