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INVERSE PROBLEM FOR SEMILINEAR EIDELMAN TYPE EQUATION

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The inverse problem for semilinear Eidelman type equation with unknown time dependent function in its right-hand side is considered in this paper. The initial, boundary and integral type overdetermination conditions are posed. The sufficient conditions of the existence and the uniqueness of weak solution for the problem are obtained.

In this paper we obtain the sufficient conditions of the unique solvability for the inverse problem for the semilinear Eidelman type equation. Unknown right-hand side time dependent function is determined from the initial, boundary and integral type overdetermination conditions. The equation contains three groups of variables with different order of differentiation of its solution with respect to these variables: there are first derivatives with respect to time variable, second derivatives with respect to the spatial variables and fourth derivatives with respect to one group of the spatial variables. If the right-hand side function of the equation is known, then the existence and the uniqueness of solution and its properties for the initial-boundary value problems for the nonlinear Eidelman type equation in bounded or unbounded domains were considered in [1, 2], for Cauchy problem in [3]–[5], for Eidelman type equation with the second time derivative in [6].

Note, that the problems of determination of a parameter in the right-hand side function of the parabolic equations were studied in [7]–[12], of the semilinear ultraparabolic equations in [13, 14, 15]. The authors used the methods of the integral equations, regularization and the Shauder principle [7, 8, 10], the methods of finite difference approximations, numerical and iterative methods [11, 12], the method of successive approximations [13]–[15].

Let $\mathcal{D}_x \subset \mathbb{R}^k$ and $\mathcal{D}_y \subset \mathbb{R}^l$ be bounded domains, their boundaries $\partial\mathcal{D}_x \in C^1$ and $\partial\mathcal{D}_y \in C^1$. Denote: $\Omega = \mathcal{D}_x \times \mathcal{D}_y$, $Q_\tau = \Omega \times (0,\tau)$, $S_\tau = \partial\Omega \times (0,\tau)$, where $\tau \in (0,T]$, $T < \infty$, $x \in \mathcal{D}_x$, $y \in \mathcal{D}_y$, $z = (x,y) \in \Omega$, $n = k + l$, ν is the outward unit normal vector to $\partial\mathcal{D}_x \times \mathcal{D}_y \times (0,T)$.

We shall introduce the space

$$V_1(\Omega) = \left\{ u\colon u \in H_0^1(\Omega), u_{x_i x_j} \in L^2(\Omega), i,j \in \{1,\dots,k\}, \left.\frac{\partial u}{\partial \nu}\right|_{\partial\mathcal{D}_x \times \mathcal{D}_y} = 0 \right\}.$$

In this paper in the domain Q_T we study the following inverse problem: find the sufficient conditions of the existence and the uniqueness of a pair of functions $(u(z,t),\ q(t))$ that

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satisfies the equation

$$u_t + \sum_{i,j=1}^{k} (a_{ij}(z,t)u_{x_ix_j})_{x_ix_j} - \sum_{i,j=1}^{n} (b_{ij}(z,t)u_{z_i})_{z_j} + c(z,t)u + g(z,t,u) = f_1(z,t)q(t) + f_0(z,t), \tag{1}$$

the initial, boundary and overdetermination conditions

$$u(z,0) = u_0(z), \quad z \in \Omega, \tag{2}$$

$$u|_{S_T} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial \mathcal{D}_x \times \mathcal{D}_y \times (0,T)} = 0, \tag{3}$$

$$\int_{\Omega} K(z)u(z,t)dz = E(t), \qquad t \in [0,T], \tag{4}$$

in the sense of definition.

Definition 1. A pair of functions $(u(z,t), q(t))$ is a *weak solution to the problem (1)–(4)*, if $u \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u_t \in L^2(Q_T)$, $q \in C([0,T])$, it satisfies the equality

$$\int_{Q_\tau} \left(u_t v + \sum_{i,j=1}^{k} a_{ij}(z,t)u_{x_ix_j}v_{x_ix_j} + \sum_{i,j=1}^{n} b_{ij}(z,t)u_{z_i}v_{z_j} + c(z,t)uv + g(z,t,u)v \right) dzdt =$$

$$= \int_{Q_\tau} (f_1(z,t)q(t) + f_0(z,t))v dzdt \tag{5}$$

for all $\tau \in (0,T]$, and all functions $v \in L^2(0,T;V_1(\Omega))$, and the conditions (2), (4) hold.

Let the coefficients of equation (1) and the initial data satisfy conditions:

(A 1): $a_{ij} \in C([0,T];L^\infty(\Omega))$, $a_{ij,t} \in L^\infty(Q_T)$,
$a_{ij}(z,t) \geq a_0 > 0$ for almost all $(z,t) \in Q_T$, $i,j \in \{1,\dots,k\}$;

(A 2): $b_{ij} \in C([0,T];L^\infty(\Omega))$, $b_{ij,t} \in L^\infty(Q_T)$, $i,j \in \{1,\dots,n\}$;
$\sum_{i,j=1}^{n} b_{ij}(z,t)\xi_i\xi_j \geq b_0|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and for almost all $(z,t) \in Q_T$, $b_0 > 0$;

(A 3): $c \in C([0,T];L^\infty(\Omega))$, $c(z,t) \geq c_0$ for almost all $(z,t) \in Q_T$,
where c_0 is a constant;

(A 4): $g(z,t,\xi)$ is measurable with respect to the variables (z,t) in Q_T for all $\xi \in \mathbb{R}^1$ and is continuous with respect to ξ for almost all $(z,t) \in Q_T$,
moreover, there exists a positive constant g_0, such that
$|g(z,t,\xi) - g(z,t,\eta)| \leq g_0|\xi - \eta|$ for almost all $(z,t) \in Q_T$ and all ξ, $\eta \in \mathbb{R}^1$;

(A 5): $f_0, \ f_1 \in C([0,T];L^2(\Omega))$;

(A 6): $u_0 \in V_1(\Omega)$;

(A 7): $K \in V_1(\Omega)$, $K_{x_ix_ix_jx_j} \in L^2(\Omega)$, $K_{z_rz_s} \in L^2(\Omega)$, $i,j \in \{1,\dots,k\}$, $r,s \in \{1,\dots,n\}$;

(A 8): $E \in H^1(0,T)$, $E(0) = \int_{\Omega} K(z)u_0(z)\,dz$.

Note, that if $q(t) = q^*(t)$, where $q^* \in C([0,T])$ is known function, then similarly as in [1] we can obtain the results of the existence and the uniqueness of the weak solution for the initial-boundary value problem (1)–(3).

Theorem 1. *Under the conditions (A 1)–(A 6) and $q^* \in C([0,T])$ there exists a unique weak solution u^* to the problem (1)–(3), i.e. $u^* \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u^*_t \in L^2(Q_T)$, it satisfies (2) and the equality*

$$\int\limits_{Q_\tau} \Big(u^*_t v + \sum_{i,j=1}^{k} a_{ij}(z,t) u^*_{x_i x_j} v_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}(z,t) u^*_{z_i} v_{z_j} + c(z,t)u^* v + g(z,t,u^*)v \Big) dzdt =$$
$$= \int\limits_{Q_\tau} (f_1(z,t)q^*(t) + f_0(z,t)) v dzdt \tag{6}$$

holds for all $\tau \in (0,T]$ and all functions $v \in L^2(0,T;V_1(\Omega))$.

The derivative u^*_t has the estimate

$$\int\limits_{Q_T} (u^*_t)^2 dzdt \le M_0 \Big(\int\limits_{\Omega} \big((u_0(z))^2 + \sum_{i=1}^{n} (u_{0,z_i}(z))^2 + \sum_{i,j=1}^{k} (u_{0,x_i x_j}(z))^2 \big) dz +$$
$$+ \int\limits_{Q_T} (f_1^2(z,t)(q^*(t))^2 + f_0^2(z,t)) dzdt \Big), \tag{7}$$

where the constant M_0 depends only on the coefficients of the left-hand side of the equation (1).

Now we shall obtain an auxiliary problem to problem (1)–(4). Denote:

$$A(t) := \int\limits_{\Omega} K(z) f_1(z,t) dz, \quad B(t) := E'(t) - \int\limits_{\Omega} K(z) f_0(z,t) dz,$$
$$C(z,t) := \sum_{i,j=1}^{k} (K_{x_i x_j}(z) a_{ij}(z,t))_{x_i x_j} - \sum_{i,j=1}^{n} (K_{z_j}(z) b_{ij}(z,t))_{z_i} + K(z)c(z,t).$$

Let $(u(z,t), q(t))$ be a weak solution to problem (1)–(4). From (4) it follows that

$$\int\limits_{\Omega} K(z) u_t(z,t) dz = E'(t), \qquad t \in [0,T]. \tag{8}$$

By using equality (5) with $v = K(z)$ and (8), we get

$$\int\limits_0^\tau E'(t)dt + \int\limits_{Q_\tau} \Big(\sum_{i,j=1}^{k} a_{ij}(z,t) K_{x_i x_j}(z) u_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}(z,t) K_{z_j}(z) u_{z_i} + c(z,t)K(z)u +$$
$$+ g(z,t,u)K(z) \Big) dzdt = \int\limits_{Q_\tau} (f_1(z,t)q(t) + f_0(z,t)) K(z) dzdt, \quad \tau \in (0,T]. \tag{9}$$

After integrating by parts in (9), in view of the condition (A 7), we obtain

$$\int_0^\tau B(t)dt + \int_{Q_\tau} \Big(C(z,t)u + g(z,t,u)K(z)\Big)dzdt = \int_0^\tau A(t)q(t)dt,$$

for all $\tau \in (0,T]$. Therefore

$$A(t)q(t) = B(t) + \int_\Omega \Big(C(z,t)u + g(z,t,u)K(z)\Big)dz, \quad t \in [0,T]. \tag{10}$$

Lemma 1. *Let the conditions (A 1)–(A 8) hold, and* $a_{ij,x_ix_j} \in C([0,T];L^2(\Omega))$, $b_{rs,z_r} \in C([0,T];L^2(\Omega))$, $i,j \in \{1,\dots,k\}$, $r,s \in \{1,\dots,n\}$. *The pair of functions* $(u(z,t),q(t))$, *where* $u \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u_t \in L^2(Q_T)$, $q \in C([0,T])$, *is a weak solution to the problem (1)—(4) if and only if it satisfies equality (5) for all* $v \in L^2(0,T;V_1(\Omega))$, $\tau \in (0,T)$ *and (2), (10) hold.*

Proof. The necessity is proved.

Let $u^* \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u_t^* \in L^2(Q_T)$, $q^* \in C([0,T])$, and they satisfy equality (5) for all $v \in L^2(0,T;V_1(\Omega))$, $\tau \in (0,T)$ and (2), (10). Then u^* is a solution to the problem (1)–(3) with q^* instead of q in (1).

We set $E^*(t) = \int_\Omega K(z)u^*(z,t)dz$, $t \in [0,T]$. In exactly the same way as in the proof of necessity, we obtain

$$\int_0^\tau (E^*(t))'dt + \int_{Q_\tau} \Bigg(\Bigg(\sum_{i,j=1}^k (a_{ij}(z,t)K_{x_ix_j}(z))_{x_ix_j} + \sum_{i,j=1}^n (b_{ij}(z,t)K_{z_j}(z))_{z_i} + c(z,t)K(z)\Bigg)u^* +$$
$$+g(z,t,u^*)K(z)\Bigg)dzdt = \int_{Q_\tau} (f_1(z,t)q^*(t) + f_0(z,t))K(z)dzdt, \quad t \in [0,T]. \tag{11}$$

On the other hand $q^*(t)$ and $u^*(z,t)$ satisfy (10), and therefore it is easy to get the following equality

$$\int_0^\tau E'(t)dt + \int_{Q_\tau} \Bigg(\Bigg(\sum_{i,j=1}^k (a_{ij}(z,t)K_{x_ix_j}(z))_{x_ix_j} + \sum_{i,j=1}^n (b_{ij}(z,t)K_{z_j}(z))_{z_i} + c(z,t)K(z)\Bigg)u^* +$$
$$+g(z,t,u^*)K(z)\Bigg)dzdt = \int_{Q_\tau} (f_1(z,t)q^*(t) + f_0(z,t))K(z)dzdt, \quad t \in [0,T]. \tag{12}$$

It follows from (11), (12) that

$$\int_0^\tau (E^*(t) - E(t))'dt = 0, \ \tau \in [0,T]. \tag{13}$$

Integrating (13) with the use of the equality $E^*(0) = E(0) = \int_\Omega K(z)u_0(z)dz$, we get $E^*(t) = E(t)$, $t \in [0,T]$. Hence, $u^*(z,t)$ satisfies (4). □

Denote:

$$f_2 := \sup_{[0,T]} \int_\Omega (f_1(z,t))^2 dz, \quad \alpha := \begin{cases} 0, & \text{if } c_0 + g_0 > 0,5; \\ 2(1 - c_0 - g_0), & \text{if } c_0 + g_0 \le 0,5, \end{cases}$$

$$\varkappa := \alpha + 2c_0 + 2g_0 - 1, \quad M_1 := f_2 e^{\alpha T}, \quad M_2 := \frac{M_1}{\min\{2a_0, 2b_0, \varkappa\}},$$

$$M_3 := \frac{2}{\min\limits_{[0,T]}(A(t))^2} \left(\sup_{[0,T]} \int_\Omega (C(z,t))^2 dz + (g_0)^2 \int_\Omega (K(z))^2 dz \right),$$

$$M_4 := M_1 M_3 \min\left\{ \frac{1}{\min\{2a_0, 2b_0, \varkappa\}}, T \right\}.$$

Theorem 2. *Let the conditions (A 1)–(A 8) hold,* $a_{ij,x_i x_j} \in C([0,T]; L^2(\Omega))$, $b_{rs,z_r} \in C([0,T]; L^2(\Omega))$, $i,j \in \{1,\dots,k\}$, $r,s \in \{1,\dots,n\}$, *and* $A(t) \neq 0$ *for all* $t \in [0,T]$. *Then there exists a unique weak solution to the problem (1)–(4) in the domain* Q_T.

Proof. **Case I.** First we consider the case, when T is such a number, that $M_4 < 1$.
Existence (case 1). In order to prove the existence result we use the method of successive approximations. We construct an approximation $(u^m(z,t), q^m(t))$ to the solution of problem (1)–(4), where the functions $q^m(t)$, $m \in \mathbb{N}$, satisfy equalities

$$q^1(t) := 0,$$

$$A(t)q^m(t) = B(t) + \int_\Omega C(z,t)u^{m-1}dz + \int_\Omega K(z)g(z,t,u^{m-1})dz, \ t \in [0,T], \ m \ge 2, \tag{14}$$

and u^m satisfies the equality

$$\int_{Q_\tau} \left(u_t^m v + \sum_{i,j=1}^k a_{ij}(z,t) u^m_{x_i x_j} v_{x_i x_j} + \sum_{i,j=1}^n b_{ij}(z,t) u^m_{z_i} v_{z_j} + c(z,t)u^m v + g(z,t,u^m)v \right) dzdt =$$

$$= \int_{Q_\tau} (f_1(z,t)q^m(t) + f_0(z,t))v dzdt, \quad m \ge 1, \ \tau \in (0,T], \tag{15}$$

for all $v \in L^2(0,T; V_1(\Omega))$, and the condition

$$u^m(z,0) = u_0(z), \quad z \in \Omega. \tag{16}$$

Theorem 1 yields that for each $m \in \mathbb{N}$ there exists a unique function $u \in L^2(0,T; V_1(\Omega)) \cap C([0,T]; L^2(\Omega))$, $u_t \in L^2(Q_T)$, that satisfies (15), (16).

Now we show that $\{(u^m(z,t), q^m(t))\}_{m=1}^\infty$ converges to the solution of the problem (1)–(4). Denote

$$w^m := w^m(z,t) = u^m(z,t) - u^{m-1}(z,t), \quad r^m(t) := q^m(t) - q^{m-1}(t), \quad m \ge 2.$$

It follows from (16) that $w^m(z,0) = 0$, $z \in \Omega$, $m \ge 2$. Hence, from (15), we get

$$\frac{1}{2} \int_\Omega (w^m(z,\tau))^2 e^{-\alpha\tau} dz + \int_{Q_\tau} \left(\frac{\alpha}{2}(w^m)^2 + \sum_{i,j=1}^k a_{ij}(z,t)(w^m_{x_i x_j})^2 + \sum_{i,j=1}^n b_{ij}(z,t) w^m_{z_i} w^m_{z_j} + \right.$$

$$+c(z,t)(w^m)^2 + (g(z,t,u^m) - g(z,t,u^{m-1}))w^m\Big) e^{-\alpha t} dzdt =$$

$$= \int\limits_{Q_\tau} f_1(z,t) r^m(t) w^m e^{-\alpha t} dzdt, \quad \tau \in (0,T], \ m \geq 2. \tag{17}$$

Then, taking into account (A 1)–(A 6), that under the hypotheses (A 4)

$$\int\limits_{Q_\tau} (g(z,t,u^{m-1}) - g(z,t,u^{m-2}))w^m dzdt \leq g_0 \int\limits_{Q_\tau} (w^{m-1})^2 dzdt, \quad \tau \in (0,T], \ m \geq 3,$$

and that

$$\int\limits_{Q_\tau} f_1(z,t) r^m(t) w^m e^{-\alpha t} dzdt \leq \frac{1}{2} \int\limits_{Q_\tau} (w^m)^2 e^{-\alpha t} dzdt + \frac{f_2}{2} \int\limits_0^T (r^m(t))^2 dt,$$

from (17) we get inequalities

$$\int\limits_\Omega (w^m(z,\tau))^2 e^{-\alpha\tau} dz + \int\limits_{Q_\tau} \Big(2a_0 \sum_{i,j=1}^k (w^m_{x_i x_j})^2 + 2b_0 \sum_{i=1}^n (w^m_{z_i})^2 + \varkappa (w^m)^2 \Big) e^{-\alpha t} dzdt \leq$$

$$\leq f_2 \int\limits_0^\tau (r^m(t))^2 e^{-\alpha t} dt, \quad \tau \in (0,T], \ m \geq 2.$$

Therefore,

$$\int\limits_\Omega (w^m(z,\tau))^2 dxdy \leq M_1 \int\limits_0^T (r^m(t))^2 dt, \ \tau \in (0,T], \ m \geq 2, \tag{18}$$

and

$$\int\limits_{Q_\tau} \Big(\sum_{i,j=1}^k (w^m_{x_i x_j})^2 + \sum_{i=1}^n (w^m_{z_i})^2 + (w^m)^2 \Big) dzdt \leq M_2 \int\limits_0^T (r^m(t))^2 dt, \ \tau \in (0,T], \ m \geq 2. \tag{19}$$

Formulae (14) for $t \in [0,T]$ and $m \geq 3$ imply the equalities

$$A(t) r^m(t) = \int\limits_\Omega C(z,t) w^{m-1} dz + \int\limits_\Omega K(z)(g(z,t,u^{m-1}) - g(z,t,u^{m-2}))dz. \tag{20}$$

We square both sides of these equalities and integrate the result with respect to t, then with the use of hypotheses (A 4) we obtain

$$\int\limits_0^T (r^m(t))^2 dt \leq M_3 \int\limits_{Q_T} (w^{m-1})^2 dzdt, \quad m \geq 3. \tag{21}$$

It follows from (21), (18) and (19) that

$$\int_0^T (r^m(t))^2 dt \le M_4 \int_0^T (r^{m-1}(t))^2 dt \le (M_4)^{m-2} \int_0^T (r^2(t))^2 dt, \quad m \ge 3. \tag{22}$$

It is easy to find the estimate

$$(r^m(t))^2 \le M_3 \int_\Omega (w^{m-1}(z,t))^2 dz, \quad t \in [0,T], m \ge 2, \tag{23}$$

from (20). Further, with the use of (18), from (23) we get

$$|r^m(t)| \le M_1^{\frac{1}{2}} M_3^{\frac{1}{2}} \left(\int_0^T (r^{m-1}(t))^2 dt \right)^{\frac{1}{2}}, \ t \in [0,T], m \ge 2. \tag{24}$$

By using (24), (22) and the assumption $M_4 < 1$ we can show that the estimate

$$\|q^{m+s}(t) - q^m(t); C([0,T])\| \le \sum_{i=m+1}^{m+s} \|r^i(t); C([0,T])\| \le M_1^{\frac{1}{2}} M_3^{\frac{1}{2}} \sum_{i=m+1}^{m+s} \|r^{i-1}(t); L^2(0,T)\| \le$$
$$\le \sum_{i=m+1}^{m+s} M_1^{\frac{1}{2}} M_3^{\frac{1}{2}} M_4^{\frac{i-3}{2}} \|r^2(t); L^2(0,T)\| \le \frac{M_1^{\frac{1}{2}} M_3^{\frac{1}{2}} M_4^{\frac{m-2}{2}}}{1 - M_4^{1/2}} \|r^2(t); L^2(0,T)\| \tag{25}$$

holds for all $s \in \mathbb{N}, \ m \ge 3$. Besides,

$$\int_{Q_T} \left(\sum_{i,j=1}^{k} (u^{m+s}_{x_i x_j} - u^m_{x_i x_j})^2 + \sum_{i=1}^{n} (u^{m+s}_{z_i} - u^m_{z_i})^2 + (u^{m+s} - u^m)^2 \right) dz dt \le$$
$$\le \sum_{p=m+1}^{m+s} \int_{Q_\tau} \left(\sum_{i,j=1}^{k} (w^p_{x_i x_j})^2 + \sum_{i=1}^{n} (w^p_{z_i})^2 + (w^p)^2 \right) dz dt \le$$
$$\le M_2 \sum_{p=m+1}^{m+s} \int_0^T (r^p(t))^2 dt \le M_2 \sum_{p=m+1}^{m+s} M_4^{p-2} \|r^2(t); L^2(0,T)\|^2 \le$$
$$\le \frac{M_2 M_4^{m-1}}{1 - M_4} \|r^2(t); L^2(0,T)\|^2, \quad s \in \mathbb{N}, \ m \ge 3 \tag{26}$$

and

$$\int_\Omega (u^{m+s}(z,\tau) - u^m(z,\tau))^2 dz \le \sum_{p=m+1}^{m+s} \int_\Omega (w^p(z,\tau))^2 dz \le M_1 \sum_{p=m+1}^{m+s} \int_0^T (r^p(t))^2 dt \le$$
$$\le \frac{M_1 M_4^{m-1}}{1 - M_4} \|r^2(t); L^2(0,T)\|^2, \quad \tau \in (0,T], \quad s \in \mathbb{N}, \ m \ge 3. \tag{27}$$

It follows from (25)–(27) that for any $\varepsilon > 0$, there exists m_0 such that for all $s, m \in \mathbb{N}$, $m > m_0$, the inequalities $\|q^{m+s}(t) - q^m(t); C([0,T])\| \le \varepsilon$, $\|u^{m+s} - u^m; L^2(0,T;V_1(\Omega))\| \le \varepsilon$ and $\|u^{m+s} - u^m; C([0,T]; L^2(\Omega))\| \le \varepsilon$ are true. Hence, the sequence $\{q^m\}_{m=1}^{\infty}$ is fundamental in $C([0,T])$, $\{u^m\}_{m=1}^{\infty}$ is fundamental in $L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$ and, therefore, as $m \to \infty$

$$u^m \to u \text{ in } L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega)), \quad q^m \to q \text{ in } C([0,T]). \tag{28}$$

Now, from (7) we obtain that

$$\int\limits_{Q_T} (u_t^m)^2 dzdt \le M_0 \Big(\int\limits_{\Omega} \big((u_0(z))^2 + \sum_{i=1}^{n} (u_{0,z_i}(z))^2 + \sum_{i,j=1}^{k} (u_{0,x_ix_j}(z))^2\big) dz + \\ + \int\limits_{Q_T} (f_1^2(z,t)(q^m(t))^2 + f_0^2(z,t)) dzdt \Big), \tag{29}$$

From (28) it follows that $\{q^m\}_{m=1}^{\infty}$ is bounded, therefore the right-hand side of estimate (29) is bounded with constant independent on m, so,

$$u_t^m \to u_t \text{ weakly in } L^2(Q_T). \tag{30}$$

Taking into account (28), (30), from (14) and (15) we get that the pair $(u(z,t), q(t))$ satisfies the equation (10) and the equality (5), and by virtue of Lemma 1 $(u(z,t), q(t))$ is a solution of the problem (1)–(4) in Q_T.

Uniqueness (case 1). Assume that $(u_{(1)}(z,t), q_{(1)}(t))$ and $(u_{(2)}(z,t), q_{(2)}(t))$ are two solutions to problem (1)–(4). Then the pair of functions $(\tilde{u}(z,t), \tilde{q}(t))$, where $\tilde{u}(z,t) = u_{(1)}(z,t) - u_{(2)}(z,t)$, $\tilde{q}(t) = q_{(1)}(t) - q_{(2)}(t)$, satisfies the condition $\tilde{u}(z,0) \equiv 0$, the equality

$$\int\limits_{Q_\tau} \big(\tilde{u}_t v + \sum_{i,j=1}^{k} a_{ij}(z,t) \tilde{u}_{x_ix_j} v_{x_ix_j} + \sum_{i,j=1}^{n} b_{ij}(z,t) \tilde{u}_{z_i} v_{z_j} + c(z,t)\tilde{u} v + \\ + (g(z,t,u_{(1)}) - g(z,t,u_{(2)}))v\big) dzdt = \int\limits_{Q_\tau} f_1(z,t)\tilde{q}(t) v dzdt, \ \tau \in [0,T], \tag{31}$$

for all $v \in V_1(Q_T)$ and the equality

$$A(t)\tilde{q}(t) = \int\limits_{\Omega} \Big(C(z,t)\tilde{u} + K(z)\big((g(z,t,u_{(1)}) - g(z,t,u_{(2)})\big) \Big) dz, \ t \in [0,T], \tag{32}$$

holds. After choosing $v = \tilde{u}$ in (31) we get

$$\int\limits_{Q_\tau} \Big(\tilde{u}_t \tilde{u} + \sum_{i,j=1}^{k} a_{ij}(z,t) (\tilde{u}_{x_ix_j})^2 + \sum_{i,j=1}^{n} b_{ij}(z,t) \tilde{u}_{z_i} \tilde{u}_{z_j} + c(z,t)(\tilde{u})^2 + \\ + (g(z,t,u_{(1)}) - g(z,t,u_{(2)}))\tilde{u} \Big) dzdt = \int\limits_{Q_\tau} f_1(z,t)\tilde{q}(t)\tilde{u} dzdt, \ \tau \in (0,T]. \tag{33}$$

It is easy to get from (32) and (A 4) inequality

$$\int_0^T (\tilde{q}(t))^2 dt \le M_3 \int_{Q_T} (\tilde{u})^2 dzdt, \tag{34}$$

From (33) by the same way as from (17) we got (18), (19), we find the following estimate:

$$\int_{Q_T} (\tilde{u})^2 dzdt \le \min\{M_1 T, M_2\} \int_0^T (\tilde{q}(t))^2 dt \tag{35}$$

and taking into account (34) from (35), we obtain $(1 - M_4) \int_{Q_T} (\tilde{u})^2 dzdt \le 0$. Since $M_4 < 1$, we conclude that $\int_{Q_T} (\tilde{u})^2 dzdt = 0$, hence, $u_{(1)} = u_{(2)}$ in Q_T. Then (34) implies $\tilde{q}(t) \equiv 0$, and, therefore, $q_{(1)}(t) \equiv q_{(2)}(t)$ in Q_T.

Case 2. Let now $T > T_1$, where T_1 is such a number, that $M_4 < 1$.
Existence (case 2). Let us divide the interval $[0, T]$ into a finite number of intervals $[0, T_1]$, $[T_1, 2T_1]$, $\ldots, [(N-2)T_1, (N-1)T_1]$, $[(N-1)T_1, T]$, where $NT_1 \ge T$. In the case 1 of this proof, we obtained that there exists a unique solution $(u_1(z,t), q_1(t))$ to the problem (1)–(4) in the domain Q_{T_1}.

Now, we shall prove that there exists a unique weak solution for the problem for equation (1) with conditions (2), (4) as $t \in [T_1, 2T_1]$ and with the initial condition $u(z, T_1) = u_1(z, T_1)$, $z \in \Omega$, in the domain $Q_{T_1, 2T_1} := \Omega \times (T_1, 2T_1)$. Let us change the variables $t = \tau + T_1$, $\tau \in [0, T_1]$ in this problem. Denote $q_0(\tau) = q(\tau + T_1)$, $U(z,\tau) = u(z, \tau + T_1)$, $a_{ij}^{(1)}(z,\tau) = a_{ij}(z, \tau + T_1)$, $b_{ij}^{(1)}(z,\tau) = b_{ij}(z, \tau + T_1)$, $c^{(1)}(z,\tau) = c(z, \tau + T_1)$, $g^{(1)}(z,\tau,U) = g(z, \tau + T_1, u(z, \tau + T_1))$, $f^{(1)}(z,\tau) = f(z, \tau + T_1)$, $E^{(1)}(\tau) = E(\tau + T_1)$. For the pair $(U(z,\tau), q_0(\tau))$ we obtain the problem

$$U_\tau + \sum_{i,j=1}^{k} (a_{ij}^{(1)}(z,\tau) U_{x_i x_j})_{x_i x_j} - \sum_{i,j=1}^{n} (b_{ij}^{(1)}(z,\tau) U_{z_i})_{z_j} + c^{(1)}(z,\tau) U +$$

$$+ g^{(1)}(z,\tau,U) = f_1^{(1)}(z,\tau) q_0(\tau) + f_0^{(1)}(z,\tau), \quad (z,\tau) \in Q_{T_1} \tag{36}$$

$$U(z,0) = u_1(z, T_1), \quad z \in \Omega, \tag{37}$$

$$U|_{\Sigma_{T_1}} = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial \mathcal{D}_x \times \mathcal{D}_y \times (0, T_1)} = 0, \tag{38}$$

$$\int_\Omega K(z) U(z,\tau) dz = E^{(1)}(\tau), \qquad \tau \in [0, T_1]. \tag{39}$$

It is obvious that all coefficients of the equation (36) and functions $f_1^{(1)}(z,\tau)$, $f_0^{(1)}(z,\tau)$, $u_1(z, T_1)$, $E^{(1)}(\tau)$ satisfy the same conditions as functions from (1) and (4). Therefore, from the result for the case 1 it follows that there exists a unique weak solution to the problem (36)–(39) in Q_{T_1}, and, thus for the problem for the equation (1) with conditions (2), (4) as $t \in [T_1, 2T_1]$ and with the initial condition $u(z, T_1) = u_1(z, T_1)$, $z \in \Omega$, in the domain $Q_{T_1, 2T_1}$. Denote it by $(u_2(z,t), q_2(t))$. Following similar reasoning on the intervals $[2T_1, 3T_1]$, $\ldots, [(N-1)T_1, NT_1]$, we prove the existence and the uniqueness of weak solutions $(u_k(z,t), q_k(t))$, $k = 3, \ldots, N$, in the domain $Q_{(k-1)T_1, kT_1} := \Omega \times ((k-1)T_1, kT_1)$, $k =$

$3, \dots, N-1$, and $Q_{(N-1)T_1,T} := \Omega \times ((N-1)T_1, T)$, of the inverse problem for the equation (1) with conditions (2), (4) as $t \in [(k-1)T_1, kT_1]$ and $t \in [(N-1)T_1, T]$, and the initial condition $u(z,(k-1)T_1) = u_{k-1}(z,(k-1)T_1)$, $z \in \Omega$. It is obvious that a pair of functions $(u(z,t), q(t))$, where

$$u(z,t) = \begin{cases} u_1(z,t), & \text{if } (z,t) \in Q_{T_1}; \\ u_2(z,t), & \text{if } (z,t) \in Q_{T_1,2T_1}; \\ \dots & \dots \\ u_{N-1}(z,t), & \text{if } (z,t) \in Q_{(N-2)T_1,(N-1)T_1}, \\ u_N(z,t), & \text{if } (z,t) \in Q_{(N-1)T_1,T}, \end{cases}$$

$$q(t) = \begin{cases} q_1(t), & \text{if } t \in [0, T_1]; \\ q_2(t), & \text{if } t \in [T_1, 2T_1]; \\ \dots & \dots \\ q_{N-1}(t), & \text{if } t \in [(N-2)T_1, (N-1)T_1], \\ q_N(t), & \text{if } t \in [(N-1)T_1, T], \end{cases}$$

is a weak solution for the problem (1)–(4) in the domain Q_T.

Uniqueness (case 2). The proving of the uniqueness of solution for the problem (1)–(4) in case 2 is based on the schemas of proof of the uniqueness (case 1) and existence (case 2). □

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