We shall introduce the space

\[ V_1(\Omega) = \left\{ u: u \in H_0^1(\Omega), u_{x_i} \in L^2(\Omega), i, j \in \{1, \ldots, k\}, \left. \frac{\partial u}{\partial \nu} \right|_{\partial \mathcal{D}_x \times \mathcal{D}_y} = 0 \right\}. \]

In this paper in the domain \( Q_T \) we study the following inverse problem: find the sufficient conditions of the existence and the uniqueness of a pair of functions \((u(z,t), q(t))\) that
satisfies the equation
\[ u_t + \sum_{i,j=1}^{k} (a_{ij}(z,t)u_{x_i x_j})_{x_i x_j} - \sum_{i,j=1}^{n} (b_{ij}(z,t)u_{z_i})_{z_j} + c(z,t)u + g(z,t,u) = f_1(z,t)q(t) + f_0(z,t), \] (1)
the initial, boundary and overdetermination conditions
\[ u(z,0) = u_0(z), \quad z \in \Omega, \] (2)
\[ u|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega \times (0,T)} = 0, \] (3)
\[ \int_{\Omega} K(z)u(z,t)dz = E(t), \quad t \in [0,T], \] (4)
in the sense of definition.

Definition 1. A pair of functions \((u(z,t), q(t))\) is a weak solution to the problem (1)–(4), if \(u \in L^2(0,T; V_1(\Omega)) \cap C([0,T]; L^2(\Omega))\), \(u_t \in L^2(Q_T), q \in C([0,T])\), it satisfies the equality
\[ \int_{Q_T} \left( u_t v + \sum_{i,j=1}^{k} a_{ij}(z,t)u_{x_i x_j},v_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}(z,t)u_{z_i}v_{z_j} + c(z,t)u v + g(z,t,u)v \right) dzdt = \int_{Q_T} (f_1(z,t)q(t) + f_0(z,t))vdzdt \] (5)
for all \(\tau \in (0,T]\), and all functions \(v \in L^2(0,T; V_1(\Omega))\), and the conditions (2), (4) hold.

Let the coefficients of equation (1) and the initial data satisfy conditions:

\(\text{A 1):} \quad a_{ij} \in C([0,T]; L^\infty(\Omega)), \quad a_{ij,t} \in L^\infty(Q_T), \quad a_{ij}(z,t) \geq a_0 > 0 \text{ for almost all } (z,t) \in QT, \quad i, j \in \{1, \ldots, k\};\)

\(\text{A 2):} \quad b_{ij} \in C([0,T]; L^\infty(\Omega)), \quad b_{ij,t} \in L^\infty(Q_T), \quad i, j \in \{1, \ldots, n\};\)

\[ \sum_{i,j=1}^{n} b_{ij}(z,t)\xi_i \xi_j \geq b_0 |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and for almost all } (z,t) \in QT, \quad b_0 > 0; \]

\(\text{A 3):} \quad c \in C([0,T]; L^\infty(\Omega)), \quad c(z,t) \geq c_0 \text{ for almost all } (z,t) \in QT, \quad \text{where } c_0 \text{ is a constant};\)

\(\text{A 4):} \quad g(z,t,\xi) \text{ is measurable with respect to the variables } (z,t) \text{ in } Q_T \text{ for all } \xi \in \mathbb{R}^1 \text{ and is continuous with respect to } \xi \text{ for almost all } (z,t) \in QT, \quad \text{and moreover, there exists a positive constant } g_0, \text{ such that } |g(z,t,\xi) - g(z,t,\eta)| \leq g_0 |\xi - \eta| \text{ for almost all } (z,t) \in QT \text{ and all } \xi, \eta \in \mathbb{R}^1;\)

\(\text{A 5):} \quad f_0, f_1 \in C([0,T]; L^2(\Omega));\)

\(\text{A 6):} \quad u_0 \in V_1(\Omega);\)

\(\text{A 7):} \quad K \in V_1(\Omega), \quad K_{x_i x_j} \in L^2(\Omega), \quad K_{z_r z_s} \in L^2(\Omega), \quad i, j \in \{1, \ldots, k\}, \quad r, s \in \{1, \ldots, n\};\)

\(\text{A 8):} \quad E \in H^1(0,T), \quad E(0) = \int_{\Omega} K(z)u_0(z)dz.\)
Note, that if \( q(t) = q^*(t) \), where \( q^* \in C([0,T]) \) is known function, then similarly as in [1] we can obtain the results of the existence and the uniqueness of the weak solution for the initial-boundary value problem (1)–(3).

**Theorem 1.** Under the conditions (A1)–(A6) and \( q^* \in C([0,T]) \) there exists a unique weak solution \( u^* \) to the problem (1)–(3), i.e. \( u^* \in L^2(0,T;V_1(\Omega)) \cap C([0,T]; L^2(\Omega)) \), \( u^*_t \in L^2(Q_T) \), it satisfies (2) and the equality

\[
\int_{Q_T} \left( u^*_t v + \sum_{i,j=1}^k a_{ij}(z,t)u^*_{x_i,x_j}v_{x_i,x_j} + \sum_{i,j=1}^n b_{ij}(z,t)u^*_z v_j + c(z,t)u^* v + g(z,t,u^*)v \right) dzdt =
\]

\[
= \int_{Q_T} (f_1(z,t)q^*(t) + f_0(z,t))v dzdt \tag{6}
\]

holds for all \( \tau \in (0,T) \) and all functions \( v \in L^2(0,T;V_1(\Omega)) \).

The derivative \( u^*_t \) has the estimate

\[
\int_{Q_T} (u^*_t)^2 dzdt \leq M_0 \left( \int_{\Omega} \left( (u_0(z))^2 + \sum_{i=1}^n (u_{0,z_i}(z))^2 + \sum_{i,j=1}^k (u_{0,x_i,x_j}(z))^2 \right) dz +
\]

\[
+ \int_{Q_T} \left( (f^2_1(z,t)(q^*(t))^2 + f^2_0(z,t)) dzdt \right) \tag{7}
\]

where the constant \( M_0 \) depends only on the coefficients of the left-hand side of the equation (1).

Now we shall obtain an auxiliary problem to problem (1)–(4). Denote:

\[
A(t) := \int_{\Omega} K(z)f_1(z,t)dz, \quad B(t) := E'(t) - \int_{\Omega} K(z)f_0(z,t)dz,
\]

\[
C(z,t) := \sum_{i,j=1}^k (K_{x_i,x_j}(z)a_{ij}(z,t))_{x_i,x_j} - \sum_{i,j=1}^n (K_{z_i}(z)b_{ij}(z,t))_{z_i} + K(z)c(z,t).
\]

Let \( (u(z,t), q(t)) \) be a weak solution to problem (1)–(4). From (4) it follows that

\[
\int_{\Omega} K(z)u_t(z,t)dz = E'(t), \quad t \in [0,T]. \tag{8}
\]

By using equality (5) with \( v = K(z) \) and (8), we get

\[
\int_0^\tau E'(t)dt + \int_{Q_T} \left( \sum_{i,j=1}^k a_{ij}(z,t)K_{x_i,x_j}(z)u_{x_i,x_j} + \sum_{i,j=1}^n b_{ij}(z,t)K_{z_i}(z)u_{z_i} + c(z,t)K(z)u +
\]

\[
+ g(z,t,u)K(z) \right) dzdt = \int_{Q_T} (f_1(z,t)q(t) + f_0(z,t))K(z)dzdt, \quad \tau \in (0,T]. \tag{9}
\]
After integrating by parts in (9), in view of the condition (A 7), we obtain

$$\int_0^\tau B(t)dt + \int_{Q_\tau} (C(z,t)u + g(z,t,u)K(z))dzdt = \int_0^\tau A(t)q(t)dt,$$

for all $\tau \in (0,T]$. Therefore

$$A(t)q(t) = B(t) + \int_{\Omega} (C(z,t)u + g(z,t,u)K(z))dz, \quad t \in [0,T]. \tag{10}$$

**Lemma 1.** Let the conditions (A 1)–(A 8) hold, and $a_{ij,x_j} \in C([0,T];L^2(\Omega))$, $b_{rs,x_j} \in C([0,T];L^2(\Omega))$, $i,j \in \{1,\ldots,k\}$, $r,s \in \{1,\ldots,n\}$. The pair of functions $(u(z,t), q(t))$, where $u \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u_t \in L^2(Q_T)$, $q \in C([0,T])$, is a weak solution to the problem (1)–(4) if and only if it satisfies equality (5) for all $v \in L^2(0,T;V_1(\Omega))$, $\tau \in (0,T)$ and (2), (10) hold.

**Proof.** The necessity is proved.

Let $u^* \in L^2(0,T;V_1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u^*_t \in L^2(Q_T)$, $q^* \in C([0,T])$, and they satisfy equality (5) for all $v \in L^2(0,T;V_1(\Omega))$, $\tau \in (0,T)$ and (2), (10). Then $u^*$ is a solution to the problem (1)–(3) with $q^*$ instead of $q$ in (1).

We set $E^*(t) = \int_{\Omega} K(z)u^*(z,t)dz$, $t \in [0,T]$. In exactly the same way as in the proof of necessity, we obtain

$$\int_0^\tau (E^*(t))'dt + \int_{Q_\tau} \left( \sum_{i,j=1}^k (a_{ij}(z,t)K_{x_jx_j}(z))_{x_ix_j} + \sum_{i,j=1}^n (b_{ij}(z,t)K_{x_j}(z))_{zi} + c(z,t)K(z) \right)u^* + g(z,t,u^*)K(z)dzdt = \int_{Q_\tau} (f_1(z,t)q^*(t) + f_0(z,t))K(z)dzdt, \quad t \in [0,T]. \tag{11}$$

On the other hand $q^*(t)$ and $u^*(z,t)$ satisfy (10), and therefore it is easy to get the following equality

$$\int_0^\tau E'(t)dt + \int_{Q_\tau} \left( \sum_{i,j=1}^k (a_{ij}(z,t)K_{x_jx_j}(z))_{x_ix_j} + \sum_{i,j=1}^n (b_{ij}(z,t)K_{x_j}(z))_{zi} + c(z,t)K(z) \right)u^* + g(z,t,u^*)K(z)dzdt = \int_{Q_\tau} (f_1(z,t)q^*(t) + f_0(z,t))K(z)dzdt, \quad t \in [0,T]. \tag{12}$$

It follows from (11), (12) that

$$\int_0^\tau (E^*(t) - E(t))'dt = 0, \quad \tau \in [0,T]. \tag{13}$$

Integrating (13) with the use of the equality $E^*(0) = E(0) = \int_{\Omega} K(z)u_0(z)dz$, we get $E^*(t) = E(t)$, $t \in [0,T]$. Hence, $u^*(z,t)$ satisfies (4).
Denote:

\[ f_2 := \sup_{[0,T]} \int_{\Omega} (f_1(z,t))^2dz, \quad \alpha := \begin{cases} 
0, & \text{if } c_0 + g_0 > 0.5; \\
2(1 - c_0 - g_0), & \text{if } c_0 + g_0 \leq 0.5, 
\end{cases} \]

\[ \varkappa := \alpha + 2c_0 + 2g_0 - 1, \quad M_1 := f_2 e^{\alpha T}, \quad M_2 := \frac{M_1}{\min\{2a_0, 2b_0, \varkappa\}}, \]

\[ M_3 := \frac{2}{\min\{A(t)\}^2} \left( \sup_{[0,T]} \int_{\Omega} (C(z,t))^2dz + (g_0)^2 \int_{\Omega} (K(z))^2dz \right), \]

\[ M_4 := M_1M_3 \min \left\{ \frac{1}{\min\{2a_0, 2b_0, \varkappa\}}, T \right\}. \]

**Theorem 2.** Let the conditions (A 1)–(A 8) hold, \( a_{ij,x,x} \in C([0,T]; L^2(\Omega)), \) \( b_{rs,zr} \in C([0,T]; L^2(\Omega)), \) \( i, j \in \{1, \ldots, k\}, \) \( r, s \in \{1, \ldots, n\}, \) and \( A(t) \neq 0 \) for all \( t \in [0,T]. \) Then there exists a unique weak solution to the problem (1)–(4) in the domain \( Q_T. \)

**Proof. Case 1.** First we consider the case, when \( M_4 < 1. \)

**Existence (case 1).** In order to prove the existence result we use the method of successive approximations. We construct an approximation \((u^m(z,t), q^m(t))\) to the solution of problem (1)–(4), where the functions \( q^m(t), m \in \mathbb{N}, \) satisfy equalities

\[ q^1(t) := 0, \]

\[ A(t)q^m(t) = B(t) + \int_{\Omega} C(z,t)u^{m-1}dz + \int_{\Omega} K(z)g(z,t, u^{m-1})dz, \quad t \in [0,T], \quad m \geq 2, \]

and \( u^m \) satisfies the equality

\[ \int_{Q_r} \left( u^m v + \sum_{i,j=1}^k a_{ij}(z,t)u^m_{x_i}v_{x_j} + \sum_{i,j=1}^n b_{ij}(z,t)u^m_{x_i}v_{x_j} + c(z,t)u^m v + g(z,t, u^m)v \right) dzdt = \]

\[ = \int_{Q_r} (f_1(z,t)q^m(t) + f_0(z,t))vdzdt, \quad m \geq 1, \quad \tau \in (0,T], \]

for all \( v \in L^2(0,T; V_1(\Omega)), \) and the condition

\[ u^m(z,0) = u_0(z), \quad z \in \Omega. \]

Theorem 1 yields that for each \( m \in \mathbb{N} \) there exists a unique function \( u \in L^2(0,T; V_1(\Omega)) \cap C([0,T]; L^2(\Omega)), \) \( u_t \in L^2(Q_T), \) that satisfies (15), (16).

Now we show that \( \{(u^m(z,t), q^m(t))\}_{m=1}^{\infty} \) converges to the solution of the problem (1)–(4).

Denote

\[ w^m := w^m(z,t) = u^m(z,t) - u^{m-1}(z,t), \quad r^m(t) := q^m(t) - q^{m-1}(t), \quad m \geq 2. \]

It follows from (16) that \( w^m(z,0) = 0, \) \( z \in \Omega, \) \( m \geq 2. \) Hence, from (15), we get

\[ \frac{1}{2} \int_{\Omega} (w^m(z,\tau))^2 e^{-\alpha \tau} dz + \int_{Q_r} \left( \frac{\alpha}{2} (w^m)^2 + \sum_{i,j=1}^k a_{ij}(z,t)(w^m_{x_i})^2 + \sum_{i,j=1}^n b_{ij}(z,t)w^m_{x_i}w^m_{x_j} \right) dzdt = \]

\[ = \int_{Q_r} (f_1(z,\tau) r^m(\tau) + f_0(z,\tau))vdzdt, \quad m \geq 1, \quad \tau \in (0,T], \]

\[ \int_{Q_r} (w^m_{x_i})(z,\tau) v_{x_i} dzdt \to 0, \quad m \to \infty, \quad v \in L^2(0,T; V_1(\Omega)), \]

\[ \int_{Q_r} (w^m_{x_i})(z,\tau) v_{x_i} dzdt = \int_{Q_r} (f_1(z,\tau) r^m(\tau) + f_0(z,\tau))vdzdt. \]
+c(z, t)(w^m)^2 + (g(z, t, u^m) - g(z, t, u^{m-1}))w^m)e^{-\alpha t}dzdt = 
= \int_{Q_r} f_1(z, t)r^m(t)w^m e^{-\alpha t}dzdt, \quad \tau \in (0, T], \ m \geq 2. \quad (17)

Then, taking into account (A 1)–(A 6), that under the hypotheses (A 4)

\int_{Q_r} (g(z, t, u^{m-1}) - g(z, t, u^{m-2}))w^mdzdt \leq g_0 \int_{Q_r} (w^{m-1})^2dzdt, \quad \tau \in (0, T], \ m \geq 3,

and that

\int_{Q_r} f_1(z, t)r^m(t)w^m e^{-\alpha t}dzdt \leq \frac{1}{2} \int_{Q_r} (w^m)^2 e^{-\alpha t}dzdt + \frac{f_2}{2} \int_0^T (r^m(t))^2 dt,

from (17) we get inequalities

\int_\Omega (w^m(z, \tau))^2 e^{-\alpha \tau} dz + \int_{Q_r} \left(2a_0 \sum_{i,j=1}^k (w^m_{x_ix_j})^2 + 2b_0 \sum_{i=1}^n (w^m_{z_i})^2 + \kappa (w^m)^2 \right) e^{-\alpha t}dzdt \leq 
\leq f_2 \int_0^T (r^m(t))^2 e^{-\alpha t} dt, \quad \tau \in (0, T], \ m \geq 2.

Therefore,

\int_\Omega (w^m(z, \tau))^2 dx dy \leq M_1 \int_0^T (r^m(t))^2 dt, \quad \tau \in (0, T], \ m \geq 2, \quad (18)

and

\int_{Q_r} \left( \sum_{i,j=1}^k (w^m_{x_ix_j})^2 + \sum_{i=1}^n (w^m_{z_i})^2 + (w^m)^2 \right) dzdt \leq M_2 \int_0^T (r^m(t))^2 dt, \quad \tau \in (0, T], \ m \geq 2. \quad (19)

Formulae (14) for \( t \in [0, T] \) and \( m \geq 3 \) imply the equalities

A(t)r^m(t) = \int_\Omega C(z, t)w^{m-1}dz + \int_\Omega K(z)(g(z, t, u^{m-1}) - g(z, t, u^{m-2}))dz. \quad (20)

We square both sides of these equalities and integrate the result with respect to \( t \), then with the use of hypotheses (A 4) we obtain

\int_0^T (r^m(t))^2 dt \leq M_3 \int_{Q_r} (w^{m-1})^2dzdt, \quad m \geq 3. \quad (21)
It follows from (21), (18) and (19) that
\[
\int_0^T (r^m(t))^2 dt \leq M_4 \int_0^T (r^{m-1}(t))^2 dt \leq (M_4)^{m-2} \int_0^T (r^2(t))^2 dt, \quad m \geq 3.
\] (22)

It is easy to find the estimate
\[
(r^m(t))^2 \leq M_3 \int_\Omega (w^{m-1}(z, t))^2 dz, \quad t \in [0, T], \ m \geq 2,
\] from (20). Further, with the use of (18), from (23) we get
\[
|r^m(t)| \leq M_1^{|i|} M_3^{1/2} \left( \int_0^T (r^{m-1}(t))^2 dt \right)^{1/2}, \quad t \in [0, T], \ m \geq 2.
\] (24)

By using (24), (22) and the assumption \( M_4 < 1 \) we can show that the estimate
\[
\|q^{m+s}(t) - q^m(t); C([0, T])\| \leq \sum_{i=m+1}^{m+s} \|r^i(t); C([0, T])\| \leq M_1^{|i|} M_3^{1/2} \sum_{i=m+1}^{m+s} \|r^{i-1}(t); L^2(0, T)\| \leq
\]
\[
\leq \sum_{i=m+1}^{m+s} M_1^{|i|} M_3^{1/2} M_4^{i-1} \|r^2(t); L^2(0, T)\| \leq \frac{M_1^2 M_3^{1/2} M_4^{m-2}}{1 - M_4^{1/2}} \|r^2(t); L^2(0, T)\|
\] (25)
holds for all \( s \in \mathbb{N}, \ m \geq 3 \). Besides,
\[
\int_{Q_T} \left( \sum_{i,j=1}^k (u_{x_i x_j}^{m+s} - u_{x_i x_j}^m)^2 + \sum_{i=1}^n (u_{z_i}^{m+s} - u_{z_i}^m)^2 + (w_{x_i}^{m+s} - w_{x_i}^m)^2 \right) dz dt \leq
\]
\[
\leq \sum_{p=m+1}^{m+s} \int_{Q_T} \left( \sum_{i,j=1}^k (u_{x_i x_j}^p)^2 + \sum_{i=1}^n (w_{z_i}^p)^2 + (w^p)^2 \right) dz dt \leq
\]
\[
\leq M_2 \sum_{p=m+1}^{m+s} \int_0^T (r^p(t))^2 dt \leq M_2 \sum_{p=m+1}^{m+s} M_4^{p-2} \|r^2(t); L^2(0, T)\|^2 \leq
\]
\[
\leq \frac{M_2 M_4^{m-1}}{1 - M_4} \|r^2(t); L^2(0, T)\|^2, \quad s \in \mathbb{N}, \ m \geq 3
\] (26)

and
\[
\int_\Omega (u^{m+s}(z, \tau) - u^m(z, \tau))^2 dz \leq \sum_{p=m+1}^{m+s} \int_\Omega (w^p(z, \tau))^2 dz \leq M_1 \sum_{p=m+1}^{m+s} \int_0^T (r^p(t))^2 dt \leq
\]
\[
\leq \frac{M_1 M_4^{m-1}}{1 - M_4} \|r^2(t); L^2(0, T)\|^2, \quad \tau \in (0, T], \quad s \in \mathbb{N}, \ m \geq 3.
\] (27)
It follows from (25), (26), (27) that for any \( \varepsilon > 0 \), there exists \( m_0 \) such that for all \( s, m \in \mathbb{N} \), \( m > m_0 \), the inequalities \( \| q^{m+s}(t) - q^m(t) \|_{C([0, T])} \leq \varepsilon, \| u^{m+s} - u^m ; L^2(0, T; V_1(\Omega)) \| \leq \varepsilon \) and \( \| u^{m+s} - u^m ; C([0, T]; L^2(\Omega)) \| \leq \varepsilon \) are true. Hence, the sequence \( \{ q^m \}_{m=1}^{\infty} \) is fundamental in \( C([0, T]) \), \( \{ u^m \}_{m=1}^{\infty} \) is fundamental in \( L^2(0, T; V_1(\Omega)) \cap C([0, T]; L^2(\Omega)) \) and, therefore, as \( m \to \infty \)

\[
u^m \to u \text{ in } L^2(0, T; V_1(\Omega)) \cap C([0, T]; L^2(\Omega)), \ q^m \to q \text{ in } C([0, T]).\]  

(28)

Now, from (7) we obtain that

\[
\int_{Q_T} (u^m_t)^2 \, dz \, dt \leq M_0 \left( \int_{\Omega} ((u_0(z))^2 + \sum_{i=1}^{n} (u_{0,z_i}(z))^2 + \sum_{i,j=1}^{k} (u_{0,x_i x_j}(z))^2) \, dz + \int_{Q_T} (f_1^2(z, t)(q^m(t))^2 + f_0^2(z, t)) \, dz \, dt \right),
\]

(29)

From (28) it follows that \( \{ q^m \}_{m=1}^{\infty} \) is bounded, therefore the right-hand side of estimate (29) is bounded with constant independent on \( m \), so,

\[
u_t^m \to u_t \text{ weakly in } L^2(Q_T).
\]

(30)

Taking into account (28), (30), from (14) and (15) we get that the pair \((u(z, t), q(t))\) satisfies the equation (10) and the equality (5), and by virtue of Lemma 1 \((u(z, t), q(t))\) is a solution of the problem (1)–(4) in \(Q_T\).

**Uniqueness (case I).** Assume that \((u_{(1)}(z, t), q_{(1)}(t))\) and \((u_{(2)}(z, t), q_{(2)}(t))\) are two solutions to problem (1)–(4). Then the pair of functions \((\tilde{u}(z, t), \tilde{q}(t))\), where \(\tilde{u}(z, t) = u_{(1)}(z, t) - u_{(2)}(z, t), \ \tilde{q}(t) = q_{(1)}(t) - q_{(2)}(t)\), satisfies the condition \(\tilde{u}(z, 0) \equiv 0\), the equality

\[
\int_{Q_T} (\tilde{u}_t v + \sum_{i,j=1}^{k} a_{ij}(z, t)\tilde{u}_{x_i x_j} v_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}(z, t)\tilde{u}_{z_i} v_{z_j} + c(z, t)\tilde{u} v + (g(z, t, u_{(1)}) - g(z, t, u_{(2)})) v) \, dz \, dt = \int_{Q_T} f_1(z, t)\tilde{q}(t) v \, dz \, dt, \ \tau \in [0, T],
\]

(31)

for all \( v \in V_1(Q_T) \) and the equality

\[
A(t)\tilde{q}(t) = \int_{\Omega} \left( C(z, t)\tilde{u} + K(z)(g(z, t, u_{(1)}) - g(z, t, u_{(2)})) \right) \, dz, \ t \in [0, T],
\]

(32)

holds. After choosing \( v = \tilde{u} \) in (31) we get

\[
\int_{Q_T} (\tilde{u}_t \tilde{u} + \sum_{i,j=1}^{k} a_{ij}(z, t)(\tilde{u}_{x_i x_j})^2 + \sum_{i,j=1}^{n} b_{ij}(z, t)\tilde{u}_{z_i} \tilde{u}_{z_j} + c(z, t)(\tilde{u})^2 + (g(z, t, u_{(1)}) - g(z, t, u_{(2)}))\tilde{u}) \, dz \, dt = \int_{Q_T} f_1(z, t)\tilde{q}(t) \tilde{u} \, dz \, dt, \ \tau \in (0, T],
\]

(33)
It is easy to get from (32) and (A 4) inequality
\[
\int_{0}^{T} (\dddot{q}(t))^{2} dt \leq M_{3} \int_{Q_{T}} (\dddot{u}(t))^{2} dz dt, \tag{34}
\]

From (33) by the same way as from (17) we got (18), (19), we find the following estimate:
\[
\int_{Q_{T}} (\dddot{u}(t))^{2} dz dt \leq \min\{M_{1}T, M_{2}\} \int_{0}^{T} (\dddot{q}(t))^{2} dt, \tag{35}
\]
and taking into account (34) from (35), we obtain \((1 - M_{4}) \int_{Q_{T}} (\dddot{u}(t))^{2} dz dt \leq 0\). Since \(M_{4} < 1\), we conclude that \(\int_{Q_{T}} (\dddot{u}(t))^{2} dz dt = 0\), hence, \(u_{1(t)} = u_{2(t)}\) in \(Q_{T}\). Then (34) implies \(\dddot{q}(t) \equiv 0\), and, therefore, \(q(t) \equiv q(2(t)\) in \(Q_{T}\).

**Case 2.** Let now \(T > T_{1}\), where \(T_{1}\) is such a number, that \(M_{4} < 1\).

**Existence (case 2).** Let us divide the interval \([0, T]\) into a finite number of intervals \([0, T_{1}], \ldots, [(N - 2)T_{1}, (N - 1)T_{1}], [(N - 1)T_{1}, T]\), where \(NT_{1} \geq T\). In the case 1 of this proof, we obtained that there exists a unique solution \((u_{1}(z, t), q_{1}(t))\) to the problem (1)–(4) in the domain \(Q_{T_{1}}\).

Now, we shall prove that there exists a unique weak solution for the problem for equation (1) with conditions (2), (4) as \(t \in [T_{1}, 2T_{1}]\) and with the initial condition \(u(z, T_{1}) = u_{1}(z, T_{1}), \ z \in \Omega\), in the domain \(Q_{T_{1}, 2T_{1}} := \Omega \times (T_{1}, 2T_{1})\). Let us change the variables \(t = \tau + T_{1}, \ \tau \in [0, T_{1}]\) in this problem. Denote \(q_{0}(\tau) = q(\tau + T_{1}), \ U(z, \tau) = u(z, \tau + T_{1}), \ a_{ij}^{(1)}(z, \tau) = a_{ij}(z, \tau + T_{1}), \ b_{ij}^{(1)}(z, \tau) = b_{ij}(z, \tau + T_{1}), \ c^{(1)}(z, \tau) = c(z, \tau + T_{1}), \ g^{(1)}(z, \tau, U) = g(z, \tau + T_{1}, u(z, \tau + T_{1})), \ f^{(1)}(z, \tau) = f(z, \tau + T_{1}), \ E^{(1)}(\tau) = E(\tau + T_{1})\). For the pair \((U(z, \tau), q_{0}(\tau))\) we obtain the problem

\[
U_{\tau} + \sum_{i,j=1}^{k} (a_{ij}^{(1)}(z, \tau)U_{x_{i}x_{j}})_{x_{i}x_{j}} - \sum_{i,j=1}^{n} (b_{ij}^{(1)}(z, \tau)U_{z_{i}})_{z_{j}} + c^{(1)}(z, \tau)U +
+ g^{(1)}(z, \tau, U) = f^{(1)}(z, \tau)q_{0}(\tau) + f_{0}^{(1)}(z, \tau), \quad (z, \tau) \in Q_{T_{1}} \tag{36}
\]

\[
U(z, 0) = u_{1}(z, T_{1}), \quad z \in \Omega, \tag{37}
\]

\[
U|_{\Sigma_{T_{1}}} = 0, \quad \left. \frac{\partial u}{\partial \nu} \right|_{\partial D_{z} \times D_{\nu} \times (0, T_{1})} = 0, \tag{38}
\]

\[
\int_{\Omega} K(z)U(z, \tau) dz = E^{(1)}(\tau), \quad \tau \in [0, T_{1}]. \tag{39}
\]

It is obvious that all coefficients of the equation (36) and functions \(f^{(1)}_{1}(z, \tau), \ f_{0}^{(1)}(z, \tau), \ u_{1}(z, T_{1}), \ E^{(1)}(\tau)\) satisfy the same conditions as functions from (1) and (4). Therefore, from the result for the case 1 it follows that there exists a unique weak solution to the problem (36)–(39) in \(Q_{T_{1}}\), and, thus for the problem for the equation (1) with conditions (2), (4) as \(t \in [T_{1}, 2T_{1}]\) and with the initial condition \(u(z, T_{1}) = u_{1}(z, T_{1}), \ z \in \Omega\), in the domain \(Q_{T_{1}, 2T_{1}}\). Denote it by \((u_{2}(z, t), q_{2}(t))\). Following similar reasoning on the intervals \([2T_{1}, 3T_{1}], \ldots, [(N - 1)T_{1}, NT_{1}]\), we prove the existence and the uniqueness of weak solutions \((u_{k}(z, t), q_{k}(t)), \ k = 3, \ldots, N, \) in the domain \(Q_{(k-1)T_{1}, kT_{1}} := \Omega \times ((k - 1)T_{1}, kT_{1})\), \( k = \ldots, N\).
3, ..., N − 1, and $Q_{(N-1)T_1,T} := \Omega \times ((N-1)T_1,T)$, of the inverse problem for the equation (1) with conditions (2), (4) as $t \in [(k-1)T_1,kT_1]$ and $t \in [(N-1)T_1,T]$, and the initial condition $u(z, (k-1)T_1) = u_{k-1}(z, (k-1)T_1)$, $z \in \Omega$. It is obvious that a pair of functions $(u(z,t), q(t))$, where
\[
\begin{align*}
  u(z,t) &= \begin{cases} 
    u_1(z,t), & \text{if } (z,t) \in Q_{T_1}; \\
    u_2(z,t), & \text{if } (z,t) \in Q_{T_1,2T_1}; \\
    \hdots & \hdots \\
    u_{N-1}(z,t), & \text{if } (z,t) \in Q_{(N-2)T_1,(N-1)T_1}; \\
    u_N(z,t), & \text{if } (z,t) \in Q_{(N-1)T_1,T},
  \end{cases} \\
  q(t) &= \begin{cases} 
    q_1(t), & \text{if } t \in [0,T_1]; \\
    q_2(t), & \text{if } t \in [T_1,2T_1]; \\
    \hdots & \hdots \\
    q_{N-1}(t), & \text{if } t \in [(N-2)T_1,(N-1)T_1], \\
    q_N(t), & \text{if } t \in [(N-1)T_1,T],
  \end{cases}
\end{align*}
\]
is a weak solution for the problem (1)–(4) in the domain $Q_T$.

Uniqueness (case 2). The proving of the uniqueness of solution for the problem (1)–(4) in case 2 is based on the schemas of proof of the uniqueness (case 1) and existence (case 2).

\[\square\]

REFERENCES


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