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ANALYTIC GAUSSIAN FUNCTIONS IN THE UNIT DISC: PROBABILITY OF ZEROS ABSENCE

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In the paper we consider a random analytic function of the form

$$f(z,\omega)=\sum_{n=0}^{+\infty}\varepsilon_n(\omega_1)\xi_n(\omega_2)a_nz^n.$$

Here (ε_n) is a sequence of independent Steinhaus random variables, (ξ_n) is a sequence of independent standard complex Gaussian random variables, and a sequence of numbers $a_n\in\mathbb{C}$ such that $a_0\neq 0$, $\varlimsup\limits_{n\to+\infty}\sqrt[n]{|a_n|}=1$, $\sup\{|a_n|\colon n\in\mathbb{N}\}=+\infty$. We investigate asymptotic estimates of the probability $p_0(r)=\ln^- P\{\omega\colon f(z,\omega)$ has no zeros inside $r\mathbb{D}\}$ as $r\uparrow 1$ outside some set E of finite logarithmic measure. Denote $N(r):=\#\{n\colon |a_n|r^n>1\}$, $s(r):=2\sum_{n=0}^{+\infty}\ln^+(|a_n|r^n)$, $\alpha:=\varliminf\limits_{r\uparrow 1}\frac{\ln N(r)}{\ln\frac{1}{1-r}}$. The article, in particular, proves the following statements:

1) if $\alpha>4$ then

$$\lim_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln N(r)}=1;$$

2) if $\alpha=+\infty$ then

$$0\le\varliminf_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)},\quad \varlimsup_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)}\le\frac{1}{2}.$$

Here E is a set of finite logarithmic measure. The obtained asymptotic estimates are in a certain sense best possible. Also we give an answer to an open question from [23, p. 119] for such random functions.

1. Introduction. One of the problems in theory of random functions is investigation of value distribution of these functions and also asymptotic properties of probability of absence zeros in a disc ("hole probability"). These problems were considered in papers of J. E. Littlewood, A. C. Offord ([1–6]), M. Sodin, B. Tsirelson ([7–9]), Yu. Peres, B. Virag ([10]), P. V. Filevych, M. P. Mahola ([11–13]), M. Sodin ([14–16]), F. Nazarov, M. Sodin, A. Volberg ([17, 18]), M. Krishnapur ([19]), A. Nishry ([20–23]), J. Buckley, A. Nishry, R. Peled, M. Sodin ([24]), A. Kiro, A. Nishry ([25]), J. Buckley, A. Nishry ([26]), A. O. Kuryliak, O. B. Skaskiv ([27,28]), H. Li, J. Wang, X. Yao, Z. Ye ([29]).

So, in [9] there was considered a random entire function of the form

$$\psi(z,\omega)=\sum_{k=0}^{+\infty}\xi_k(\omega)\frac{z^k}{\sqrt{k!}},\tag{1}$$

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where $\{\xi_k(\omega)\}$ are independent complex valued random variables with a density function

$$p_{\xi_k}(z) = \frac{1}{\pi}e^{-|z|^2},\ z\in\mathbb{C},\ k\in\mathbb{Z}_+.$$

From now, we denote such a distribution by $\mathcal{N}_{\mathbb{C}}(0,1)$.

Let us denote by $n_\psi(r,\omega)$ the number of zeros of the function $\psi(z,\omega)$ in $r\mathbb{D}=\{z\colon |z|<r\}$. Then ([9]) for any $\delta\in(0,1/4]$ and all $r\geq 1$ one has

$$P\Big\{\omega\colon \Big|\frac{n(r,\omega)}{r^2}-1\Big|\geq\delta\Big\}\leq \exp(-c(\delta)r^4),$$

where the constant $c(\delta)$ depends only on δ.

In [9, 20] also investigated the probability of the absence zeros of the function $\psi(z,\omega)$,

$$P_0(r)=P\{\omega\colon \psi(z,\omega)\neq 0 \text{ inside } r\mathbb{D}\}.$$

For the function of the form (1) it is possible to fix the disc of radius r and investigate the asymptotic behavior of $P\{\omega\colon n_\psi(r,\omega)\geq m\}$ as $m\to+\infty$. So in [19] there was proved that for any $r>0$ we get

$$\ln P\{\omega\colon n_\psi(r,\omega)\geq m\} = -\frac{1}{2}m^2\ln m(1+o(1)),\ m\to+\infty.$$

Very large deviations of zeros of function (1) also was considered in [18].

There was considered [22, 23] a more general Gaussian entire functions of the form

$$f(z,\omega)=\sum_{n=0}^{+\infty}\xi_k(\omega)a_nz^n,$$

where $a_0\neq 0,\ n\in\mathbb{Z}_+,\ \overline{\lim\limits_{n\to+\infty}}\sqrt[n]{a_n}=0$. If $\varepsilon>0$, then there exists ([22, 23]) a set of finite logarithmic measure $E\subset(1,+\infty)$ ($\int_E\frac{dr}{r}<+\infty$) such that for all $r\in(1,+\infty)\backslash E$ we obtain

$$s(r)-s^{1/2+\varepsilon}(r)\leq p_0(r)\leq s(r)+s^{1/2+\varepsilon}(r),\ s(r)=2\sum_{n=0}^{+\infty}\ln^+(a_nr^n). \tag{2}$$

It is proved that the exceptional set in this statement is necessary. There was constructed a Gaussian entire function and a set E of infinite Lebesgue's measure such that $p_0(r)\geq s(r)-c\sqrt{s(r)}, r\in E$. Also there was formulated the following question. *Is the error term in inequality (2) optimal for a regular sequence of the coefficients $\{a_n\}$?*

The authors [28] found an answer to this question. We proved

$$0\leq \varliminf_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)},\ \varlimsup_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)}\leq\frac{1}{2},$$

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln N(r)}=1,$$

where E is a set of finite logarithmic measure. There was constructed a Gaussian entire function $f(z,\omega)$ for which

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)}=\frac{1}{2}$$

and a Gaussian entire function $g(z, \omega)$ such that

$$\lim_{\substack{r\to+\infty \\ r\notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} = 0.$$

In [19], there was considered a function of the form

$$f_\rho(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) \binom{-\rho}{n}^{1/2} z^n, \ \rho > 0,$$

which is almost surely analytic function in the unit disc.

In [10], [24] there was considered a random analytic function of the form

$$f_L(z, \omega) = \sum_{k=0}^{+\infty} \xi_k(\omega) \sqrt{\frac{L(L+1)\dots(L+k-1)}{k!}} z^k, \ L > 0, \tag{3}$$

where $\xi_k(\omega) \in \mathcal{N}_{\mathbb{C}}(0, 1), \ k \in \mathbb{Z}_+$.

Remark that if $L = 1$ then $f_1(z, \omega) = \sum_{k=0}^{+\infty} \xi_k(\omega) z^k$. For this function we find the following result ([10]). Denote by $\mathbf{E}\eta$ and $\mathbf{D}\eta$ the mean value and variance of a random variable η, respectively. The mean value and variance of random variable $n_{f_1}(r, \omega)$ (number of zeros of function $f_1(z, \omega)$ inside $r\mathbb{D}$) are equal

$$\mathbf{E}(n_{f_1}(r, \omega)) = \frac{r^2}{1 - r^2}, \quad \mathbf{D}(n_{f_1}(r, \omega)) = \frac{r^2}{1 - r^4}, \quad p_0(r) = \frac{\pi^2 + o(1)}{1 - r}, \ r \uparrow 1.$$

For $f_L(z, \omega)$ the following estimates can be found in [24]: for $0 < L < 1$

$$\frac{1 - L - o(1)}{2^{L+1}} \frac{1}{(1-r)^L} \ln \frac{1}{1-r} \le p_0(r) \le \frac{1 - L + o(1)}{2^L} \frac{1}{(1-r)^L} \ln \frac{1}{1-r} \ (r \uparrow 1)$$

and for $L > 1$

$$p_0(r) = \frac{(L-1)^2 + o(1)}{4} \frac{1}{1-r} \ln^2 \frac{1}{1-r} \quad (r \uparrow 1).$$

Remark that for $f_L(z, \omega)$ in the case when $L > 1$ we have

$$\alpha := \varliminf_{r\uparrow 1} \frac{\ln N(r)}{\ln \frac{1}{1-r}} = 1.$$

The **aim** of this paper is to obtain analogs of these relations for a "wide" class of random analytic functions.

This problem was already considered in [27] for some class of functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n,$$

where $a_n > 0, \ \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = 1$, $\{a_n\}$ are log-concave and

$$\varliminf_{r\uparrow 1} (1 - r) \ln \ln \left(\sum_{n=0}^{+\infty} a_n r^n \right) = +\infty. \tag{4}$$

Then for all $\varepsilon > 0$ there exists a set $E(\varepsilon, f) = E_1 \subset (0,1)$ such that

$$DE_1 = \overline{\lim_{r\uparrow 1}} \frac{1}{1-r}\text{meas}(E_1 \cap [r,1)) = 0$$

and for all $r \in (0,1)\backslash E_1$ we get $p_0(r) = s(r) + o(s(r))$. In particular,

$$s(r) - s^{9/10}(r)\ln^{18/5+\varepsilon} s(r) \le p_0(r) \le s(r) + \sqrt{s(r)}\ln^{3+\varepsilon} s(r), \quad s(r) = 2\sum_{n=0}^{+\infty} \ln^+(a_n r^n).$$

2. Notations. In this paper, we consider random analytic functions of the form

$$f(z,\omega) = f(z,\omega_1,\omega_2) = \sum_{n=0}^{+\infty} \varepsilon_n(\omega_1)\xi_n(\omega_2)a_n z^n. \tag{5}$$

Here $\varepsilon_n(\omega_1) = e^{i\theta_n(\omega_1)}$, (θ_n) is a sequence of the independent random variables uniformly distributed on $[-\pi,\pi)$, $\big(\xi_n(\omega_2)\big) \in \mathcal{N}_{\mathbb{C}}(0,1)$ and $a_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$ such that

$$a_0 \ne 0, \quad \overline{\lim_{n\to+\infty}} \sqrt[n]{|a_n|} = 1, \quad \sup\{|a_n|\colon n \in \mathbb{N}\} = +\infty.$$

Denote the class of such random analytic functions by $\mathcal{A}$.

In this paper we study asymptotic behavior of

$$p_0(r) = \ln^- P\{\omega\colon n_f(r,\omega) = 0\}$$

as $r \uparrow 1$ for random analytic functions of the form (5).

We denote

$$\mathcal{N}(r) = \{n\colon \ln(|a_n|r^n) > 0\}, \quad \exp_2\{x\} = e^{e^x}, \quad N(r) = \#\mathcal{N}(r),$$
$$\mathcal{N}_1(r) = \{n\colon \ln(|a_n|r_1^n) > 0\}, \quad N(r) = \#\mathcal{N}(r), \quad N_1(r) = \#\mathcal{N}_1(r),$$
$$r_1 = 1-(1-r)\exp\Big\{-\frac{1}{\ln^2 N(r)}\Big\}, \quad s(r) = 2\sum_{n\in\mathcal{N}(r)} \ln(|a_n|r^n) = 2\sum_{n=0}^{+\infty} \ln^+(|a_n|r^n),$$
$$\mu_f(r) = \max\{|a_n|r^n\colon n \in \mathbb{Z}_+\}, \quad \nu_f(r) = \max\{n\colon \mu_f(r) = |a_n|r^n\},$$
$$M_f(r) = \max\{|f(z)|\colon |z| \le r\}, \quad S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2 r^{2n}.$$

3. Auxiliary lemmas.

Lemma 1 ([12])**.** *Let $r_0 \in [0;1)$, $u(r)$ be a non-decreasing unbounded function on $[r_0;1)$, $x_0 = u(r_0)$. Also, the function $\varphi(u)$ is a positive continuous increasing to $+\infty$ on $[x_0;+\infty)$ function defined on $[u_0;+\infty)$ such that*

$$\int_{u_0}^{+\infty} \frac{du}{\varphi(u)} < +\infty.$$

Then for all $r \ge r_0$ outside the set E of finite logarithmic measure ($\int_E \frac{dr}{1-r} < +\infty$) we have

$$u\Big(1-(1-r)\exp\Big\{-\frac{1}{\varphi(\ln u(r))}\Big\}\Big) < eu(r).$$

Remark that if $f \in \mathcal{A}$, then $\lim\limits_{r\uparrow 1} N(r) = +\infty$ and we can choose in Lemma 1 $u(r) = N(r)$ and $\varphi(x) = x^2,\ x \to +\infty$. So, we have the following statement.

Lemma 2. *Let $f \in \mathcal{A}$. There exists a set $E \subset (0,1)$ of finite logarithmic measure such that for all $r \in [0;1)\backslash E$ we have*
$$N_1(r) < eN(r).$$

Lemma 3. *Let $f \in \mathcal{A}$. There exists a set $E \subset [0;1)$ of finite logarithmic measure such that for all $r \in [0;1)\backslash E$ we have*

$$N(r) < \frac{2}{\sqrt{1-r}} \sqrt{s(r)} \ln s(r).$$

Proof. Remark that outside a set of the countable number of points we have

$$s'(r) = 2 \sum_{n \in \mathcal{N}(r)} \frac{n}{r} \geq 2 \sum_{n \in \mathcal{N}(r)} n \geq 2 \sum_{n=0}^{N(r)-1} n = (N(r)-1)N(r) > \frac{N^2(r)}{4},\ r \uparrow 1.$$

Define
$$E = \Big\{ r \in [0;1) \colon s'(r) \geq \frac{1}{1-r} s(r) \ln^2 s(r),\ s(r) > 2 \Big\}.$$
This set has a finite logarithmic measure, i.e.

$$\int\limits_E \frac{dr}{1-r} < \int\limits_E \frac{s'(r)dr}{s(r)\ln^2 s(r)} = \int\limits_{s(E)} \frac{dt}{t \ln^2 t} \leq \int\limits_2^{+\infty} \frac{dt}{t\ln^2 t} < +\infty.$$

Therefore, for all $r \in [0;1)\backslash E$ we obtain

$$N(r) < 2\sqrt{s'(r)} \leq \frac{2}{\sqrt{1-r}} \sqrt{s(r)} \ln s(r).$$

□

Lemma 4 ([28]). *Let $\big(\eta_n(\omega)\big)$ be a sequence of independent non-negative identically distributed random variables, such that $\mathbf{E}\eta_n < +\infty$ and $\mathbf{E}(\frac{1}{\eta_n}) < +\infty,\ n \in \mathbb{Z}_+$. Then*

$$P\Big\{\omega \colon (\exists N^*(\omega))(\forall n > N^*(\omega))\ \Big[\frac{1}{n} \leq \eta_n(\omega) \leq n\Big]\Big\} = 1.$$

4. Upper and lower bounds for $p_0(r)$.

Theorem 1. *Let $f \in \mathcal{A}$ and*
$$\alpha = \varliminf_{r\uparrow 1} \frac{\ln N(r)}{\ln \frac{1}{1-r}} > 4.$$
There exists a set $E \subset [0;1)$ of finite logarithmic measure such that for all $r \in [0;1)\backslash E$ we have
$$p_0(r) \leq s(r) + (1+e)N(r)\ln N(r) + C_0 N(r), \tag{6}$$
where $C_0 = 3 + 9/|a_0|$.

Proof of theorem 1. Let $\alpha = 4+5\varepsilon,\ \varepsilon>0$. Then

$$N(r) > \frac{1}{(1-r)^{4+4\varepsilon}},\ r\uparrow 1. \tag{7}$$

By definition of $N_1(r)$ and r_1 we get

$$N_1(r)=\#\{n\colon \ln(|a_n|r_1^n)>0\}=\#\Big\{n\colon |a_n|\Big(1-(1-r)\exp\Big\{-\frac{1}{\ln^2 N(r)}\Big\}\Big)^n>1\Big\}=$$
$$=\{n\colon |a_n|r^n>q^n(r)\},\ q(r)=\frac{r}{1-(1-r)\exp\Big\{-\frac{1}{\ln^2 N(r)}\Big\}}<1.$$

Let us consider the event $B=\bigcap_{i=1}^4 A_i$, where

$$A_1\colon |\xi_0(\omega_1)|\geq \frac{3}{|a_0|}\sqrt{N(r)},$$
$$A_2\colon |\xi_n(\omega_1)|\leq \frac{1}{|a_n|r^n\sqrt{N(r)}}\ \text{for all}\ n\in\mathcal{N}(r)\backslash\{0\},$$
$$A_3\colon |\xi_n(\omega_1)|\leq \frac{1}{\sqrt{N(r)}}\ \text{for all}\ n\in\mathcal{N}_1(r)\backslash(\mathcal{N}(r)\cup\{0\}),$$
$$A_4\colon |\xi_n(\omega_1)|\leq \sqrt{n}\ \text{for all}\ n\notin\mathcal{N}_1(r)\cup\{0\}.$$

Remark that

$$\int\limits_1^{+\infty}\sqrt{x}q^x(r)dx=\frac{1}{\ln q(r)}\Big(\sqrt{x}q^x(r)\big|_1^{+\infty}-\int\limits_1^{+\infty}q^x(r)\frac{1}{2\sqrt{x}}dx\Big)=$$
$$=\frac{1}{\ln\frac{1}{q(r)}}\Big(q(r)+\int\limits_1^{+\infty}q^x(r)\frac{1}{2\sqrt{x}}dx\Big)<\frac{1}{\ln\frac{1}{q(r)}}\Big(q(r)+\int\limits_1^{+\infty}q^x(r)dx\Big)=$$
$$=\frac{1}{\ln\frac{1}{q(r)}}\Big(q(r)+\frac{q^x(r)}{\ln q(r)}\Big|_1^{+\infty}\Big)=\frac{1}{\ln\frac{1}{q(r)}}\Big(q(r)-\frac{q(r)}{\ln q(r)}\Big)=$$
$$=\frac{1}{\ln\frac{1}{q(r)}}\Big(q(r)+\frac{q(r)}{\ln\frac{1}{q(r)}}\Big)<\frac{1}{\ln^2\frac{1}{q(r)}}<\frac{1}{(1-q(r))^2},\quad r\uparrow 1. \tag{8}$$

If B occurs, then using lemma 2 and inequalities (7), (8) we obtain for $r\notin E$

$$|\varepsilon_0(\omega_1)\xi_0(\omega_2)a_0|-\Big|\sum_{n=1}^{+\infty}\varepsilon_n(\omega_1)\xi_n(\omega_2)a_nr^n\Big|\geq 3\sqrt{N(r)}-$$
$$-\sum_{n\in\mathcal{N}(r)}\frac{|a_n|r^n}{|a_n|r^n\sqrt{N(r)}}-\sum_{n\in\mathcal{N}_1(r)\backslash\mathcal{N}(r)}\frac{|a_n|r^n}{\sqrt{N(r)}}-\sum_{n\notin\mathcal{N}_1(r)\cup\{0\}}\sqrt{n}\cdot q^n(r)>$$
$$>3\sqrt{N(r)}-\sqrt{N(r)}-\frac{N_1(r)-N(r)}{\sqrt{N(r)}}-\frac{1}{(1-q(r))^2}\geq$$
$$\geq 3\sqrt{N(r)}-e\sqrt{N(r)}-\left(\frac{1-(1-r)\exp\Big\{-\frac{1}{\ln^2 N(r)}\Big\}}{(1-r)\Big(1-\exp\Big\{-\frac{1}{\ln^2 N(r)}\Big\}\Big)}\right)^2\geq$$

$$\geq (3-e)\sqrt{N(r)} - \ln^5 N(r)\frac{1}{(1-r)^2} = \ln^5 N(r)\Big((3-e)\frac{\sqrt{N(r)}}{\ln^5 N(r)} - \frac{1}{(1-r)^2}\Big)\geq$$
$$\geq \ln^5 N(r)\Big(\frac{1}{(1-r)^{2+\varepsilon}} - \frac{1}{(1-r)^2}\Big) > 0, \quad r\uparrow 1.$$

Thus, we proved that the first term dominants the sum of all the other terms inside $r\mathbb{D}$, i.e.

$$|\varepsilon_0(\omega_1)\xi_0(\omega_2)a_0| > \left|\sum_{n=1}^{+\infty}\varepsilon_n(\omega_1)\xi_n(\omega_2)a_nr^n\right|. \tag{9}$$

If B occurs then the function $f(z,\omega)$ almost surely has no zeros inside $r\mathbb{D}$. Now we find the lower bound for the probability of the event B.

$$P(A_1) = \exp\Big\{-\frac{9N(r)}{|a_0|^2}\Big\},$$
$$P(A_2) \geq \prod_{n\in\mathcal{N}(r)}\frac{1}{2|a_n|^2r^{2n}N(r)} = \exp\Big\{-s(r) - N(r)\ln N(r) - N(r)\ln 2\Big\},$$
$$P(A_3) \geq \prod_{n\in\mathcal{N}_1(r)}\frac{1}{2N(r)} \geq \exp\Big\{-N_1(r)\ln(2N(r))\Big\}\geq \exp\Big\{-eN(r)(\ln N(r)+\ln 2)\Big\},$$
$$P(A_4) = P\{\omega\colon (\forall n\notin\mathcal{N}_1(r)\cup\{0\})|\xi_n(\omega)|<\sqrt{n}\}\geq$$
$$\geq 1 - \sum_{n\notin\mathcal{N}_1(r)\cup\{0\}}e^{-n} \geq 1 - \sum_{n=1}^{+\infty}e^{-n} = 1 - \frac{1}{e-1} > \frac{2}{5},\ r\uparrow 1.$$

It follows from $B\subset\{\omega\colon n(r,\omega)=0\}$ that

$$p_0(r) = \ln^- P\{\omega\colon n(r,\omega)=0\} \leq \ln^- P(B) = \sum_{n=1}^{4}\ln^- P(A_n) \leq$$
$$\leq \ln\frac{5}{2} + \frac{9N(r)}{|a_0|^2} + s(r) + N(r)\ln N(r) + N(r)\ln 2 + eN(r)\ln N(r) + e\ln 2\cdot N(r) \leq$$
$$\leq s(r) + \frac{9N(r)}{|a_0|^2} + (1+e)N(r)\ln N(r) + 3N(r) = s(r) + (1+e)N(r)\ln N(r) + C_0N(r).$$

□

There was considered in [13] a random analytic function of the form

$$g(z,\omega_1) = \sum_{n=0}^{+\infty}e^{i\theta_n(\omega_1)}a_nz^n, \tag{10}$$

where $a_0\neq 0$, $\sup\{|a_n|\colon n\in\mathbb{N}\} = +\infty$ and independent random variables $\theta_n(\omega_1)$ are uniformly distributed on $[-\pi,\pi)$. For such functions there were proved the following statements.

Theorem 2 ([11])**.** *Let $g(z,\omega_1)$ be a random analytic function of the form (10). Then for $r>r_0$ and all ω_1 we obtain*

$$N_g(r,\omega_1) \leq \frac{1}{2e} + \ln\mathfrak{M}_g(r),$$

where

$$N_g(r,\omega_1)=\frac{1}{2\pi}\int\limits_0^{2\pi}\ln|g(re^{i\alpha},\omega_1)|d\alpha-\ln|a_0|,\quad \mathfrak{M}_g(r)=\sum_{n=0}^{+\infty}|a_n|r^n.$$

Theorem 3 ([13])**.** *There exists an absolute constant $C>0$ such that for a function $g(z,\omega_1)$ of the form (10) we P_1-almost surely have*

$$\ln\mathfrak{M}_g(r)\le N_g(r,\omega_1)+C\ln N_g(r,\omega_1),\ r_0(\omega_1)\le r<+\infty. \tag{11}$$

Let $P=P_1\times P_2$ be a direct product of the probability measures P_1 and P_2 defined on $(\Omega_1\times\Omega_2,\mathcal{A}_1\times\mathcal{A}_2)$. Here $\mathcal{A}_1\times\mathcal{A}_2$ is the minimal σ-algebra, which contains all $A_1\times A_2$ such that $A_1\in\mathcal{A}_1$ and $A_2\in\mathcal{A}_2$. Let $\varepsilon_n(\omega_1)=e^{i\theta_n(\omega_1)}$, (θ_n) is a sequence of the independent random variables uniformly distributed on $[-\pi,\pi)$ on $(\Omega_1,\mathcal{A}_1,P_1)$, $\xi_n(\omega_2)\in\mathcal{N}_{\mathbb{C}}(0,1)$ on $(\Omega_2,\mathcal{A}_2,P_2)$, where $(\Omega_1,\mathcal{A}_1,P_1)$, $(\Omega_2,\mathcal{A}_2,P_2)$ are two probability spaces.

Corollary 1. *Let $(\zeta_n(\omega_2))$ be a sequence of independent identically distributed random variables such that for any $n\in\mathbb{Z}_+$ the density function of the distribution of the random variable $\eta=\zeta_n$ has the form $p_\eta(z)=h(|z|)$ and $\mathbf{E}|\eta|<+\infty$, $\mathbf{E}(\frac{1}{|\eta|})<+\infty$. There exist an absolute constant $C>0$ and a set $B\in\mathcal{A}$: $P(B)=1$ such that for the functions*

$$f(z,\omega)=\sum_{n=0}^{+\infty}\varepsilon_n(\omega_1)\zeta_n(\omega_2)a_nz^n,\quad a_0\ne0,\quad \sup\{a_n\colon n\in\mathbb{N}\}=+\infty,\quad \overline{\lim_{n\to+\infty}}\sqrt[n]{|a_n|}=1,$$

and for all $\omega\in B$ and all $r\in[r_0(\omega);+\infty)$ we get

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\ln|f(re^{i\alpha},\omega)|d\alpha-\ln|a_0\varepsilon_0(\omega_1)\zeta_0(\omega_2)|\ge\ln\mathfrak{M}_f(r,\omega_2)-(C+1)\ln\ln\mathfrak{M}_f(r,\omega_2),$$

where $\mathfrak{M}_f(r,\omega_2)=\sum\limits_{n=1}^{+\infty}|a_n||\zeta_n(\omega_2)|r^n$.

Remark that if the density function of $\zeta_n(\omega_1)$ has the following form $p_{\zeta_n}(z)=q(|z|), n\in\mathbb{N}$, then $\arg\zeta_n(\omega_1)$ are uniformly distributed on $[-\pi,\pi)$. Indeed, from the equality

$$1=P_1\{\omega_1\colon\zeta_n(\omega_1)\in\mathbb{C}\}=\int\limits_{-\pi}^{\pi}d\varphi\int\limits_0^{+\infty}rh(r)dr=2\pi\int\limits_0^{+\infty}rh(r)dr,$$

for any $\alpha,\beta\in[-\pi,\pi)\colon\ \alpha<\beta$ it follows

$$P_1\{\omega_1\colon\ \arg\zeta_n(\omega_1)\in(\alpha,\beta)\}=\int\limits_\alpha^\beta d\varphi\int\limits_0^{+\infty}rh(r)dr=\frac{\beta-\alpha}{2\pi}.$$

Since the random variables $\xi_n(\omega_1)$ satisfy the conditions of Corollary 1 (here $p_{\xi_k}(z)=h(|z|)=\frac{1}{\pi}e^{-|z|^2},\ z\in\mathbb{C},\ k\in\mathbb{Z}_+$), we have the following statement for the functions of the form (5).

Corollary 2. *There exist an absolute constant $C > 0$ and a set $B\colon P(B) = 1$ such that for the functions of the form (5) and for all $\omega \in B$ and all $r \in [r_0(\omega); +\infty)$ we obtain*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta - \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \geq$$
$$\geq \ln \mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r,\omega_2).$$

Proof of corollary 2. It follows from Theorem 2 that $\ln N_g(r,\omega_1) \leq 1 + \ln\ln \mathfrak{M}_g(r)$ and by Theorem 3 we ω_1-almost surely have

$$N_g(r,\omega_1) \geq \ln \mathfrak{M}_g(r) - C\ln N_g(r,\omega_1) \geq \ln \mathfrak{M}_g(r) - (C+1)\ln\ln \mathfrak{M}_g(r),$$

for $r_0(\omega_1) \leq r < +\infty$. Therefore,

$$P_1\{\omega\colon (\exists r_0(\omega_1))(\forall r > r_0(\omega_1))\ [N_g(r,\omega_1) \geq \ln \mathfrak{M}_g(r) - (C+1)\ln\ln \mathfrak{M}_g(r)]\} = 1.$$

Let us consider a random function $f(z,\omega_1,\omega_2)$ of the form (5). Define

$$A_f = \{(\omega_1,\omega_2)\colon (\exists r_0(\omega_1,\omega_2))(\forall r > r_0(\omega_1,\omega_2))$$
$$[N_f(r,\omega_1,\omega_2) \geq \ln \mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln \mathfrak{M}_f(r,\omega_2)]\},$$

where

$$\mathfrak{M}_f^2(r,\omega_2) = \sum_{n=0}^{+\infty} |\varepsilon_n(\omega_1)|^2|\zeta_n(\omega_2)|^2|a_n|^2 r^{2n} = \sum_{n=0}^{+\infty} |\zeta_n(\omega_2)|^2|a_n|^2 r^{2n}.$$

Let us consider the events

$$F = \{\omega_2\colon (\forall n \in \mathbb{N})\ [\zeta_n(\omega_2) \neq 0]\},\ H = \Big\{\omega_2\colon \overline{\lim_{n\to+\infty}} \sqrt[n]{|a_n||\zeta_n(\omega_2)|} = 1\Big\}.$$

Then by Lemma 4 for $\eta_n = |\zeta_n|$ one has $P_2(H) = 1$. Since $\mathbf{E}(\frac{1}{\zeta_n}) < +\infty$, the probability of the event F

$$1 \geq P_2(F) \geq 1 - \sum_{n=0}^{+\infty} P_2\{\omega_2\colon \zeta_n(\omega_2) = 0\} = 1.$$

Denote $G = F \cap H$. So, $P_2(G) = 1$. Then for fixed $\omega_2^0 \in G$

$$P_1(A_f(\omega_2^0)) := P_1\{\omega_1\colon (\exists r_0(\omega_1,\omega_2^0))(\forall r > r_0(\omega_1,\omega_2^0))$$
$$[N_f(r,\omega_1,\omega_2^0) \geq \ln \mathfrak{M}_f(r,\omega_2^0) - (C+1)\ln\ln \mathfrak{M}_f(r,\omega_2^0)]\} = 1.$$

It remains to use Fubini's Theorem

$$P(A_f) = \int_{\Omega_2}\Bigg(\int_{A_f(\omega_2)} dP_1(\omega_1)\Bigg)dP_2(\omega_2) \geq \int_G\Bigg(\int_{A_f(\omega_2)} dP_1(\omega_1)\Bigg)dP_2(\omega_2) =$$
$$= \int_G dP_2(\omega_2) = P_2(G) = 1.$$

□

Theorem 4. *Let $f \in \mathcal{A}$. Then there P_1-almost surely exists $r_0(\omega) > 0$ such that for all $r \in (r_0(\omega); 1)$ we get*
$$p_0(r) \geq s(r) + N(r)\ln N(r) - 4N(r).$$

Proof of Theorem 4. By Jensen's formula we almost surely get
$$0 = \int\limits_0^r \frac{n(t,\omega)}{t}dt = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta - \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|,$$
$$\ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta.$$

Therefore,
$$P\{\omega\colon n(r,\omega)=0\} \leq P\Big\{\omega\colon \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta\Big\}.$$

For $r > r_0(\omega)$ we define
$$A = \Big\{\omega\colon \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \geq$$
$$\geq \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) + \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$G_1(r) = \{\omega\colon \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \geq \ln\gamma(\omega_2)\},$$
$$G_2(r) = \Big\{\omega\colon \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \leq \ln\gamma(\omega_2)\Big\},$$

where $r_0(\omega)$ is chosen from Corollary 1 and $\gamma(\omega_2) > 1$. By this corollary we obtain that $P(A) = 1$.

Then for $r > r_0(\omega)$
$$\overline{G_1}(r)\bigcap\overline{G_2}(r) = \Big\{\omega\colon \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta > \ln\gamma(\omega_2) > \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$\overline{G_1}(r)\bigcap\overline{G_2}(r) \subset \Big\{\omega\colon \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \neq \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\},$$
$$G_1(r)\bigcup G_2(r) = \overline{\overline{G_1}\bigcap\overline{G_2}} \supset \Big\{\omega\colon \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta = \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\}.$$

Hence, for $r > r_0(\omega)$
$$P\{\omega\colon n(r,\omega)=0\} \leq P(G_1\cup G_2) \leq P(G_1) + P(G_2),\ r\to+\infty. \tag{12}$$

Put $\gamma(\omega_2) = C_1 \cdot |a_0| \cdot |\xi_0(\omega_2)|,\ C_1 > 1$. Then we may calculate the probability of the event G_1

$$P(G_1) = P\Big\{\omega\colon\ \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \geq \ln C_1 + \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\} = P\Big\{\omega\colon\ \ln C_1 \leq 0\Big\} = 0$$

and estimate the probability of the event G_2 as $r > r_0(\omega)$

$$\begin{gathered}P(G_2) = P(G_2 \cap A) + P(G_2 \cap \overline{A}) \leq P(G_2 \cap A) + P(\overline{A}) = P(G_2 \cap A) = \\ = P\Bigg\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) + \\ + \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \leq \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta},\omega)|d\theta \leq \ln\gamma(r,\omega)\Bigg\} = \\ = P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) + \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)| \leq \\ \leq \ln C_1 + \ln|a_0\varepsilon_0(\omega_1)\xi_0(\omega_2)|\Big\} = \\ = P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) - (C+1)\ln\ln\mathfrak{M}_f(r,\omega_2) \leq \ln C_1\Big\} \leq \\ \leq P\Big\{\omega\colon\ \ln\mathfrak{M}_f(r,\omega_2) \leq 2\ln C_1\Big\} = P\Big\{\omega\colon \mathfrak{M}_f(r,\omega_2) \leq C_1^2\Big\} = \\ \leq P\Big\{\omega\colon\ \sum_{n\in\mathcal{N}(r)} |\xi_n(\omega_2)|^2|a_n|^2r^{2n} \leq C_1^4\Big\},\ r \uparrow 1. \qquad (13)\end{gathered}$$

The distribution function of the random variable $|\xi_n(\omega_2)|$

$$\begin{gathered}F_{|\xi_n|}(x) = 1 - \exp\{-x^2\},\ F_{|\xi_n|^2}(x) = F_{|\xi_n|}(\sqrt{x}) = 1 - \exp\{-x\}, \\ F_{|\xi_n|^2|a_n|^2r^{2n}}(x) = F_{|\xi_n|^2}\Big(\frac{x}{|a_n|^2r^{2n}}\Big) = 1 - \exp\Big\{-\frac{x}{|a_n|^2r^{2n}}\Big\}\end{gathered}$$

for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}\colon a_n \neq 0$. Then for the random vector

$$\eta(\omega_2) = (|\xi_1(\omega_2)|a_1r^{j_1}, \ldots, |\xi_{j_k}(\omega_2)|a_{j_k}r^{j_k}),\ j_k \in \mathcal{N}(r),$$

the density function is the following

$$p_\eta(x) = \begin{cases} \prod\limits_{n\in\mathcal{N}(r)} \frac{1}{|a_n|^2r^{2n}}\exp\Big\{-\frac{x_n}{|a_n|^2r^{2n}}\Big\}, & x \in \mathbb{R}_+^{\mathcal{N}(r)}; \\ 0, & x \notin \mathbb{R}_+^{\mathcal{N}(r)}. \end{cases}$$

So, for $r > r_0(\omega)$ we obtain

$$\begin{gathered}P\Big\{\omega\colon\ \sum_{n\in\mathcal{N}(r)} |\xi_n(\omega_2)|^2|a_n|^2r^{2n} \leq C_1^4\Big\} = P\{\omega\colon \eta(\omega_2) \in W(r)\} = \\ = \prod_{n\in\mathcal{N}(r)} \frac{1}{|a_n|^2r^{2n}} \cdot \int\cdots\int_{W(r)} \prod_{n\in\mathcal{N}(r)} \exp\Big\{-\frac{x_n}{|a_n|^2r^{2n}}\Big\}dx_1\ldots dx_{N(r)} \leq \\ \leq \exp(-s(r)) \cdot \operatorname{meas}_{N(r)}W(r), \qquad (14)\end{gathered}$$

where

$$W(r)=\Big\{x\in\mathbb{R}_+^{N(r)}\colon \sum_{n\in\mathcal{N}(r)} x_n\le C_1^4\Big\}.$$

For $C>0$ by elementary calculation we get

$$\operatorname{meas}_n\Big\{x\in\mathbb{R}_+^n\colon \sum_{i=1}^n x_i\le C\Big\}=\frac{C^n}{n!}.$$

From this equality and Stirling's formula

$$n!=\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\cdot\exp\Big\{-\frac{\theta_n}{12n}\Big\},\ \theta_n\in[0,1],\ n\in\mathbb{N},$$

it follows that the volume of the set $B(r)$

$$\ln\Big(\operatorname{meas}_{N(r)}W(r)\Big)\le -\frac{1}{2}\ln(2\pi)-\frac{1}{2}\ln N(r)-N(r)\ln N(r)+\frac{1}{12N(r)}+$$
$$+N(r)+4N(r)\ln C_1\le -N(r)(\ln N(r)-1-4\ln C_1).$$

Let us choose $C_1=2$. From (14) we obtain $p_0(r)\ge s(r)+N(r)\ln N(r)-4N(r)$, for $r>r_0(\omega)$. □

From Lemma 3 and Theorems 1 and 4 it follows such a statement.

Theorem 5. *Let $\varepsilon>0$ and $f\in\mathcal{A}$ such that*

$$\alpha=\varliminf_{r\uparrow 1}\frac{\ln N(r)}{\ln\frac{1}{1-r}}>4. \tag{15}$$

Then there P-almost surely exist a nonrandom set E of finite logarithmic measure and $r_0(\omega)>0$ such that for all $r\in(r_0(\omega),1)\setminus E$ we obtain

$$(1-\varepsilon)N(r)\ln N(r)\le p_0(r)-s(r)\le(1+e+\varepsilon)N(r)\ln N(r), \tag{16}$$

in particular,

$$0\le\varliminf_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)},\ \varlimsup_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln s(r)}\le\frac{1}{2-\alpha^{-1}} \tag{17}$$

and

$$\lim_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln N(r)}=1.$$

Proof. It follows from Theorems 1 and 4 inequality (16). Also, from (16) we deduce for $r\in(r_0(\omega);1)\setminus E$

$$\frac{\ln N(r)+\ln\ln N(r)-1}{\ln N(r)}\le\frac{\ln(p_0(r)-s(r))}{\ln N(r)}\le\frac{\ln N(r)+\ln\ln N(r)+2}{\ln N(r)},$$
$$\lim_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln(p_0(r)-s(r))}{\ln N(r)}=1.$$

By Lemma 3, we obtain for arbitrary $\delta > 0$ as $r \uparrow 1$

$$s^{1+\delta}(r) > s(r)\ln^2 s(r) > N^2(r)(1-r), \quad \ln s(r) > \frac{1}{1+\delta}\Big(2\ln N(r) - \ln\frac{1}{1-r}\Big),$$

$$\varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} = \varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln N(r)}\cdot\frac{\ln N(r)}{\ln s(r)} =$$

$$= \varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln N(r)}{\ln s(r)} \le (1+\delta)\varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln N(r)}{2\ln N(r) - \ln\frac{1}{1-r}} = (1+\delta)\varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{1}{2-\frac{\ln\frac{1}{1-r}}{\ln N(r)}} =$$

$$= (1+\delta)\frac{1}{2-\varliminf\limits_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln\frac{1}{1-r}}{\ln N(r)}} = (1+\delta)\frac{1}{2-\Bigg(\varlimsup\limits_{\substack{r\uparrow 1\\ r\notin E}}\frac{\ln N(r)}{\ln\frac{1}{1-r}}\Bigg)^{-1}} = (1+\delta)\frac{1}{2-\alpha^{-1}}.$$

Since δ is arbitrary then

$$\varlimsup_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} \le \frac{1}{2-\alpha^{-1}}.$$

From $N(r) > 1$ and $s(r) > 1$, $r \uparrow 1$, it follows that

$$\varliminf_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} = \varliminf_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln N(r)}\cdot\frac{\ln N(r)}{\ln s(r)} = \varliminf_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln N(r)}{\ln s(r)} \ge 0.$$

□

5. Examples on the sharpness of inequalities (17).

Theorem 6. *There exist random analytic function $f \in \mathcal{A}$: $\alpha = +\infty$, a set E of finite logarithmic measure such that*

$$\lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} = \frac{1}{2}.$$

Proof. Denote

$$h(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad h(z,\omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n, \tag{18}$$

where $a_n = \exp\{\frac{n+5}{\ln\ln(n+5)}\}$, $n \in \mathbb{Z}_+$. Then

$$M_h(r) = \sum_{n=0}^{+\infty} a_n r^n = \sum_{n=0}^{+\infty} \exp\Big\{\frac{n+5}{\ln\ln(n+5)} - n\ln\frac{1}{r}\Big\}.$$

Let us consider the function $g(x) = \frac{x+5}{\ln\ln(x+5)} - x\ln\frac{1}{r}$, $x \ge 0$. Then

$$g'(x) = \frac{\ln\ln(x+5) - \frac{1}{\ln(x+5)}}{\ln^2\ln(x+5)} - \ln\frac{1}{r} =$$

$$= \frac{1}{\ln\ln(x+5)}\Big(1 - \frac{1}{\ln(x+5)\ln\ln(x+5)}\Big) - \ln\frac{1}{r} = 0.$$

Let $x_{\max}(r)$ be a point of maximum of the function $g(x)$. Then $\lim\limits_{r\uparrow 1} x_{\max}(r) = +\infty$ and for $r \uparrow 1$ we get

$$\frac{1}{\ln\ln(x_{\max}(r)+5)} < 2\ln\frac{1}{r}, \quad x_{\max}(r) > \exp_2\Big\{\frac{1}{2\ln\frac{1}{r}}\Big\} - 5,$$
$$\nu_h(r) > \exp_2\Big\{\frac{1}{2\ln\frac{1}{r}}\Big\} - 5 > \exp_2\Big\{\frac{1}{3(1-r)}\Big\}, \ r\uparrow 1. \tag{19}$$

Therefore,

$$\exp\{g(x_{\max})\} \geq \mu_h(r) \geq \exp\Big\{g\Big(\exp_2\Big\{\frac{1}{2\ln\frac{1}{r}}\Big\} - 5\Big)\Big\} \geq$$
$$\geq \exp\Big\{\exp_2\Big\{\frac{1}{2\ln\frac{1}{r}}\Big\}\Big(2\ln\frac{1}{r} - \ln\frac{1}{r}\Big)\Big\} = \exp\Big\{\exp_2\Big\{\frac{1}{2\ln\frac{1}{r}}\Big\}\ln\frac{1}{r}\Big\} > \exp_3\Big\{\frac{1}{3}\frac{1}{1-r}\Big\},$$
$$\ln s(r) > \ln\ln\mu_h(r) > \exp\Big\{\frac{1}{3}\frac{1}{1-r}\Big\},$$

and

$$\alpha = \varliminf_{r\uparrow 1}\frac{\ln N(r)}{\ln\frac{1}{1-r}} \geq \varliminf_{r\uparrow 1}\frac{\ln\nu_h(r)}{\ln\frac{1}{1-r}} \geq \varliminf_{r\uparrow 1}\frac{\exp\{\frac{1}{3(1-r)}\}}{\ln\frac{1}{1-r}} = +\infty. \tag{20}$$

By Lemma 3 there exists a set $E \subset [0;1)$ of finite logarithmic measure such that for all $r \in [0;1)\backslash E$ we have

$$N(r) < \frac{2}{\sqrt{1-r}}\sqrt{s(r)}\ln s(r) \leq \sqrt{s(r)}\ln^2 s(r). \tag{21}$$

From

$$\ln\mu_h(r) - \ln\mu_h(r_0) = \int\limits_{r_0}^{r}\frac{\nu_h(t)dt}{t} \leq \nu_h(r)(\ln r - \ln r_0),$$

it follows that for any $r > r_2 > r_0$ there exists a constant $c > 0$ such that

$$\nu_h(r) \geq \frac{\ln\mu_h(r) - \ln\mu_h(r_0)}{\ln r - \ln r_0} \geq \frac{c\ln\mu_h(r)}{-\ln r_0}.$$

Remark that the sequence $a_n = \exp\{\frac{n+5}{\ln\ln(n+5)}\}$, $n \in \mathbb{Z}_+$, is log-concave. Then for any $r \in [0;1)$: $N(r) > \nu_h(r)$. So,

$$s(r) < (N(r)+1)\ln\mu_h(r) < 2(N(r)+1)\frac{1}{c}\ln\frac{1}{r_0}\nu_h(r) \leq$$
$$\leq \frac{1}{c}\ln\frac{1}{r_0}(N(r)+1)N(r) < 4N^2(r), \ r\uparrow 1. \tag{22}$$

From (20) it follows that the function $h(z,\omega)$ satisfies the conditions of Theorem 5 ($\alpha = +\infty$). Taking into account (21) and (22) we deduce for $r \in [0;1)\backslash E$

$$2\sqrt{s(r)} < N(r) < \sqrt{s(r)}\ln^2 s(r),$$

$$\lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} = \lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln N(r)} \cdot \frac{\ln N(r)}{\ln s(r)} = \frac{1}{2}.$$

□

Theorem 7. *There exist random analytic function $f \in \mathcal{A}$: $\alpha = +\infty$, a set E of zero density, that is (here* meas *is Lebesgue's measure on the line)*

$$DE = \overline{\lim_{r\to 1-0}} \frac{1}{1-r} \operatorname{meas}(E \cap [r,1)) = 0,$$

such that

$$\lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r)-s(r))}{\ln s(r)} = 0.$$

Proof. Let us consider random analytic functions

$$h(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad g(z) = \sum_{n\in\mathcal{N}^*} a_n z^n,$$

where $a_n = \exp\{\frac{n+5}{\ln\ln(n+5)}\}$, $\mathcal{N}^* = \{n\colon n = [e^k]+1 \text{ for some } k \in \mathbb{Z}_+\}$. Here $[e^k]$ means the integer part of the real number e^k. We denote

$$\mathcal{N}_h(r) = \{n \in \mathbb{Z}_+\colon \ \ln(|a_n|r^n) > 0\}\backslash\{0\}, \ \mathcal{N}_g(r) = \{n \in \mathcal{N}^*\colon \ \ln(|a_n|r^n) > 0\},$$
$$s_h(r) = 2\sum_{n\in\mathcal{N}_h(r)} \ln(|a_n|r^n), \ s_g(r) = 2\sum_{n\in\mathcal{N}_g(r)} \ln(|a_n|r^n).$$

Remark that the sequence $a_n = \exp\{\frac{n+5}{\ln\ln(n+5)}\}$, $n \in \mathbb{Z}_+$, is log-concave and $\mathcal{N}_h(r) = \{0,\dots,N_h(r)-1\}$. Then by the definition of $N_g(r)$ we get $N_g(r) \le 2\ln N_h(r)$, $r \uparrow 1$. Since $h(z)$ satisfies condition (4), then ([27]) there exists a set E of zero density such that for $r \in [0;1)\backslash E$ we have

$$N_g(r) \le 2\ln N_h(r) \le 2\ln(\ln\mu_h(r)\ln^5(\ln\mu_h(r))) < 4\ln\ln\mu_h(r).$$

On the other hand

$$N_g(r) \ge \frac{1}{2}\ln N_h(r) \ge \frac{1}{2}\ln\nu_h(r) \ge \frac{1}{2}\ln\Big(\frac{c\ln\mu_h(r)}{-\ln r_0}\Big) \ge \frac{1}{3}\ln\ln\mu_h(r) \ge$$
$$\ge \frac{1}{3}\exp\Big\{\frac{1}{3}\frac{1}{1-r}\Big\} \ge \exp\Big\{\frac{1}{4}\frac{1}{1-r}\Big\}, \quad r\uparrow 1.$$

Therefore,

$$\alpha = \lim_{r\uparrow 1} \frac{\ln N(r)}{\ln\frac{1}{1-r}} \ge \frac{1}{4}\lim_{r\uparrow 1}\frac{\frac{1}{1-r}}{\ln\frac{1}{1-r}} = +\infty, \quad r\uparrow 1.$$

Remark that

$$\min\{n \in \mathcal{N}^*\colon n > \nu_g(r)\} \le [e\nu_g(r)] + 1 < (e+1)\ln\nu_g(r).$$

Fix $r > 0$. Let us consider the function

$$y(t) = \ln(a(t)r^t) = \frac{t+5}{\ln\ln(t+5)} - t\ln\frac{1}{r}.$$

The graph of the function $y(t)$ passes through the points $(\frac{5}{\ln\ln 5}; 0)$ and $(\nu_g(r), \ln\mu_g(r))$. It follows from log-concavity of the function $y(t)$ that the point $(\nu_h(r), \ln\mu_h(r))$ belongs to the triangle with the vertices $(\nu_g(r), \ln\mu_g(r))$, $((e+1)\nu_g(r), \ln\mu_g(r))$ and $((e+1)\nu_g(r), (e+1)\ln\mu_g(r))$. Then

$$\ln\mu_h(r) < (e+1)\ln\mu_g(r) < 4\ln\mu_g(r), \quad s_g(r) \geq 2\ln\mu_g(r) \geq \frac{\ln\mu_h(r)}{2}, \quad r\uparrow 1.$$

For the function $g(z)$ and $r \in [0;1)\backslash E$ we get

$$0 \leq \lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} = \lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln N_g(r)}{\ln s_g(r)} \leq \lim_{\substack{r\uparrow 1\\ r\notin E}} \frac{\ln(4\ln\ln\mu_h(r))}{\ln(\frac{\ln\mu_h(r)}{2})} = 0.$$

□

Remark that for $f \in \mathcal{A}$ and $r > r_2 > r_1$ there exist $C_1 > 0$ such that

$$\ln\mu_f(r) - \ln\mu(r_1) = \int_{r_1}^{r} \frac{\nu_f(t)dt}{t} \leq \nu_f(r)(\ln r - \ln r_1),$$

$$\nu_f(r) \geq \frac{\ln\mu_f(r) - \ln\mu(r_1)}{\ln r - \ln r_1} \geq \frac{C_1\ln\mu_f(r)}{-\ln r_1}.$$

If we additionally suppose that a sequence (a_n) is log-concave then

$$N(r) > \nu_f(r) > \frac{C_1\ln\mu_f(r)}{-\ln r_1}, \quad r\uparrow 1.$$

Therefore, if a sequence (a_n) is log-concave then condition (15) in Theorem 5 can be replaced by

$$\varliminf_{r\uparrow 1} \frac{\ln\ln\mu_f(r)}{\ln\frac{1}{1-r}} > 4.$$

The last fact leads us to the following conjecture.

Conjecture 1. *If a sequence* (a_n) *is log-concave then condition* (15) *in Theorem 5 can be replaced by the condition*

$$\varliminf_{r\uparrow 1} \frac{\ln\ln\mu_f(r)}{\ln\frac{1}{1-r}} > 1.$$

Conjecture 2. *Condition* (15) *in Theorem 5 can by replaced by the condition* $\alpha > 1$.

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