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ASYMPTOTIC SOLUTIONS OF SINGULARLY PERTURBED LINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH PERIODIC COEFFICIENTS

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The paper deals with the problem of constructing asymptotic solutions for singular perturbed linear differential-algebraic equations with periodic coefficients. The case of multiple roots of a characteristic equation is studied. It is assumed that the limit pencil of matrices of the system has one eigenvalue of multiplicity n, which corresponds to two finite elementary divisors and two infinite elementary divisors whose multiplicity is greater than 1.

A technique for finding the asymptotic solutions is developed and n formal linearly independent solutions are constructed for the corresponding differential-algebraic system. The developed algorithm for constructing formal solutions of the system is a nontrivial generalization of the corresponding algorithm for constructing asymptotic solutions of a singularly perturbed system of differential equations in normal form, which was used in the case of simple roots of the characteristic equation.

The modification of the algorithm is based on the equalization method in a special way the coefficients at powers of a small parameter in algebraic systems of equations, from which the coefficients of the formal expansions of the searched solution are found. Asymptotic estimates for the terms of these expansions with respect to a small parameter are also given.

For an inhomogeneous differential-algebraic system of equations with periodic coefficients, existence and uniqueness theorems for a periodic solution satisfying some asymptotic estimate are proved, and an algorithm for constructing the corresponding formal solutions of the system is developed. Both critical and non-critical cases are considered.

1. Introduction. The systematic study of differential-algebraic equations (DAEs)

$$A(t)\frac{dx}{dt} = B(t)x + f(t),\ t \in [0;T], \tag{1}$$

was begun in the second half of the past century. In particular, Campbell has proposed the notion of a standard canonical form of system (1) that was represented as

$$\begin{pmatrix} I_{n-s} & 0 \\ 0 & N_s(t) \end{pmatrix} \frac{dx}{dt} = \begin{pmatrix} M(t) & 0 \\ 0 & I_s \end{pmatrix} x + h(t), \tag{2}$$

where I_s and I_{n-s} are identity matrices of orders s and $n-s$, respectively, and $N_s(t)$ is a nilpotent lower (or upper) triangular matrix [4]. Note that, when the matrix $N_s(t)$ is

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additionally constant, system (2) is called the strong standard canonical form of system (1) [10, 24].

Later Campbell and Petzold have found sufficient conditions of reduction of system (1) to its Kronecker's form [6, 24]. It allows to find the general solution of system (1) and to study then Cauchy problem, boundary value problems, and others [6, 5, 1, 18, 19].

Another way to deal with DAEs is to decouple them by means of canonical projectors. Using a concept of the tractability index, the numerical methods for solving differential algebraic system were developed in papers by Gear and Petzold [10], Griepentrog and März [11], Brenan, Campbell and Petzold [2], J. Pade and C. Tischendorf [23], A. Dick, O. Koch, R. März and E. Weinmuller [7].

Effective methods of transforming a special class of index-1 tractable DAE to a standard canonical form are presented by Boyarintsev [1], Samoilenko, Shkil' and Yakovets [25]. Also it was assumed that the matrix $A(t)$ had a constant rank on the interval $[0;T]$.

Based on the notion of index, Lamour, Marz and Winkler [20, 21] have extended the Floquet theory for DAEs with periodic coefficients. They have defined the monodromy matrix for the differential algebraic system, studied the stability of the corresponding homogeneous system, found the sufficient conditions for the existence of a unique periodic solution of system (1).

One of the efficient methods of integration of DAEs is the perturbation method [22, 13, 14, 1] according to which in a perturbed system

$$(A(t) + \varepsilon A_1(t))\frac{dx}{dt} = (B(t) + \varepsilon B_1(t))x + f(t), \tag{3}$$

where ε is a small parameter, matrices $A_1(t)$ and $B_1(t)$ are chosen in such a way that the index of system (3) should be smaller than that of initial system (1). Then under certain conditions, the solutions of system (3) converge to the corresponding solutions of system (1), as $\varepsilon \to 0$ [30, 31].

Although the systems of linear differential equations (3) are studied since the 70s of the last century there are many questions of theoretical and practical interest unanswered. The exception is only the systems with constant coefficients for which on the condition $\det(A_0 + \varepsilon A_1) \neq 0$ the fundamental matrix solutions can be represented by convergent power series in ε [3].

Consider a more general system than (3), namely,

$$\varepsilon A(t,\varepsilon)\frac{dx}{dt} = B(t,\varepsilon)x + f(t,\varepsilon),\ t \in [0;T], \tag{4}$$

where $A(t,\varepsilon)$, $B(t,\varepsilon)$ are $n\times n$–matrices, $f(t,\varepsilon)$ is an n-dimensional vector possessing uniform asymptotic expansions of the following form

$$A(t,\varepsilon) = \sum\nolimits_{k\geq 0} \varepsilon^k A_k(t),\ B(t,\varepsilon) = \sum\nolimits_{k\geq 0} \varepsilon^k B_k(t),\ f(t,\varepsilon) = \sum\nolimits_{k\geq 0} \varepsilon^k f_k(t)$$

with real or complex-valued infinitely differentiable coefficients. The solutions of such singular perturbed DAEs have some specific features in comparison with the solutions of system (1). Samoilenko, Shkil' and Yakovets have shown that under certain conditions for perturbed matrices a homogeneous system

$$\varepsilon A(t,\varepsilon)\frac{dx}{dt} = B(t,\varepsilon)x \tag{5}$$

has two types of linear independent formal solutions corresponding to finite or infinite elementary divisors of a pencil $B_0(t) - \lambda A_0(t)$ [25]. Moreover, their linear combination is a formal general solution of system (5). It should be noted that in the case of multiple divisors of a pencil $B_0(t) - \lambda A_0(t)$ asymptotic expansions of solutions of system (5) can be constructed in some fractional powers of small parameter ε, where values of powers of ε depends on the multiplicity of elementary divisors of the pencil $B_0(t) - \lambda A_0(t)$, as well as on perturbed coefficients of system (5).

The obtained results have been used to find solutions of singularly perturbed periodic differential algebraic systems [28, 32].

This paper deals with the DAEs (4) with periodic coefficients in case of multiple spectrum of the main pencil $B_0(t) - \lambda A_0(t)$. We have found the conditions under which system (4) has the only periodic solution with prescribed asymptotic expansion.

It is well known [25], that in the case of multiple eigenvalues of $B_0(t) - \lambda A_0(t)$ the technique of constructing asymptotic expansions is tedious and quite complicated. That is why we propose in present paper a modified approach for finding formal asymptotic solutions to system (5). Namely, our technique is based on transformation of the system with multiple spectrum of the main pencil of matrices into system whose main pencil of matrices has a simple spectrum [28]. On our opinion, the proposed approach is more rational one than others.

2. Homogeneous DAEs. We assume that the following conditions are satisfied:

1. The matrices $A(t, \varepsilon)$ and $B(t, \varepsilon)$ are periodic in t of period T.
2. The pencil of matrices $B_0(t) - \lambda A_0(t)$ is regular for all $t \in [0; T]$.
3. The pencil $B_0(t) - \lambda A_0(t)$ has one eigenvalue $\lambda_0(t)$, two finite elementary divisors $(\lambda - \lambda_0(t))^{p_1}$, $(\lambda - \lambda_0(t))^{p_2}$, $2 \le p_1 < p_2$, and two infinite elementary divisors of multiplicity q_1 and q_2, $2 \le q_1 < q_2$; furthermore $p_1 + p_2 + q_1 + q_2 = n$.

Then there exist periodic nonsingular sufficiently smooth matrices $P(t, \varepsilon)$, $Q(t, \varepsilon)$ such that

$$P(t,\varepsilon)A(t,\varepsilon)Q(t,\varepsilon) = E(t,\varepsilon) \equiv \operatorname{diag}\{N_q(t,\varepsilon), I_p(t,\varepsilon)\},$$
$$P(t,\varepsilon)B(t,\varepsilon)Q(t,\varepsilon) = \Omega(t,\varepsilon) \equiv \operatorname{diag}\{I_q(t,\varepsilon), W_p(t,\varepsilon)\},$$

where $I_q(t, 0) = I_q$, $I_p(t, 0) = I_p$, I_q and I_p are the identity matrices of orders q ($q = q_1 + q_2$) and p ($p = p_1 + p_2$), respectively;

$$N_q(t, 0) = N_q \equiv \operatorname{diag}\{N_{q_1}, N_{q_2}\},\ W_p(t, 0) = \operatorname{diag}\{W_{p_1}(t), W_{p_2}(t)\},\ p = p_1 + p_2,$$

N_{q_i} is the square matrix of order q_i such that

$$N_{q_i} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ i = 1, 2,$$

and $W_{p_i}(t) = \lambda_0(t) I_{p_i} + N_{p_i}$, $i = 1, 2$ [29, 26].

We set $x(t, \varepsilon) = Q(t, \varepsilon) y(t, \varepsilon)$. Then system (5) can be written in the form

$$\varepsilon E(t,\varepsilon)\frac{dy}{dt} = (\Omega(t,\varepsilon) - \varepsilon E(t,\varepsilon)Q^{-1}(t,\varepsilon)Q'(t,\varepsilon))y, \tag{6}$$

where $Q'(t, 0) \equiv 0$, $t \in [0; T]$.

The transformation $y(t,\varepsilon)=\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_0(t)dt\right)z(t,\varepsilon)$ changes system (6) into

$$\varepsilon E(t,\varepsilon)\frac{dz}{dt}=(\Omega(t,\varepsilon)-\lambda_0(t)E(t,\varepsilon)-\varepsilon E(t,\varepsilon)Q^{-1}(t,\varepsilon)Q'(t,\varepsilon))z. \tag{7}$$

It should be noted that the pencils $\Omega(t,0)-\lambda_0(t)E(t,0)-\lambda E(t,0)$ and $\operatorname{diag}\{I_q,N_p\}-\lambda\operatorname{diag}\{N_q,I_p\}$ have the same Kronecker normal form [9]. Then there exist periodic nonsingular quasi-diagonal sufficiently smooth matrices $P(t)$, $Q(t)$ such that

$$P(t)E(t,0)Q(t)=\operatorname{diag}\{N_q,I_p\},\quad P(t)(\Omega(t,0)-\lambda_0(t)E(t,0))Q(t)=\operatorname{diag}\{I_q,N_p\}.$$

We set $z(t,\varepsilon)=Q(t)u(t,\varepsilon)$. Then system (7) can be written in the form

$$\varepsilon H(t,\varepsilon)\frac{du}{dt}=C(t,\varepsilon)u, \tag{8}$$

where $H(t,\varepsilon)=P(t)E(t,\varepsilon)Q(t)$,

$$C(t,\varepsilon)=P(t)(\Omega(t,\varepsilon)-\lambda_0(t)E(t,\varepsilon)-\varepsilon E(t,\varepsilon)Q^{-1}(t,\varepsilon)Q'(t,\varepsilon))Q(t)-\varepsilon P(t)E(t,\varepsilon)Q'(t).$$

Let us define the matrices

$$H(t,\varepsilon)=\sum_{k\geq 0}\varepsilon^kH_k(t),\ C(t,\varepsilon)=\sum_{k\geq 0}\varepsilon^kC_k(t),$$

where $H_0(t)\equiv H_0=\operatorname{diag}\{N_q,I_p\}$, $C_0(t)\equiv C_0=\operatorname{diag}\{I_q,N_p\}$, $H_1(t)=\operatorname{diag}\{H_{1q}(t),H_{1p}(t)\}$, $C_1(t)=\operatorname{diag}\{C_{1q}(t),C_{1p}(t)\}$; here, $C_{1q}(t)$, $H_{1q}(t)$ are square matrices of order q.

We seek a formal solution of system (8) in the following form

$$u_i(t,\varepsilon)=v_i(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_i(t,\varepsilon)dt\right),\ i=\overline{1,n}, \tag{9}$$

where $v_i(t,\varepsilon)$ are n-dimensional vectors, $\lambda_i(t,\varepsilon)$ are scalar functions, and [25, 27]

$$v_i(t,\varepsilon)=\sum_{k\geq 0}\varepsilon^k\widetilde{v}_i^{(k)}(t,\varepsilon),\ \lambda_i(t,\varepsilon)=\sum_{k\geq 0}\varepsilon^k\widetilde{\lambda}_i^{(k)}(t,\varepsilon),\ i=\overline{1,n}. \tag{10}$$

Substituting representation (9) into system (8), we get

$$(C_0+\varepsilon C_1(t))v_i(t,\varepsilon)-(H_0+\varepsilon H_1(t))v_i(t,\varepsilon)\lambda_i(t,\varepsilon)=\varepsilon H(t,\varepsilon)v_i'(t,\varepsilon)- \\ -\sum_{k\geq 2}\varepsilon^kC_k(t)v_i(t,\varepsilon)+\sum_{k\geq 2}\varepsilon^kH_k(t)v_i(t,\varepsilon)\lambda_i(t,\varepsilon). \tag{11}$$

Let $K(t,\varepsilon)$ be defined by $K(t,\varepsilon)=\operatorname{diag}\{I_q+\varepsilon C_{1q}(t),I_p+\varepsilon H_{1p}(t)\}$. Then

$$K^{-1}(t,\varepsilon)=\sum_{k\geq 0}\varepsilon^kM_k(t)\equiv\operatorname{diag}\{I_q+\sum_{k\geq 1}\varepsilon^kM_{kq}(t),I_p+\sum_{k\geq 1}\varepsilon^kM_{kp}(t)\}$$

and $M_1(t)=\operatorname{diag}\{-C_{1q}(t),-H_{1p}(t)\}$.

Multiplying both sides of relation (11) on the left by $K^{-1}(t,\varepsilon)$, we obtain

$$(D_0+\varepsilon D_1(t))v_i(t,\varepsilon)-(F_0+\varepsilon F_1(t))v_i(t,\varepsilon)\lambda_i(t,\varepsilon)=\varepsilon\sum_{k\geq 0}\varepsilon^kF_k(t)v_i'(t,\varepsilon)- \\ -\sum_{k\geq 2}\varepsilon^kD_k(t)v_i(t,\varepsilon)+\sum_{k\geq 2}\varepsilon^kF_k(t)v_i(t,\varepsilon)\lambda_i(t,\varepsilon), \tag{12}$$

where $D_0 = C_0, F_0 = H_0,\ \ D_1(t) = \mathrm{diag}\{0, D_{1p}(t)\} \equiv \mathrm{diag}\{0, C_{1p}(t) - H_{1p}(t)N_p\}$,

$$F_1(t) = \mathrm{diag}\{F_{1q}(t), 0\} \equiv \mathrm{diag}\{H_{1q}(t) - C_{1q}(t)N_q, 0\},$$

$$D_2(t) = C_2(t) + \mathrm{diag}\{0, M_{2p}(t)N_p - H_{1p}(t)C_{1p}(t)\},$$

$$D_k(t) = \sum_{i=0}^{k-2} M_i(t)C_{k-i}(t) + \mathrm{diag}\{0, M_{kp}(t)N_p + M_{k-1,p}C_{1p}(t)\},\ k \geq 3,$$

$$F_2(t) = H_2(t) + \mathrm{diag}\{M_{2q}(t)N_q - C_{1q}(t)H_{1q}(t), 0\},$$

$$F_k(t) = \sum_{i=0}^{k-2} M_i(t)H_{k-i}(t) + \mathrm{diag}\{M_{kq}(t)N_q + M_{k-1,q}H_{1q}(t), 0\},\ k \geq 3.$$

We modify the standard procedure for deriving the functions $\widetilde{v}_i^{(s)}(t,\varepsilon)$ and $\widetilde{\lambda}_i^{(s)}(t,\varepsilon)$. We do it in the following way: if we compare the coefficients belonging to ε^s, we take in the left-hand side of (12) also the higher order terms $D_1(t)\widetilde{v}_i^{(s)}(t,\varepsilon)\varepsilon^{s+1}$ and $\widetilde{\lambda}_i^{(0)}(t,\varepsilon)F_1(t)\widetilde{v}_i^{(s)}(t,\varepsilon)\varepsilon^{s+1}$. Of course, these terms will then be neglected in deriving the functions $\widetilde{v}_i^{(s+1)}(t,\varepsilon)$. Thus, we have

$$(D_0 + \varepsilon D_1(t) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)(F_0 + \varepsilon F_1(t)))\widetilde{v}_i^{(0)}(t,\varepsilon) = 0, \tag{13}$$

$$(D_0 + \varepsilon D_1(t) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)(F_0 + \varepsilon F_1(t)))\widetilde{v}_i^{(s)}(t,\varepsilon) = d_i^{(s)}(t,\varepsilon),\ s \in N, \tag{14}$$

where

$$d_i^{(s)}(t,\varepsilon) = \sum_{k=0}^{s} F_k(t)(\widetilde{v}_i^{(s-k-1)}(t,\varepsilon))' - \sum_{k=2}^{s} D_k(t)\widetilde{v}_i^{(s-k)}(t,\varepsilon) +$$
$$+ \sum_{k=2}^{s}\sum_{j=0}^{k} F_k(t)\widetilde{v}_i^{(j)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k-j)}(t,\varepsilon) + (F_0 + \varepsilon F_1(t))\sum_{k=0}^{s-1}\widetilde{v}_i^{(k)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k)}(t,\varepsilon).$$

We write equation (13) as

$$(I_q - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)(N_q + \varepsilon F_{1q}(t)))\widetilde{v}_{i1}^{(0)}(t,\varepsilon) = 0, \tag{15}$$

$$(N_p + \varepsilon D_{1p}(t) - \widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)\widetilde{v}_{i2}^{(0)}(t,\varepsilon) = 0, \tag{16}$$

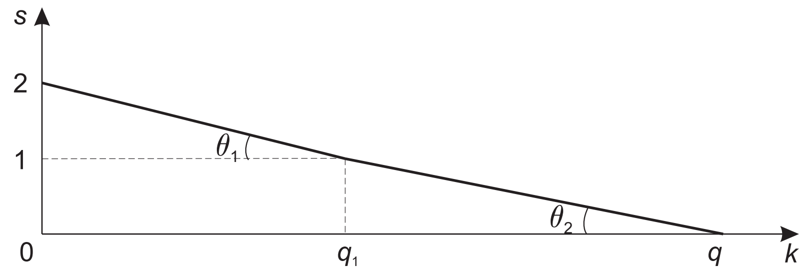
where $\widetilde{v}_i^{(0)}(t,\varepsilon) = \begin{pmatrix} \widetilde{v}_{i1}^{(0)}(t,\varepsilon) \\ \widetilde{v}_{i2}^{(0)}(t,\varepsilon) \end{pmatrix}$.

Consider equation (15). The characteristic equation of $N_q + \varepsilon F_{1q}(t)$ has the form

$$w^q + \beta_1(t,\varepsilon)w^{q-1} + ... + \beta_{q-1}(t,\varepsilon)w + \beta_q(t,\varepsilon) = 0, \tag{17}$$

where $\beta_j(t,\varepsilon) = O(\varepsilon),\ \varepsilon \to 0+,\ j = \overline{1, q_2 - 1},\ \beta_{q_2}(t,\varepsilon) = -h_{q,q_1+1}^{(1)}(t)\varepsilon + O(\varepsilon^2),\ \varepsilon \to 0+,$ $\beta_j(t,\varepsilon) = O(\varepsilon^2),\ \varepsilon \to 0+,\ i = \overline{q_2 + 1, q - 1},\ \beta_q(t,\varepsilon) = h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^2 + O(\varepsilon^3),\ \varepsilon \to 0+,$ for all $t \in [0;T]$. Here, $h_{ij}^{(1)}(t)$ is the element of the matrix $H_1(t)$. Let the following condition be satisfied.

4. $h_{q_1,1}^{(1)}(t) \neq 0,\ h_{q,q_1+1}^{(1)}(t) \neq 0,\ h(t) \neq 0,\ t \in [0;T]$, where
$$h(t) = h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t) - h_{q_1,q_1+1}^{(1)}(t)h_{q1}^{(1)}(t).$$



We construct the characteristic polygon for the equation (17) [15, 25]. Its vertices are the points $(q;0)$, $(q_1;1)$, and $(0;2)$.

Since $\operatorname{tg}\theta_1 = \frac{1}{q_1}$, $\operatorname{tg}\theta_2 = \frac{1}{q_2}$, the solutions of equation (17) have the following form

$$w_j(t,\varepsilon) = \sqrt[q_1]{h^{(1)}_{q_1,1}(t)}\varepsilon^{\frac{1}{q_1}} + O(\varepsilon^{\frac{2}{q_1}}),\ j=\overline{1,q_1},$$
$$w_j(t,\varepsilon) = \sqrt[q_2]{h^{(1)}_{q,q_1+1}(t)}\varepsilon^{\frac{1}{q_2}} + O(\varepsilon^{\frac{2}{q_2}}),\ j=\overline{q_1+1,q}.$$

Let $\varphi_j(t,\varepsilon)$, $j=\overline{1,q}$, be the columns of the matrix $T_q(t,\varepsilon)$, where

$$T_q^{-1}(t,\varepsilon)(N_q+\varepsilon F_{1q}(t))T_q(t,\varepsilon) = \widetilde{W}_q(t,\varepsilon) \equiv \operatorname{diag}\{w_1(t,\varepsilon), w_2(t,\varepsilon), ..., w_q(t,\varepsilon)\}.$$

Then

$$(N_q+\varepsilon F_{1q}(t) - w_j(t,\varepsilon)I_q)\varphi_j(t,\varepsilon) = 0,\ j=\overline{1,q}.$$

The components of the vectors $\varphi_j(t,\varepsilon)$, $j=\overline{1,q_1}$, and $\varphi_j(t,\varepsilon)$, $j=\overline{q_1+1,q}$ can be taken as the cofactors of the first and (q_1+1)th rows of the matrix $N_q+\varepsilon F_{1q}(t)-w_j(t,\varepsilon)I_q$, respectively. Hence it follows that

$$\varphi_j(t,\varepsilon) = \begin{pmatrix} a_j^{q_1-1}(t)h^{(1)}_{q,q_1+1}(t) + O(\varepsilon^{\frac{1}{q_1}}) \\ h(t)\varepsilon^{\frac{1}{q_1}} + O(\varepsilon^{\frac{2}{q_1}}) \\ a_j(t)h(t)\varepsilon^{\frac{2}{q_1}} + O(\varepsilon^{\frac{3}{q_1}}) \\ \dots \\ a_j^{q_1-2}(t)h(t)\varepsilon^{\frac{q_1-1}{q_1}} + O(\varepsilon) \\ -a_j^{q_1-1}(t)h^{(1)}_{q1}(t) + O(\varepsilon^{\frac{1}{q_1}}) \\ -a_j^{q_1}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{1}{q_1}} + O(\varepsilon^{\frac{2}{q_1}}) \\ \dots \\ -a_j^{2(q_1-1)}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{q_1-1}{q_1}} + O(\varepsilon) \\ O(\varepsilon) \\ \dots \\ O(\varepsilon) \end{pmatrix},\ j=\overline{1,q_1},$$

$$\varphi_j(t,\varepsilon) = \begin{pmatrix} b_j^{q_2-1}(t)h^{(1)}_{q_1,q_1+1}(t)\varepsilon^{\frac{q_2-q_1}{q_2}} + O(\varepsilon^{\frac{q_2-q_1+1}{q_2}}) \\ b_j^{q_2}(t)h^{(1)}_{q_1,q_1+1}(t)\varepsilon^{\frac{q_2-q_1+1}{q_2}} + O(\varepsilon^{\frac{q_2-q_1+2}{q_2}}) \\ \dots \\ b_j^{q_2+q_1-2}(t)h^{(1)}_{q_1,q_1+1}(t)\varepsilon^{\frac{q_2-1}{q_2}} + O(\varepsilon) \\ b_j^{q_1+q_2-1}(t) + O(\varepsilon^{\frac{1}{q_2}}) \\ b_j^{q_1}(t)h^{(1)}_{q,q_1+1}(t)\varepsilon^{\frac{1}{q_2}} + O(\varepsilon^{\frac{2}{q_2}}) \\ \dots \\ b_j^{q_1+q_2-2}(t)h^{(1)}_{q,q_1+1}(t)\varepsilon^{\frac{q_2-1}{q_2}} + O(\varepsilon) \end{pmatrix},\ j=\overline{q_1+1,q},$$

where

$$a_j(t) = \sqrt[q_1]{|a_j(t)|}\left(\cos\frac{\arg a_j(t)+2\pi j}{q_1} + i\sin\frac{\arg a_j(t)+2\pi j}{q_1}\right), j=\overline{1,q_1},$$
$$b_j(t) = \sqrt[q_2]{|b_j(t)|}\left(\cos\frac{\arg b_j(t)+2\pi j}{q_2} + i\sin\frac{\arg b_j(t)+2\pi j}{q_2}\right), j=\overline{1,q_2}.$$

It is easy to see that the matrix $T_q(t,\varepsilon)$ is nonsingular. Indeed, let us define the matrix

$$T_q(t,\varepsilon) = \begin{pmatrix} T_1(t,\varepsilon) & T_2(t,\varepsilon) \\ T_3(t,\varepsilon) & T_4(t,\varepsilon)\end{pmatrix},$$

where $T_1(t,\varepsilon)$ is a square matrix of order q_1. Then

$$\det\, T_q(t,\varepsilon) = \det\, T_1(t,\varepsilon)\, \det(T_4(t,\varepsilon) - T_3(t,\varepsilon)T_1^{-1}(t,\varepsilon)T_2(t,\varepsilon)) =$$
$$= \det V_1(t)\, \det V_2(t)(h^{(1)}_{q,q_1+1}(t))^{q_2}h^{q_1-1}(t)\prod_{i=1}^{q_2} b^{q_1}_{q_1+i}(t)\varepsilon^{\frac{q-2}{2}} + O(\varepsilon^{\frac{q-2}{2}+\gamma}),\ \gamma>0,$$

where

$$V_1(t) = \begin{pmatrix} a_1^{q_1-1}(t) & a_2^{q_1-1}(t) & \dots & a_{q_1}^{q_1-1}(t) \\ 1 & 1 & \dots & 1 \\ a_1(t) & a_2(t) & \dots & a_{q_1}(t) \\ \dots & \dots & \dots & \dots \\ a_1^{q_1-2}(t) & a_2^{q_1-2}(t) & \dots & a_{q_1}^{q_1-2}(t)\end{pmatrix}, \quad V_2(t) = \begin{pmatrix} b_{q_1+1}^{q_2-1}(t) & b_{q_1+2}^{q_2-1}(t) & \dots & b_{q}^{q_2-1}(t) \\ 1 & 1 & \dots & 1 \\ b_{q_1+1}(t) & b_{q_1+2}(t) & \dots & b_{q}(t) \\ \dots & \dots & \dots & \dots \\ b_{q_1+1}^{q_2-2}(t) & b_{q_1+2}^{q_2-2}(t) & \dots & b_{q}^{q_2-2}(t)\end{pmatrix},$$

because [9]

$$\det\, T_1(t,\varepsilon) = \det V_1(t) h^{(1)}_{q,q_1+1}(t)h^{q_1-1}(t)\varepsilon^{\frac{q_1-1}{2}} + O(\varepsilon^{\frac{q_1-1}{2}+\frac{1}{q_1}}).$$

Further, $\det V_1(t)$ and $\det V_2(t)$ are Vandermonde determinants up to a sign. Thus, $\det V_1(t)\neq 0$, $\det V_2(t)\neq 0$, $t\in[0;T]$, and $\det\, T_q(t,\varepsilon)\neq 0$, $t\in[0;T]$.

Substituting the representation

$$\widetilde{v}^{(0)}_{i1}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}(t,\varepsilon), \tag{18}$$

into equation (15), we get

$$(I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}^{(0)}_{i1}(t,\varepsilon) = 0. \tag{19}$$

Therefore,

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = \frac{1}{w_i(t,\varepsilon)},\quad \{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j = 0,\ t\in[0;T],\ i\neq j,\ i,j=\overline{1,q},$$

where $\{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j$ is the jth component of $\widetilde{q}^{(0)}_{i1}(t,\varepsilon)$.

Consider now equation (16). Suppose the following.

5. $c^{(1)}_{q+p_1,q+1}(t)\neq 0$, $c^{(1)}_{n,q+p_1+1}(t)\neq 0$, $c(t)\neq 0$, $t\in[0;T]$, where $c^{(1)}_{ij}(t)$ is the element of the matrix $C_1(t)$, $c(t) = c^{(1)}_{q+p_1,q+1}(t)c^{(1)}_{n,q+p_1+1}(t) - c^{(1)}_{q+p_1,q+p_1+1}(t)c^{(1)}_{n,q+1}(t)$.

Then the solutions of the equation $\det(N_p + \varepsilon D_{1p}(t) - wI_p) = 0$ have the following form

$$w_j(t,\varepsilon) = \sqrt[p_1]{c^{(1)}_{q+p_1,q+1}(t)}\varepsilon^{\frac{1}{p_1}} + O(\varepsilon^{\frac{2}{p_1}}),\ j = \overline{q+1, q+p_1},$$
$$w_j(t,\varepsilon) = \sqrt[p_2]{c^{(1)}_{n,q+p_1+1}(t)}\varepsilon^{\frac{1}{p_2}} + O(\varepsilon^{\frac{2}{p_2}}),\ j = \overline{q+p_1+1, n}.$$

Let $T_p(t,\varepsilon)$ be the square matrix of order p such that

$$T_p^{-1}(t,\varepsilon)(N_p + \varepsilon D_{1p}(t))T_p(t,\varepsilon) = \widetilde{W}_p(t,\varepsilon) \equiv \operatorname{diag}\{w_{q+1}(t,\varepsilon), w_{q+2}(t,\varepsilon), ..., w_n(t,\varepsilon)\}.$$

Substituting the representation $\widetilde{v}^{(0)}_{i2}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2}(t,\varepsilon)$ into (16), we obtain

$$(\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{q}^{(0)}_{i2}(t,\varepsilon) = 0.$$

Thus

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = w_i(t,\varepsilon),\ \ \{\widetilde{q}^{(0)}_{i2}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}^{(0)}_{i2}(t,\varepsilon)\}_j = 0,\ t\in[0;T],\ i\neq j,\ i,j = \overline{q+1,n},$$

and $\widetilde{q}^{(0)}_{i1}(t,\varepsilon) = 0,\ t\in[0;T],\ i=\overline{q+1,n},\ \widetilde{q}^{(0)}_{i2}(t,\varepsilon) = 0,\ t\in[0;T],\ i=\overline{1,q}$.

We write equation (14) for $s = 1$ as follows

$$(I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)(N_q + \varepsilon F_{1q}(t)))\widetilde{v}^{(1)}_{i1}(t,\varepsilon) = d^{(1)}_{i1}(t,\varepsilon), \tag{20}$$
$$(N_p + \varepsilon D_{1p}(t) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{v}^{(1)}_{i2}(t,\varepsilon) = d^{(1)}_{i2}(t,\varepsilon), \tag{21}$$

where $\widetilde{v}^{(1)}_i(t,\varepsilon) = \begin{pmatrix}\widetilde{v}^{(1)}_{i1}(t,\varepsilon)\\ \widetilde{v}^{(1)}_{i2}(t,\varepsilon)\end{pmatrix}$, and $\widetilde{d}^{(1)}_i(t,\varepsilon) = \begin{pmatrix}\widetilde{d}^{(1)}_{i1}(t,\varepsilon)\\ \widetilde{d}^{(1)}_{i2}(t,\varepsilon)\end{pmatrix}$.

The transformation $\widetilde{v}^{(1)}_{i1}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}^{(1)}_{i1}(t,\varepsilon)$, $\widetilde{v}^{(1)}_{i2}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}^{(1)}_{i2}(t,\varepsilon)$ changes (20), (21) into the equations

$$(I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}^{(1)}_{i1}(t,\varepsilon) = g^{(1)}_{i1}(t,\varepsilon), \tag{22}$$
$$(\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{q}^{(1)}_{i2}(t,\varepsilon) = g^{(1)}_{i2}(t,\varepsilon), \tag{23}$$

where $g^{(1)}_{i1}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)d^{(1)}_{i1}(t,\varepsilon) \equiv T_q^{-1}(t,\varepsilon)N_qT'_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1} + \widetilde{W}_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}\widetilde{\lambda}^{(1)}_i(t,\varepsilon)$,
$g^{(1)}_{i2}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)d^{(1)}_{i2}(t,\varepsilon) \equiv T_p^{-1}(t,\varepsilon)T'_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2} + \widetilde{q}^{(0)}_{i2}\widetilde{\lambda}^{(1)}_i(t,\varepsilon)$.

Therefore,

$$\widetilde{\lambda}^{(1)}_i(t,\varepsilon) = -\frac{\{f^{(1)}_{i1}(t,\varepsilon)\}_i}{w_i(t,\varepsilon)},\ \{\widetilde{q}^{(1)}_{i1}(t,\varepsilon)\}_i \equiv 0,\ t\in[0;T],$$
$$\{\widetilde{q}^{(1)}_{i1}(t,\varepsilon)\}_j = \frac{\{g^{(1)}_{i1}(t,\varepsilon)\}_j w_i(t,\varepsilon)}{w_i(t,\varepsilon) - w_j(t,\varepsilon)},\ i\neq j,\ i,j = \overline{1,q},$$
$$\widetilde{q}^{(1)}_{i2}(t,\varepsilon) = (\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)^{-1}g^{(1)}_{i2}(t,\varepsilon),\ i = \overline{1,q},$$

where $f^{(1)}_{i1}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)N_qT'_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}$, and

$$\widetilde{\lambda}^{(1)}_i(t,\varepsilon) = -\{f^{(1)}_{i2}(t,\varepsilon)\}_i,\ \{\widetilde{q}^{(1)}_{i2}(t,\varepsilon)\}_i \equiv 0,\ t\in[0;T],$$
$$\{\widetilde{q}^{(1)}_{i2}(t,\varepsilon)\}_j = \frac{\{g^{(1)}_{i2}(t,\varepsilon)\}_j}{w_j(t,\varepsilon) - w_i(t,\varepsilon)},\ i\neq j,\ i,j = \overline{q+1,n},$$
$$\widetilde{q}^{(1)}_{i1}(t,\varepsilon) = (I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))^{-1}g^{(1)}_{i1}(t,\varepsilon),\ i = \overline{q+1,n},$$

where $f_{i2}^{(1)}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)T_p'(t,\varepsilon)\widetilde{q}_{i2}^{(0)}$.

Let $T_q^{-1}(t,\varepsilon)$ be defined by

$$T_q^{-1}(t,\varepsilon) = \begin{pmatrix} V_1(t,\varepsilon) & V_2(t,\varepsilon) \\ V_3(t,\varepsilon) & V_4(t,\varepsilon) \end{pmatrix},$$

where $V_1(t,\varepsilon)$ is a square matrix of order q_1.

Using the Frobenius formula for the inverse of the block matrix [9], we find

$$V_i(t,\varepsilon) = O\left(\varepsilon^{\delta_{2i}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\right)\begin{pmatrix} O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ \dots & \dots & \dots & \dots \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \end{pmatrix},$$

$t \in [0;T]$, $\varepsilon \to 0+$; $\gamma_i = q_{\delta_{1i}} + q_{2\delta_{2i}} + q_{\delta_{3i}} + q_{2\delta_{4i}}$, $i = \overline{1,4}$; δ_{ij} is the Kronecker delta.

It should be noted that the matrix $T_p^{-1}(t,\varepsilon)$ has the same form as $T_q^{-1}(t,\varepsilon)$.

For the sequel we assume the following.

6. $q_2 < p_1$.

Then

$$\widetilde{v}_i^{(1)}(t,\varepsilon) = O(\varepsilon^{\frac{1}{q_2}}),\ \widetilde{\lambda}_i^{(1)}(t,\varepsilon) = O(1),\ i = \overline{1,q},$$
$$\widetilde{v}_i^{(1)}(t,\varepsilon) = O(\varepsilon^{-\frac{1}{p_1}}),\ \widetilde{\lambda}_i^{(1)}(t,\varepsilon) = O(1),\ i = \overline{q+1,n},$$

$t \in [0;T]$, $\varepsilon \to 0+$. In the same way we define the functions $\widetilde{v}_i^{(s)}(t,\varepsilon)$, $\widetilde{\lambda}_i^{(s)}(t,\varepsilon)$, $i = \overline{1,n}$, $s = 2, 3....$ In addition,

$$\widetilde{v}_i^{(s)}(t,\varepsilon) = O(\varepsilon^{-\left[\frac{s}{2}\right]}),\ i = \overline{1,n},\quad \widetilde{\lambda}_i^{(s)}(t,\varepsilon) = O(\varepsilon^{-\left[\frac{s}{2}\right]-\frac{1}{q_1}}),\ i = \overline{1,q},$$
$$\widetilde{\lambda}_i^{(s)}(t,\varepsilon) = O(\varepsilon^{-\left[\frac{s}{2}\right]-\frac{2}{p_2}\left(\frac{s}{2}-\left[\frac{s}{2}\right]\right)+\frac{1}{p_2}}),\ i = \overline{q+1,n},\ s = 2,3,...,\ t \in [0;T],\ \varepsilon \to 0+,$$

where $\left[\frac{s}{2}\right]$ is the integer part of $\frac{s}{2}$. Moreover, the functions $\widetilde{v}_i^{(s)}(t,\varepsilon)$, $\widetilde{\lambda}_i^{(s)}(t,\varepsilon)$, $i = \overline{1,n}$, $s \in N$, are periodic in t of period T.

The main results this section can be formulated as a theorem.

Theorem 1. *If $A_s(t)$, $B_s(t) \in C^{m+1}[0;T]$, $s \geq 0$, and the assumptions 1–6 are satisfied, then system (5) has n formal solutions of the form (9).*

Remark 1. If the pencil $B_0(t) - \lambda A_0(t)$ has more than one distinct eigenvalue, then system (5) can be reduced to a set of systems of lower order in each of which the corresponding characteristic equation has only one eigenvalue [8, 15, 16].

3. Nonhomogeneous DAEs. Consider now system (4). We seek a formal solutions of system (4) in the following form

$$x(t,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s x_s(t). \tag{24}$$

Substituting representation (24) into system (4) and equating the coefficients of like powers of ε, we get

$$B_0(t)x_s = d_s(t),\ s = 0,1,...,$$

where $d_s(t) = \sum_{k=0}^{s} A_k(t)x'_{s-k-1}(t) - \sum_{k=1}^{s} B_k(t)x_{s-k}(t) - f_s(t)$, $x_i(t) \equiv 0$, $t \in [0;T]$, $i < 0$.

Suppose that the following condition is satisfied.

7. $\lambda_0(t) \neq 0,\ t \in [0;T]$.

Hence, it follows that $x_s(t) = (B_0(t))^{-1} d_s(t),\ s = 0, 1, ...,$ because $\det B_0(t) \neq 0,\ t \in [0;T]$. Moreover, the functions $x_s(t)$ are periodic of period T. Thus, expression (24) is a formal solution of system (4) of period T.

Let us show that the constructed formal solution of (4) has asymptotic character. Substituting relation

$$x(t,\varepsilon) = y(t,\varepsilon) + x_m(t,\varepsilon),\ x_m(t,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s x_s(t),$$

into (4), we have the following system

$$\varepsilon A(t,\varepsilon)\frac{dy}{dt} = B(t,\varepsilon)y + g(t,\varepsilon), \tag{25}$$

where $g(t,\varepsilon) = O(\varepsilon^{m+1}),\ t \in [0;T],\ \varepsilon \to 0+$.

We construct a square matrix $V_m(t,\varepsilon)$ of order n consisting of the first m terms of expressions (10)

$$V_m(t,\varepsilon) = \sum_{k=0}^{m} \varepsilon^k \widetilde{V}^{(k)}(t,\varepsilon),\ \widetilde{V}^{(k)}(t,\varepsilon) = [\widetilde{v}_1^{(k)}(t,\varepsilon), \widetilde{v}_2^{(k)}(t,\varepsilon), ..., \widetilde{v}_n^{(k)}(t,\varepsilon)].$$

We set $y(t,\varepsilon) = \exp(\frac{1}{\varepsilon}\int_0^t \lambda_0(t)dt)Q(t,\varepsilon)Q(t)V_m(t,\varepsilon)z(t,\varepsilon)$ and multiply both sides of (25) on the left by $P(t)P(t,\varepsilon)$. Then system (25) can be written as

$$\varepsilon H(t,\varepsilon)V_m(t,\varepsilon)\frac{dz}{dt} = (C(t,\varepsilon)V_m(t,\varepsilon) - \varepsilon H(t,\varepsilon)V_m'(t,\varepsilon))z + h(t,\varepsilon), \tag{26}$$

where $h(t,\varepsilon) = \exp(-\frac{1}{\varepsilon}\int_0^t \lambda_0(t)dt)P(t)P(t,\varepsilon)g(t,\varepsilon)$.

Observe that the matrices $H(t,\varepsilon)$ and $V_m(t,\varepsilon)$ are nonsingular for all $t \in [0;T]$ and for all small positive ε.

Since

$$C(t,\varepsilon)V_m(t,\varepsilon) - \varepsilon H(t,\varepsilon)V_m'(t,\varepsilon) = H(t,\varepsilon)V_m(t,\varepsilon)(\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}S_m(t,\varepsilon)),$$

where $\Lambda_m(t,\varepsilon) = \sum_{k=0}^{m} \varepsilon^k \widetilde{\Lambda}^{(k)}(t,\varepsilon),\ \widetilde{\Lambda}^{(k)}(t,\varepsilon) = \mathrm{diag}\{\widetilde{\lambda}_1^{(k)}(t,\varepsilon), \widetilde{\lambda}_2^{(k)}(t,\varepsilon), ..., \widetilde{\lambda}_n^{(k)}(t,\varepsilon)\}$,

$$S_m(t,\varepsilon) = \left(I_n + O(\varepsilon^{1-\frac{1}{p_1}})\right)^{-1} \begin{pmatrix} \widetilde{W}_q(t,\varepsilon) + O(\varepsilon^{1+\frac{1}{q_2}}) & 0 \\ 0 & I_p + O(\varepsilon^{1+\frac{1}{p_2}}) \end{pmatrix}^{-1} T^{-1}(t,\varepsilon)G(t,\varepsilon),$$

$$T(t,\varepsilon) = \mathrm{diag}\{T_q(t,\varepsilon), T_p(t,\varepsilon)\},$$

$$G(t,\varepsilon) = \sum_{k=2}^{m+1} D_k(t)\widetilde{V}^{(m+1-k)}(t,\varepsilon) + \sum_{k\geq 1}\varepsilon^k \sum_{s=k+1}^{m+k+1} D_s(t)\widetilde{V}^{(m+1+k-s)}(t,\varepsilon) -$$

$$-\sum_{k\geq 0}\varepsilon^k \sum_{s=k}^{m+k} F_s(t)(\widetilde{V}^{(m+k-s)}(t,\varepsilon))' - \sum_{k\geq 0}\varepsilon^k \sum_{s\geq 2} F_s(t)\sum_{j\geq 0}\widetilde{V}^{(j)}(t,\varepsilon)\widetilde{\Lambda}^{(m+1+k-s-j)}(t,\varepsilon) -$$

$$-(F_0 + \varepsilon F_1(t))\sum_{k=0}^{m-1}\varepsilon^k \sum_{s=k+1}^{m} \widetilde{V}^{(s)}(t,\varepsilon)\widetilde{\Lambda}^{(m+1+k-s)}(t,\varepsilon),$$

$\widetilde{V}^{(s)}(t,\varepsilon) \equiv 0,\ \widetilde{\Lambda}^{(s)}(t,\varepsilon) \equiv 0,\ t \in [0;T],\ s < 0,\ s > m$, the system (26) can be written in the form

$$\varepsilon\frac{dz}{dt} = (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}S_m(t,\varepsilon))z + (H(t,\varepsilon)V_m(t,\varepsilon))^{-1}h(t,\varepsilon). \tag{27}$$

In addition,

$$\varepsilon^{m+1}S_m(t,\varepsilon)=O(\varepsilon^{m-[\frac{m}{2}]-\frac{1}{q_1}}),\ t\in[0;T],\ \varepsilon\to 0+.$$

The transformation $z(t,\varepsilon)=\exp(-\frac{1}{\varepsilon}\int_0^t\lambda_0(t)dt)u(t,\varepsilon)$ changes system (27) into

$$\varepsilon\frac{du}{dt}=(\Lambda_m(t,\varepsilon)+\lambda_0(t)I_n+\varepsilon^{m+1}S_m(t,\varepsilon))u+q(t,\varepsilon), \tag{28}$$

where $q(t,\varepsilon)=(H(t,\varepsilon)V_m(t,\varepsilon))^{-1}P(t)P(t,\varepsilon)g(t,\varepsilon)$. Observe that $q(t,\varepsilon)=O(\varepsilon^m)$, $t\in[0;T]$, $\varepsilon\to 0+$.

Suppose the following.

8. $\mathrm{Re}(\widetilde{\lambda}_i^{(0)}(t,\varepsilon)+\lambda_0(t))\neq 0$, $t\in[0;T]$, $i=\overline{1,n}$.

Then, without loss of generality, we can assume that

$$\Lambda_m(t,\varepsilon)+\lambda_0(t)I_n=\mathrm{diag}\{\Lambda_{m-}(t,\varepsilon),\Lambda_{m+}(t,\varepsilon)\},$$

where $\Lambda_{m-}(t,\varepsilon)$ and $\Lambda_{m+}(t,\varepsilon)$ are the diagonal matrices whose eigenvalues are the eigenvalues of $\Lambda_m(t,\varepsilon)+\lambda_0(t)I_n$ with negative and positive real parts, respectively.

Let us write a system of integral equations

$$u(t,\varepsilon)=\varepsilon^m\int_t^{t+T}\left(\Psi_1(s,\varepsilon)\left(\Psi_1^{-1}(T,\varepsilon)-I_n\right)\Psi_1^{-1}(t,\varepsilon)\right)^{-1}S_m(s,\varepsilon)u(s,\varepsilon)ds, \tag{29}$$

where $\Psi_1(t,\varepsilon)=\exp(\frac{1}{\varepsilon}\int_0^t(\Lambda_m(t,\varepsilon)+\lambda_0(t)I_n)dt)$ is a fundamental matrix solutions of the corresponding homogeneous system $\varepsilon\frac{du}{dt}=(\Lambda_m(t,\varepsilon)+\lambda_0(t)I_n)u$, $\Psi_1(0,\varepsilon)=I_n$. Note that system (29) is equivalent to (28) [12] on the set $P=\{u(t,\varepsilon)\in C[0;T]:u(t+T,\varepsilon)=u(t,\varepsilon)\}$.

System (29) can be written in the form

$$u_-(t,\varepsilon)=\varepsilon^m\Big(I_--\exp\Big(\frac{1}{\varepsilon}\int_0^T\Lambda_{m-}(t,\varepsilon)dt\Big)\Big)^{-1}\times$$

$$\times\int_0^T\exp\Big(\frac{1}{\varepsilon}\int_{t+s-T}^t\Lambda_{m-}(t,\varepsilon)dt\Big)(S_{m1}(t+s,\varepsilon)u_-(t+s,\varepsilon)+S_{m2}(t+s,\varepsilon)u_+(t+s,\varepsilon))ds, \tag{30}$$

$$u_+(t,\varepsilon)=\varepsilon^m\Big(\exp\Big(-\frac{1}{\varepsilon}\int_0^T\Lambda_{m+}(t,\varepsilon)dt\Big)-I_+\Big)^{-1}\times$$

$$\times\int_0^T\exp\Big(\frac{1}{\varepsilon}\int_{t+s}^t\Lambda_{m+}(t,\varepsilon)dt\Big)(S_{m3}(t+s,\varepsilon)u_-(t+s,\varepsilon)+S_{m4}(t+s,\varepsilon)u_+(t+s,\varepsilon))ds. \tag{31}$$

Here $u(t,\varepsilon)=\begin{pmatrix}u_-(t,\varepsilon),\\ u_+(t,\varepsilon)\end{pmatrix}$, $S_m(t,\varepsilon)=\begin{pmatrix}S_{m1}(t,\varepsilon) & S_{m2}(t,\varepsilon)\\ S_{m3}(t,\varepsilon) & S_{m4}(t,\varepsilon)\end{pmatrix}$, and the dimensions of vectors $u_-(t,\varepsilon)$ and $u_+(t,\varepsilon)$ coincide with the orders of the matrices $\Lambda_{m-}(t,\varepsilon)$, $S_{m1}(t,\varepsilon)$ and $\Lambda_{m+}(t,\varepsilon)$, $S_{m4}(t,\varepsilon)$, respectively; I_- and I_+ are the identity matrices of appropriate order.

Let us consider the mapping $\varphi=Au$ of the set P into itself given by system (30), (31). The mapping $\varphi=Au$ is a contraction mapping. Thus, the operator equation $r=Ar$ (and consequently system (30), (31) also) has one and only one solution [17]. Note that $u(t,\varepsilon)=O(\varepsilon^{m-[\frac{m}{2}]-\frac{1}{p_1}-\frac{1}{q_1}})$, $t\in[0;T]$, $\varepsilon\to 0+$.

Theorem 2. *If $A_s(t)$, $B_s(t) \in C^{m+1}[0;T]$, $s \geq 0$, and the assumptions 1 – 8 are satisfied, then for sufficiently small ε, $\varepsilon \in (0;\varepsilon_0]$, and for each fixed $m \geq 1$ system (4) has a unique solution $x = x(t,\varepsilon)$ of period T such that such the estimate*

$$x(t,\varepsilon) - x_m(t,\varepsilon) = O(\varepsilon^{m-[\frac{m}{2}]-\frac{1}{p_1}-\frac{1}{q_1}}),\ t \in [0;T],\ \varepsilon \to 0+$$

is valid.

Further, assume that the following condition is satisfied.

9. $\lambda_0(t) \equiv 0$, $t \in [0;T]$.

The transformation $x(t,\varepsilon) = Q(t,\varepsilon)y(t,\varepsilon)$ changes system (4) into

$$\varepsilon H(t,\varepsilon)\frac{dy}{dt} = C(t,\varepsilon)y + r(t,\varepsilon), \tag{32}$$

where $r(t,\varepsilon) = P(t,\varepsilon)f(t,\varepsilon)$.

Then, multiplying both sides of system (32) on the left by $K^{-1}(t,\varepsilon)$, we get

$$\varepsilon F(t,\varepsilon)\frac{dy}{dt} = D(t,\varepsilon)y + l(t,\varepsilon), \tag{33}$$

where $l(t,\varepsilon) = K^{-1}(t,\varepsilon)r(t,\varepsilon) \equiv \sum_{k\geq 0}\varepsilon^k l_k(t)$.

We seek a formal solutions of system (33) in the following form

$$y(t,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s y_s(t,\varepsilon). \tag{34}$$

Next, we substitute representation (34) into system (33) and compare the coefficients of powers of ε in such a way that

$$(D_0 + \varepsilon D_1(t))y_s = d_s(t,\varepsilon),\ s = 0,1,..., \tag{35}$$

where

$$d_s(t,\varepsilon) = \sum_{k=0}^{s}F_k(t)y'_{s-k-1}(t,\varepsilon) - \sum_{k=2}^{s}D_k(t)y_{s-k}(t,\varepsilon) - l_s(t), \quad y_i(t,\varepsilon) \equiv 0,\ t \in [0;T],\ i < 0.$$

The transformation $y_s(t,\varepsilon) = R(t,\varepsilon)z_s(t,\varepsilon)$, $R(t,\varepsilon) = \mathrm{diag}\{I_q, T_p(t,\varepsilon)\}$, changes system (35) into $\Phi(t,\varepsilon)z_s(t,\varepsilon) = q_s(t,\varepsilon)$. Here $\Phi(t,\varepsilon) = \mathrm{diag}\{I_q, \widetilde{W}_p(t,\varepsilon)\}$,

$$q_s(t,\varepsilon) = R^{-1}(t,\varepsilon)\Big(\sum_{k=0}^{s}F_k(t)(R'(t,\varepsilon)z_{s-k-1}(t,\varepsilon) + R(t,\varepsilon)z'_{s-k-1}(t,\varepsilon)) - \\ - \sum_{k=2}^{s}D_k(t)R(t,\varepsilon)z_{s-k}(t,\varepsilon) - l_s(t)\Big).$$

So, we have $z_s(t,\varepsilon) = \Phi^{-1}(t,\varepsilon)q_s(t,\varepsilon)$, $s = 0,1,\dots$.

Let $z_{s1}(t,\varepsilon)$ be a q-dimensional vector whose components are the first q components of $z_s(t,\varepsilon)$, and let $z_{s2}(t,\varepsilon)$ be a p-dimensional vector whose components are the last p components of $z_s(t,\varepsilon)$. Then

$$z_{s1}(t,\varepsilon) = O(\varepsilon^{-[\frac{s}{2}]-\frac{2}{p_1}(\frac{s}{2}-[\frac{s}{2}])}), \quad z_{s2}(t,\varepsilon) = O(\varepsilon^{-[\frac{s}{2}]-1-\frac{2}{p_1}(\frac{s}{2}-[\frac{s}{2}])}),$$

$s = 0,1,2,...$, $t \in [0;T]$, $\varepsilon \to 0+$.

In the similar manner, we obtain system (28), where now $q(t,\varepsilon) = O(\varepsilon^{m-[\frac{m}{2}]-1-\frac{2}{p_1}(\frac{m}{2}-[\frac{m}{2}])})$, $t \in [0;T]$, $\varepsilon \to 0+$.

Repeating the preceding argument, we have the following result.

Theorem 3. *If $A_s(t)$, $B_s(t) \in C^{m+1}[0;T]$, $s \geq 0$, and the assumptions 1–6, 8, 9 are satisfied, then for sufficiently small ε, $\varepsilon \in (0;\varepsilon_0]$, and for each fixed $m \geq 3$ system (4) has a unique solution $x = x(t,\varepsilon)$ of period T such that*

$$x(t,\varepsilon) - x_m(t,\varepsilon) = O(\varepsilon^{m-\left[\frac{m}{2}\right]-1-\frac{2}{p_1}}),\ t \in [0;T],\ \varepsilon \to 0+,$$

where $x_m(t,\varepsilon) = Q(t,\varepsilon)y_m(t,\varepsilon)$.

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