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TWO-POINT BOUNDARY VALUE PROBLEM FOR A PARTIAL DIFFERENTIAL EQUATION IN SPACES OF PERIODIC FUNCTIONS

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We investigate the two-point in time boundary value problem for the partial differential equations of the second-order with one spatial variable and constant coefficients. The problem is considered in in the spaces of functions which Fourier coefficients are characterized by exponential behavior on the Cartesian product of the time interval and spatial domain $\mathbb{R}/2\pi\mathbb{Z}$.

The correct solvability of the problem is established, the formulas for solutions are presented, the kernel is described and the smoothness of the solution is established in the spaces of functions that are periodic in one spatial variable. We have established the conditions which are close to the necessary conditions of solvability of the problem in scale of spaces of functions with exponentially increasing (or decreasing) Fourier coefficients. We also found the asymptotic estimates demonstrating the absence of the problem of small denominators, which arises of many spatial variables and makes the boundary value problem incorrect. We have established sufficient conditions of the finite-dimensionality of the kernel of the problem and found upper bounds for its dimension. The results are obtained under the condition of minimum smoothness on the right-hand sides of two-point conditions, which is close to the necessary condition.

1. Introduction. The problem of finding a solution $y(x)$ of the ordinary differential equation of the nth order which satisfies the n-point conditions

$$y(x_1) = y_1, \ \ldots, \ y(x_n) = y_n,$$

is called the *Vallée-Poussin problem* ([53]), but it was considered earlier in [44, 51]. It arises naturally in studying many physical, economic, demographic and other processes. This problem was investigated in many papers, in particular, the linear case in [1, 2, 35], the nonlinear case in [18, 20, 40], the degenerate case in [34, 36] etc. Some efficient methods of constructing the approximate solutions were elaborated in [11, 52].

Recently much attention has been paid to the problems for the partial differential equations with multipoint in time conditions. The solution of such a problem is not unique unless some additional conditions with respect to spatial variables are imposed. Solvability and smoothness of the solutions are connected with the problem of small denominators. The problems of this type are the examples of the ill-posed problems in sense of Hadamard.

The metric approach to studying the multipoint problems for the partial differential equations was developed by B. Yo. Ptashnyk et al. [45, 46, 47]. The problems with the local

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multipoint conditions were considered in [48, 50], with the nonlocal conditions in [22, 23, 27, 47] (in the complex domain in [28]), with the integral conditions in [25, 26]. The question of how to estimate the small denominators with complicated nonlinear structure is common for all types of multipoint problems ([24, 27, 47]).

The ill-posedness, in Hadamard's sense, of multipoint problems for partial differential equations is caused by the non-triviality (even by infinite-dimensionality) of the kernels. The kernels of the problems for equation of the second order in time with two-point in time conditions in the unbounded spatial domains are studied in [3, 8] (in the case of equations of finite order in the spatial variables) and in [41, 42] (in the case of equations of infinite order).

The unique solvability of the problems for partial differential equations in unbounded domains with multipoint in time conditions in the spaces of functions of exponential growth is established in [9, 38]. The operator method of studying the two-point problem in the strip in the Sobolev spaces is described in [12]. The two-point nonlocal problems for the weakly nonlinear differential-operator equations in the complex domain in the scales and specified scales of Sobolev spaces as well as in the scales of Dirichlet–Taylor spaces are studied by the Nash–Moser method in [29], [31] and [30], respectively. The solvability of the multipoint problems for the systems of quasilinear hyperbolic equations is proved by applying some fixed point theorems in [4, 5, 7, 10, 49].

The differential-symbol method of solving the Cauchy problem for partial differential equation of finite or infinite order in spatial variables was proposed in [33, 39]. Afterwards this method was used to solve the partial differential equation of the second-order in time and of the infinite order in the spatial variables, see [43].

In the present paper we consider the two-point in time boundary value problem for the partial differential equations of the second-order with one spatial variable. The correct solvability of the problem is established, the formulas for solutions are presented, the kernel is described and the smoothness of the solution is established in the spaces of functions that are periodic in one spatial variable. We also found the asymptotic estimates demonstrating the absence of the problem of small denominators, which arises in the case of many spatial variables and makes the boundary value problem incorrect [45, 47]. The results are obtained under the condition of minimum smoothness on the right-hand sides of two-point conditions, which is close to the necessary condition.

2. Basic notations and statement of the problem. Let $\mathcal{D} = (0,h) \times \Omega$ and let $\overline{\mathcal{D}} = [0,h] \times \Omega$, where $h > 0$, $\Omega = \mathbb{R}/2\pi\mathbb{Z}$.

Denote by $\mathbf{W}$ the linear space of finite sums $v = v(x) = \sum_k v_k e^{ikx}$, where v_k are the complex coefficients and $k \in \mathbb{Z}$. We call $\mathbf{W}$ a space of test functions.

The space $\mathbf{W}'$ is dual to $\mathbf{W}$; this is the space of generalized trigonometric functions (the linear continuous functionals $V\colon \mathbf{W} \to \mathbb{C}$), that are the formal trigonometric series $V(z) = \sum_{k\in\mathbb{Z}} V_k e^{ikx}$ acting on the function $v \in \mathbf{W}$ by the rule $\langle V, v\rangle = \sum_k V_k \bar{v}_k$, where the number $\bar{v}_k$ is the complex conjugate of the number v_k ([17, p. 59–61]).

For real numbers q, α, positive integer n and a function $\beta\colon [0,h] \to \mathbb{R}$ we consider the following weighted function spaces:

- the Hilbert space $\mathbf{E}^q_\alpha = \mathbf{E}^q_\alpha(\Omega)$ of periodic functions $v = v(x) = \sum_{k\in\mathbb{Z}} v_k e^{ikx}$ endowed with the scalar product $(v,w)_{\mathbf{E}^q_\alpha} = \sum_{k\in\mathbb{Z}} (1+k^2)^q e^{2\alpha|k|} v_k \bar{w}_k$, $w = w(x) = \sum_{k\in\mathbb{Z}} w_k e^{ikx}$;
- the Banach space $\mathbf{E}^{n,q}_\beta = \mathbf{E}^{n,q}_\beta(\overline{\mathcal{D}})$ of functions $u = u(t,x)$ on $\overline{\mathcal{D}}$ such that their partial derivatives $\partial^r_t u(t,\cdot)$ (where $\partial_t = \partial/\partial t$) defined for $r = 0,1,\ldots,n$ by the formula

$\partial_t^r u(t,x) = \sum_{k\in\mathbb{Z}} u_k^{(r)}(t)e^{ikx}$ belong to the Hilbert spaces $\mathbf{E}^{q-r}_{\beta(t)}(\Omega)$ for every t, respectively, and are continuous in t in these spaces. The squared norm of the function u in the space $\mathbf{E}^{n,q}_{\beta}$ is calculated by the formula

$$\|u\|^2_{\mathbf{E}^{n,q}_{\beta}} = \sum_{r=0}^{n} \max_{t\in[0,h]} \left\|\partial_t^r u(t,\cdot)\right\|^2_{\mathbf{E}^{q-r}_{\beta(t)}} .$$

Note that if $\psi \in \mathbf{E}^q_\alpha$, then $\partial_x^s\psi \in \mathbf{E}^{q-s}_\alpha$ for all $s \in \mathbb{N}$ (where $\partial_x = \partial/\partial x$).

Consider in the domain $\mathcal{D}$ the following problem with two-point boundary conditions:

$$L(\partial_t,\partial_x)u \equiv \partial_t^2 u - 2a(\partial_x)\partial_t u + b(\partial_x)u = 0, \tag{1}$$

$$L_1(\partial_t,\partial_x)u\big|_{t=0} \equiv a_1(\partial_x)\partial_t u + b_1(\partial_x)u\big|_{t=0} = \varphi_1, \tag{2}$$

$$L_2(\partial_t,\partial_x)u\big|_{t=h} \equiv a_2(\partial_x)\partial_t u + b_2(\partial_x)u\big|_{t=h} = \varphi_2, \tag{3}$$

where

$$\begin{aligned} a(\partial_x) =& a_0\partial_x + a_1, & b(\partial_x) =& b_0\partial_x^2 + b_1\partial_x + b_2, & \{a_0,a_1,b_0,b_1,b_2\} \subset \mathbb{C} \\ a_\alpha(\partial_x) =& a_{\alpha 0}\partial_x + a_{\alpha 1}, & b_\alpha(\partial_x) =& b_{\alpha 0}\partial_x^2 + b_{\alpha 1}\partial_x + b_{\alpha 2}, & \{a_{\alpha 0},a_{\alpha 1},b_{\alpha 0},b_{\alpha 1},b_{\alpha 2}\} \subset \mathbb{C}, \quad \alpha = 1,2, \end{aligned}$$

$\varphi_\alpha = \varphi_\alpha(x)$ are the given functions defined on Ω and $u = u(t,x)$ is the unknown function.

The conditions (1) and (2) can be interpreted as fixing the spatial tensions for a one-dimensional periodically oscillating body at two different moments of time [6, 13, 16, 37]. These oscillation processes are described by the wave equation and some other equations [19, p. 25], [32, p. 77] of the form (1). Problems with similar conditions were studied also in [14, 15].

In view of the form of $\mathcal{D}$ and Ω, we impose the conditions of 2π–periodicity with respect to the variable x on the function u and functions φ_1, φ_2, and since

$$L(\partial_t,\partial_x)u \in \mathbf{E}^{0,q-2}_{\beta}(\overline{\mathcal{D}}), \quad L_1(\partial_t,\partial_x)u\big|_{t=0} \in \mathbf{E}^{q-2}_{\beta(0)}(\Omega), \quad L_2(\partial_t,\partial_x)u\big|_{t=h} \in \mathbf{E}^{q-2}_{\beta(h)}(\Omega)$$

for any element $u \in \mathbf{E}^{3,q}_{\beta}(\overline{\mathcal{D}})$, it is naturally to use the spaces $\mathbf{E}^q_\alpha(\Omega)$ and $\mathbf{E}^{2,q}_\beta(\overline{\mathcal{D}})$.

Definition 1. A function $u \in \mathbf{C}^2([0,h];\mathbf{W}')$ is called a *solution of the problem* (1)–(3) if it satisfies the equation (1) on $[0,T]$, the conditions (2), (3) on the space $\mathbf{W}'$ and belongs to the space $\mathbf{E}^{2,q}_\beta(\overline{\mathcal{D}})$.

The solution u depends on two vectors

$$\vec{a} = (a_0,a_1,a_{10},a_{11},a_{20},a_{21}), \quad \vec{b} = (b_0,b_1,b_2,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22}).$$

The components of these vectors will be treated as the parameters of the problem which change in the bounded fixed domain. For the existence of a solution of the problem in the space $\mathbf{E}^{2,q}_\beta(\overline{\mathcal{D}})$ it is necessary that the right-hand sides of conditions (2), (3) have the following smoothness: $\varphi_1 \in \mathbf{E}^{q-2}_{\beta(0)}(\Omega)$ and $\varphi_2 \in \mathbf{E}^{q-2}_{\beta(h)}(\Omega)$.

3. Construction of solution and asymptotic estimates. We find a solution of problem (1), (2) in the form of series in the space $\mathbf{C}^2([0,h];\mathbf{W}')$

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}, \tag{4}$$

where the coefficients $u_k(t)$ are the unknown functions, which will be determined by the method of separation of variables.

From the definition of solution of problem (1)–(3) it follows that the function $u_k = u_k(t)$ is the solution of the corresponding two-point problem for the ordinary differential equation

$$L\Big(\frac{d}{dt}, ik\Big)u_k \equiv u_k'' - 2a(ik)u_k' + b(ik)u_k = 0, \qquad a(ik) = ika_0 + a_1, \tag{5}$$

$$L_1\Big(\frac{d}{dt}, ik\Big)u_k\big|_{t=0} \equiv a_1(ik)u_k'(0) + b_1(ik)u_k(0) = \hat{\varphi}_{1k}, \quad a_1(ik) = ika_{10} + a_{11}, \tag{6}$$

$$L_2\Big(\frac{d}{dt}, ik\Big)u_k\big|_{t=h} \equiv a_2(ik)u_k'(h) + b_2(ik)u_k(h) = \hat{\varphi}_{2k}, \quad a_2(ik) = ika_{20} + a_{21}, \tag{7}$$

where $b(ik) = (ik)^2 b_0 + ikb_1 + b_2$,

$$b_1(ik) = (ik)^2 b_{10} + ikb_{11} + b_{12}, \quad b_2(ik) = (ik)^2 b_{20} + ikb_{21} + b_{22},$$

and the complex numbers $\hat{\varphi}_{1k}$, $\hat{\varphi}_{2k}$ are the Fourier coefficients for the functions φ_1, φ_2.

The uniqueness of the solution u_k of the problem (5)–(7) for the ordinary differential equation in the space $\mathbf{C}^2[0,h]$ for all $k \in \mathbb{Z}$ is necessary and sufficient condition for the uniqueness of the solution of the original problem in the space $\mathbf{C}^2([0,h];\mathbf{W}')$.

Denote by $\mathcal{O}_A$ $(A > 0)$ the closed disk $\{z\colon |z| \le A\} \subset \mathbb{C}$ of some radius A with center at the origin of the the complex plane and assume for convenience that $\vec{a} \in \mathcal{O}_1^6$ and $\vec{b} \in \mathcal{O}_1^9$.

In order to construct and estimate the solutions of problem (5)–(7) for $k \ne 0$ and $\alpha = 1, 2$, we introduce the following notations:

$$\begin{aligned} \tilde{a}(k) &= (ik)^{-1}a(ik), & a_0^*(k) &= k(\tilde{a}(k) - a_0), \\ \tilde{b}(k) &= (ik)^{-2}b(ik), & b_0^*(k) &= k(\tilde{b}(k) - b_0), \\ \tilde{a}_\alpha(k) &= (ik)^{-1}a_\alpha(ik), & a_{\alpha 0}^*(k) &= k(\tilde{a}_\alpha(k) - a_{\alpha 0}), \\ \tilde{b}_\alpha(k) &= (ik)^{-2}b_\alpha(ik), & b_{\alpha 0}^*(k) &= k(\tilde{b}_\alpha(k) - b_{\alpha 0}). \end{aligned}$$

Then we have $\{a_0^*(k), a_{\alpha 0}^*(k)\} \subset \mathcal{O}_1$ for all $k \in \mathbb{Z}$ and

$$\{\tilde{a}(k), \tilde{a}_\alpha(k), b_0^*(k), b_{\alpha 0}^*(k)\} \subset \mathcal{O}_{3/2}, \qquad \{\tilde{b}(k), \tilde{b}_\alpha(k)\} \subset \mathcal{O}_{7/4}$$

for all $|k| \ge 2$ and $\alpha = 1, 2$.

The solutions $\lambda_1(k)$, $\lambda_2(k)$ of the quadratic equation

$$\tilde{L}_k(\lambda) \equiv \lambda^2 - 2\tilde{a}(k)\lambda + \tilde{b}(k) = 0$$

are determined by the formula $\lambda_{1,2}(k) = \tilde{a}(k) \pm \sqrt{D(k)}$, where $D(k) = \tilde{a}^2(k) - \tilde{b}(k)$ and the square root is chosen so that $\nu_1(k) \ge \nu_2(k)$, where $\nu_1(k) = \operatorname{Im}\lambda_1(k)$, $\nu_2(k) = \operatorname{Im}\lambda_2(k)$.

If $|k| \ge 2$, then the roots of the polynomial $\tilde{L}_k$ satisfy [21, p. 365] the estimate

$$|\lambda_\alpha(k)| \le \max\big\{1 + 2|\tilde{a}(k)|, |\tilde{b}(k)|\big\} \le 4.$$

Since $D(k) = D_0 + (1/k)D_0^*(k)$, where

$$D_0 = a_0^2 - b_0, \qquad D_0^*(k) = 2a_0 a_0^*(k) + a_0^{*2}(k)/k - b_0^*(k),$$

we have $\lambda_{1,2}(k) = \lambda_{1,2;0} + \frac{1}{k}\lambda^*_{1,2;0}$, where

$$\lambda_{1,2;0} = a_0 \pm \sqrt{D_0}, \qquad \operatorname{Im}\lambda_{10} \geq \operatorname{Im}\lambda_{20}, \qquad |\lambda_{1,2;0}| \leq 1+\sqrt{2},$$

and if $D_0 \neq 0$, then $z = \frac{1}{k}\frac{D^*_0(k)}{D_0}$, where $\sqrt{1+z}\,\big|_{z=0} = 1$, and

$$\lambda^*_{1,2;0} = a^*_0(k) \pm \frac{D^*_0(k)}{\sqrt{D(k)}+\sqrt{D_0}} = a^*_0(k) \pm \frac{D^*_0(k)}{\sqrt{D_0}\big(\sqrt{1+z}+1\big)}.$$

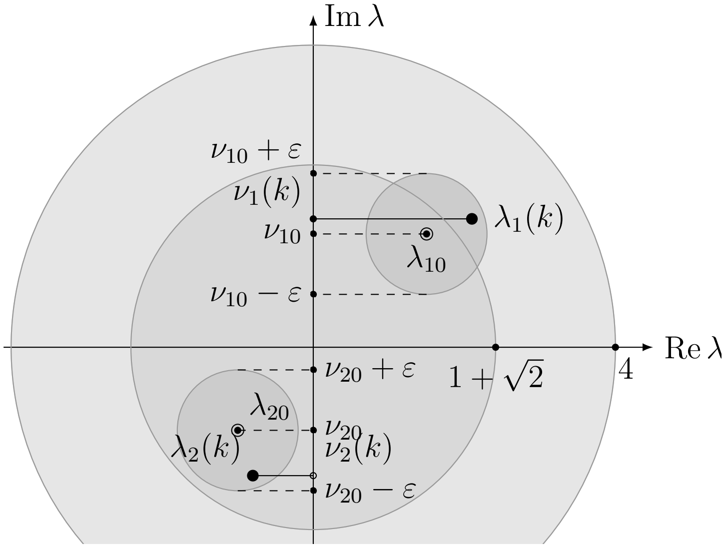

Figure 1: The location of roots of quadratic equations: λ_{10}, λ_{20} are located in the middle disk, $\lambda_1(k)$, $\lambda_2(k)$ are in the big disk and if $|k| \geq K_1(\varepsilon)$, then $\lambda_1(k)$ is located on the top small disk and $\lambda_2(k)$ is on the bottom small disk.

It is obvious that $|\sqrt{1+z}+1| \geq 3/2$ for all $z \in \mathcal{O}_{3/4}$ and that $|D^*_0(k)| \leq 4$ for all $k \in \mathbb{Z}\setminus\{0\}$. Then

$$|\lambda^*_{\alpha 0}| \leq 1 + \frac{4}{\sqrt{|D_0|}\,\big|\sqrt{1+z}+1\big|} \leq 1 + \frac{8/3}{\sqrt{|D_0|}} \quad \text{for} \quad |k| \geq K_1 = \max\Big\{2, \frac{16/3}{|D_0|}\Big\} \geq 2.$$

From this estimate it follows that (see Figure 1) for any $\varepsilon > 0$

$$|\lambda_\alpha(k) - \lambda_{\alpha 0}| = \frac{1}{|k|}|\lambda^*_{\alpha 0}| \leq \varepsilon, \qquad \alpha = 1,2, \tag{8}$$

if $|k| \geq K_1(\varepsilon) = \max\Big\{\frac{1}{\varepsilon}\Big(1+\frac{8/3}{\sqrt{|D_0|}}\Big), K_1\Big\}$.

Let $L_{\alpha k}(\lambda) \equiv L_\alpha(\lambda, ik) = -k^2\big((ik)^{-1}\tilde{a}_\alpha(k)\lambda + \tilde{b}_\alpha(k)\big)$, where $\alpha = 1,2$, and let $\mathcal{K}$ be the set of integers k for which the polynomial $\tilde{L}_k$ has a multiple root. Then $D(k) \neq 0$ and $\lambda_1(k) \neq \lambda_2(k)$ for $k \in \mathbb{Z}\setminus\mathcal{K}$.

If we take $0 < \varepsilon < (\nu_{10} - \nu_{20})/2$, then we obtain, after using (8), that

$$\mathcal{K} \subset \mathcal{K}_1(\varepsilon) = \{k \in \mathbb{Z} \colon |k| < K_1(\varepsilon)\}, \quad |\lambda_1(k) - \lambda_2(k)| \geq \nu_1(k) - \nu_2(k) \geq \nu_{10} - \nu_{20} - 2\varepsilon > 0,$$

where $\nu_{10} = \operatorname{Im}\lambda_{10}$, $\nu_{20} = \operatorname{Im}\lambda_{20}$.

Let $\mathcal{K}^0$ be the set of $k \in \mathbb{Z}$ for which $\det\delta_k = 0$, where

$$\det\delta_k = \big(a_2(ik)b_1(ik) - a_1(ik)b_2(ik) + hL_{1k}(a(ik))L_{2k}(a(ik))\big)e^{ha(ik)}.$$

For $k \in \mathcal{K}\setminus\mathcal{K}^0$ the solution of the problem (5)–(7) is given by the formula

$$u_k(t) = \frac{\delta_{1k}(t)}{\det\delta_k}\hat\varphi_{1k} + \frac{\delta_{2k}(t)}{\det\delta_k}\hat\varphi_{2k},\ \delta_k = \begin{pmatrix} L_{1k}(a(ik)) & a_1(ik) \\ L_{2k}(a(ik))e^{ha(ik)} & \big(a_2(ik) + hL_{2k}(a(ik))\big)e^{ha(ik)} \end{pmatrix}, \quad (9)$$

where $\delta_{1k}(t) = \big(a_2(ik) + (h-t)L_{2k}\big(a(ik)\big)e^{(h+t)a(ik)}$, $\delta_{2k}(t) = \big(tL_{1k}\big(a(ik) - a_1(ik)\big)e^{ta(ik)}$. For $k \in (\mathbb{Z}\setminus\mathcal{K})\setminus\mathcal{K}^0$ the formula for solution has the form

$$u_k(t) = \frac{\delta_{1k}(t)}{\det\delta_k}\hat\varphi_{1k} + \frac{\delta_{2k}(t)}{\det\delta_k}\hat\varphi_{2k},\ \delta_k = \begin{pmatrix} L_{1k}(ik\lambda_1(k)) & L_{1k}(ik\lambda_2(k)) \\ L_{2k}(ik\lambda_1(k))e^{ikh\lambda_1(k)} & L_{2k}(ik\lambda_2(k))e^{ikh\lambda_2(k)} \end{pmatrix}, \quad (10)$$

where

$$\det\delta_k = L_{1k}\big(ik\lambda_1(k)\big)L_{2k}\big(ik\lambda_2(k)e^{ikh\lambda_2(k)} - L_{1k}\big(ik\lambda_2(k)\big)L_{2k}\big(ik\lambda_1(k)e^{ikh\lambda_1(k)},$$
$$\delta_{1k}(t) = L_{2k}\big(ik\lambda_2(k)e^{ik(t\lambda_1(k)+h\lambda_2(k))} - L_{2k}\big(ik\lambda_1(k)e^{ik(h\lambda_1(k)+t\lambda_2(k))},$$
$$\delta_{2k}(t) = L_{1k}\big(ik\lambda_1(k)\big)e^{ikt\lambda_2(k)} - L_{1k}\big(ik\lambda_2(k)e^{ikt\lambda_1(k)}.$$

If $k \in \mathbb{Z}\setminus\mathcal{K}$ and is such that $kR_1(k)R_2(k)\tilde b(k) \neq 0$, where for $\alpha = 1, 2$,

$$R_\alpha(k) = L_{\alpha k}\big(ik\lambda_1(k)\big)L_{\alpha k}\big(ik\lambda_2(k)\big) = k^4\big(\tilde a_\alpha^2(k)\tilde b(k) + 2\tilde a(k)\tilde a_\alpha(k)\tilde b_\alpha(k) + \tilde b_\alpha^2(k)\big)$$

is the resultant of the polynomials $L_{\alpha k}$ and $L(\cdot, ik)$, then in the case $k > 0$ the formula for solution (10) and its derivatives has the following form:

$$u_k^{(j)}(t) = \frac{e^{ik\lambda_1(k)t}}{\lambda_1^{-j}(k)}\frac{L_{1k}(ik\lambda_2(k))}{R_1(k)}\frac{\Delta_{1k}^{j+}(t)}{\Delta_k^+}\frac{\hat\varphi_{1k}}{(ik)^{-j}} + \frac{e^{-ik\lambda_2(k)(h-t)}}{\lambda_2^{-j}(k)}\frac{L_{2k}(ik\lambda_1(k))}{R_2(k)}\frac{\Delta_{2k}^{j+}(t)}{\Delta_k^+}\frac{\hat\varphi_{2k}}{(ik)^{-j}}, \quad (11)$$

where

$$\Delta_{1k}^{j+}(t) = 1 - \frac{L_{2k}^2(ik\lambda_1(k))}{R_2(k)}\frac{\lambda_2^{2j}(k)}{\tilde b^j(k)}e^{i2kD(k)(h-t)}, \quad j = 1, 2,$$
$$\Delta_{2k}^{j+}(t) = 1 - \frac{L_{1k}^2(ik\lambda_2(k))}{R_1(k)}\frac{\lambda_1^{2j}(k)}{\tilde b^j(k)}e^{i2kD(k)t}, \quad j = 1, 2,$$
$$\Delta_k^+ = 1 - \frac{L_{1k}^2(ik\lambda_2(k))L_{2k}^2(ik\lambda_1(k))}{R_1(k)R_2(k)}e^{i2kD(k)h} \neq 0$$

and in the case $k < 0$

$$u_k^{(j)}(t) = \frac{e^{ik\lambda_2(k)t}}{\lambda_2^{-j}(k)}\frac{L_{1k}(ik\lambda_1(k))}{R_1(k)}\frac{\Delta_{1k}^{j-}(t)}{\Delta_k^-}\frac{\hat\varphi_{1k}}{(ik)^{-j}} + \frac{e^{-ik\lambda_1(k)(h-t)}}{\lambda_1^{-j}(k)}\frac{L_{2k}(ik\lambda_2(k))}{R_2(k)}\frac{\Delta_{2k}^{j-}(t)}{\Delta_k^-}\frac{\hat\varphi_{2k}}{(ik)^{-j}}, \quad (12)$$

where

$$\Delta_{1k}^{j-}(t) = 1 - \frac{L_{2k}^2(ik\lambda_2(k))}{R_2(k)}\frac{\lambda_1^{2j}(k)}{\tilde b^j(k)}e^{-i2kD(k)(h-t)}, \qquad j = 1, 2,$$

$$\Delta_{2k}^{j-}(t) = 1 - \frac{L_{1k}^2(ik\lambda_1(k))}{R_1(k)} \frac{\lambda_2^{2j}(k)}{\tilde{b}^j(k)} e^{-i2kD(k)t}, \qquad j = 1, 2,$$
$$\Delta_k^- = 1 - \frac{L_{1k}^2(ik\lambda_1(k)) L_{2k}^2(ik\lambda_2(k))}{R_1(k)R_2(k)} e^{-i2kD(k)h} \neq 0.$$

Let us estimate derivatives of the solution $u_k(t)$ of the problem (5)–(7). We use the following relations:

$$L_{\alpha k}\big(ik\lambda_\beta(k)\big) = -k^2\big(L_{\alpha\beta 0} + (1/k)L_{\alpha\beta 0}^*\big), \qquad R_\alpha(k) = k^4\big(R_{\alpha 0} + (1/k)R_{\alpha 0}^*\big), \quad \{\alpha, \beta\} \subset \{1, 2\},$$

where

$$L_{\alpha\beta 0} = a_{\alpha 0}\lambda_{\beta 0} + b_{\alpha 0}, L_{\alpha\beta 0}^* = a_{\alpha 0}^*\lambda_{\beta 0} + a_{\alpha 0}\lambda_{\beta 0}^* + (1/k)a_{\alpha 0}^*\lambda_{\beta 0}^*, R_{\alpha 0} = a_{\alpha 0}^2 b_0 + 2a_0 a_{\alpha 0} b_{\alpha 0} + b_{\alpha 0}^2,$$
$$R_{\alpha 0}^* = \big(a_{\alpha 0} + \tilde{a}(k)\big)a_{\alpha 0}^* b_0 + \tilde{a}^2(k)b_0^* + 2(a_0^* a_{\alpha 0} + \tilde{a}(k)a_{\alpha 0}^*)b_{\alpha 0} + \big(2\tilde{a}(k)\tilde{a}_\alpha(k) + b_{\alpha 0} + \tilde{b}(k)\big)b_{\alpha 0}^*,$$

from which it follows that $\big|L_{\alpha k}\big(ik\lambda_\beta(k)\big)\big| \leq 8k^2$ for $|k| \geq 2$, $|R_{\alpha 0}^*| < 21$, and also that $|R_\alpha(k)| \geq k^4|R_{\alpha 0}|/2$ for $|k| \geq 42$.

Using the obtained estimates and the inequality

$$\nu_{20} - \varepsilon < \nu_2(k) < \nu_{20} + \varepsilon < \nu_{10} - \varepsilon < \nu_1(k) < \nu_{10} + \varepsilon, \qquad |k| \geq K_1(\varepsilon),$$

for an arbitrary $0 < \varepsilon < (\nu_{10} - \nu_{20})/2$ and letting $K_2(\varepsilon) = \max\{K_1(\varepsilon), 42\}$, we find that

$$|u_k^{(j)}(t)| \leq \frac{4^{j+2}}{k^{2-j}}\Big(e^{-k(\nu_{10}-\varepsilon)t}\frac{|\Delta_{1k}^{j+}(t)\hat{\varphi}_{1k}|}{|\Delta_k^+ R_{10}|} + e^{k(\nu_{20}+\varepsilon)(h-t)}\frac{|\Delta_{2k}^{j+}(t)\hat{\varphi}_{2k}|}{|\Delta_k^+ R_{20}|}\Big) \quad \text{for} \quad k \geq K_2(\varepsilon), \quad (13)$$
$$|u_k^{(j)}(t)| \leq \frac{4^{j+2}}{k^{2-j}}\Big(e^{-k(\nu_{20}+\varepsilon)t}\frac{|\Delta_{1k}^{j-}(t)\hat{\varphi}_{1k}|}{|\Delta_k^- R_{10}|} + e^{k(\nu_{10}-\varepsilon)(h-t)}\frac{|\Delta_{2k}^{j-}(t)\hat{\varphi}_{2k}|}{|\Delta_k^- R_{20}|}\Big) \quad \text{for} \quad k \leq -K_2(\varepsilon). \quad (14)$$

Basing on the estimates $|\Delta_{1k}^{\pm}(t)| \leq 1 + 2^{7+5j}/|R_{20}b_0^j|$, $|\Delta_{2k}^{\pm}(t)| \leq 1 + 2^{7+5j}/|R_{10}b_0^j|$, we can rewrite formulas (13) and (14) for the case

$$|k| \geq K_3(\varepsilon) = \max\{K_2(\varepsilon), 8\ln 2/(\nu_{10} - \nu_{20} - 2\varepsilon)\}.$$

Letting $C_1 \equiv 2^{22}/|R_{10}R_{20}|$, we get

$$|u_k^{(j)}(t)|^2 \leq \frac{2C_1^2}{|b_0|^{2j}}(1 + k^2)^{j-2}\big(e^{2(\varepsilon-\nu_{10})t|k|}|\hat{\varphi}_{1k}|^2 + e^{2(\nu_{20}+\varepsilon)(h-t)|k|}|\hat{\varphi}_{2k}|^2\big) \quad \text{for} \quad k \geq K_3(\varepsilon), \quad (15)$$

$$|u_k^{(j)}(t)|^2 \leq \frac{2C_1^2}{|b_0|^{2j}}(1 + k^2)^{j-2}\big(e^{2(\nu_{20}+\varepsilon)t|k|}|\hat{\varphi}_{1k}|^2 + e^{2(\varepsilon-\nu_{10})(h-t)|k|}|\hat{\varphi}_{2k}|^2\big) \quad \text{for } k \leq -K_3(\varepsilon). \quad (16)$$

4. Solvability of the problem. The last step in solving the problem (1)–(3) is to determine the solutions of the problem (5)–(7) for $k \in \mathcal{K}^0$, i.e., in the case of the degenerate matrix δ_k which rank is either zero ($\delta_k = 0$, $k \in \mathcal{K}^{00}$), or one ($k \in \mathcal{K}^{10}$). In this case the solution does not exist or is not unique. To describe all solutions, we let $\delta_k = \left(\begin{smallmatrix} \Gamma_{11}(k) & \Gamma_{12}(k) \\ \Gamma_{21}(k) & \Gamma_{22}(k) \end{smallmatrix}\right)$, and denote by $G_{ij}(k)$ the conjugate of the complex number $\Gamma_{ij}(k)$, and by $(\gamma_{1k}, \gamma_{2k})$ an arbitrary vector in the space $\mathbb{C}^2$.

If $\delta_k = 0$ ($k \in \mathcal{K}^{00}$), then the necessary and sufficient condition for the existence of solution is that

$$\hat{\varphi}_{1k} = \hat{\varphi}_{2k} = 0. \tag{17}$$

The space of solutions (the kernel of problem (5)–(7)), in this case, is two-dimensional and consists of all the functions

$$u_k(t) = e^{ta(ik)}(\gamma_{1k} + t\gamma_{2k}), \qquad k \in \mathcal{K}, \tag{18}$$

$$u_k(t) = e^{ikt\lambda_1(k)}\gamma_{1k} + e^{ikt\lambda_2(k)}\gamma_{2k}, \qquad k \notin \mathcal{K}. \tag{19}$$

If δ_k has rank one ($k \in \mathcal{K}^{10}$), then it also has nonzero row and nonzero column. The necessary and sufficient condition for the solvability is the following condition of proportionality:

$$\Gamma_{21}(k)\hat{\varphi}_{1k} = \Gamma_{11}(k)\hat{\varphi}_{2k}, \qquad \Gamma_{22}(k)\hat{\varphi}_{1k} = \Gamma_{12}(k)\hat{\varphi}_{2k}. \tag{20}$$

The solutions of the problem in the case $S_1(k) \equiv |\Gamma_{11}(k)|^2 + |\Gamma_{12}(k)|^2 > 0$ have the form:

$$u_k(t) = e^{ta(ik)} \begin{pmatrix} 1 & t \end{pmatrix} \left(\begin{pmatrix} G_{11}(k) \\ G_{12}(k) \end{pmatrix} \frac{\hat{\varphi}_{1k}}{S_1(k)} + \begin{pmatrix} |\Gamma_{12}(k)|^2 & -G_{11}(k)\Gamma_{12}(k) \\ -\Gamma_{11}(k)G_{12}(k) & |\Gamma_{11}(k)|^2 \end{pmatrix} \begin{pmatrix} \gamma_{1k} \\ \gamma_{2k} \end{pmatrix} \right) \tag{21}$$

for $k{\in}\mathcal{K}$. If $S_1(k) = 0$, then $S_2(k) \equiv |\Gamma_{21}(k)|^2 + |\Gamma_{22}(k)|^2 > 0$ and

$$u_k(t) = e^{ta(ik)} \begin{pmatrix} 1 & t \end{pmatrix} \left(\begin{pmatrix} G_{21}(k) \\ G_{22}(k) \end{pmatrix} \frac{\hat{\varphi}_{2k}}{S_2(k)} + \begin{pmatrix} |\Gamma_{22}(k)|^2 & -G_{21}(k)\Gamma_{22}(k) \\ -\Gamma_{21}(k)G_{22}(k) & |\Gamma_{21}(k)|^2 \end{pmatrix} \begin{pmatrix} \gamma_{1k} \\ \gamma_{2k} \end{pmatrix} \right) \tag{22}$$

for $k{\in}\mathcal{K}$.

In the case $k \in \mathcal{K}^0 \setminus \mathcal{K}$, we have the analogical formulas:

$$u_k(t) = \begin{pmatrix} e^{ikt\lambda_1(k)} & e^{ikt\lambda_2(k)} \end{pmatrix} \left(\begin{pmatrix} G_{11}(k) \\ G_{12}(k) \end{pmatrix} \frac{\hat{\varphi}_{1k}}{S_1(k)} + \begin{pmatrix} |\Gamma_{12}(k)|^2 & -G_{11}(k)\Gamma_{12}(k) \\ -\Gamma_{11}(k)G_{12}(k) & |\Gamma_{11}(k)|^2 \end{pmatrix} \begin{pmatrix} \gamma_{1k} \\ \gamma_{2k} \end{pmatrix} \right), \tag{23}$$

$$u_k(t) = \begin{pmatrix} e^{ikt\lambda_1(k)} & e^{ikt\lambda_2(k)} \end{pmatrix} \left(\begin{pmatrix} G_{21}(k) \\ G_{22}(k) \end{pmatrix} \frac{\hat{\varphi}_{2k}}{S_2(k)} + \begin{pmatrix} |\Gamma_{22}(k)|^2 & -G_{21}(k)\Gamma_{22}(k) \\ -\Gamma_{21}(k)G_{22}(k) & |\Gamma_{21}(k)|^2 \end{pmatrix} \begin{pmatrix} \gamma_{1k} \\ \gamma_{2k} \end{pmatrix} \right). \tag{24}$$

Theorem 1 (Existence of formal solution)**.** *Let the functions φ_1 and φ_2 belong to the space $\mathbf{W}'$ and satisfy the conditions* (20) *for $k \in \mathcal{K}^0$ as well as the condition* (17) *for $k \in \mathcal{K}^{00}$, then a solution of the problem* (1)–(3) *exists in the space $\mathbf{C}^2([0,h]; \mathbf{W}')$. The solution is unique, if $\mathcal{K}^0 = \varnothing$.*

Proof. The solution has the form (4) with the coefficients u_k calculated by formulas (9) and (10), if $k \in \mathbb{Z} \setminus \mathcal{K}^0$, and by formulas (21)–(24), if $k \in \mathcal{K}^0$.

Assume that $\mathcal{K}^0 \neq \varnothing$. Then for any $k \in \mathcal{K}^0$ the corresponding homogeneous problem (1)–(3) has the nontrivial solution, that belongs to the set

$$\bigl\{e^{ta(ik)}(|\Gamma_{12}(k)|^2 - tG_{11}(k)\Gamma_{12}(k)), \quad e^{ta(ik)}(|\Gamma_{22}(k)|^2 - tG_{21}(k)\Gamma_{22}(k))\bigr\},$$

if $k \in \mathcal{K}$, or belongs to the set

$$\bigl\{e^{ikt\lambda_1(k)}(|\Gamma_{12}(k)|^2 - e^{ikt\lambda_2(k)}G_{11}(k)\Gamma_{12}(k)), \quad e^{ikt\lambda_1(k)}|\Gamma_{22}(k)|^2 - e^{ikt\lambda_1(k)}G_{21}(k)\Gamma_{22}(k)\bigr\}, \tag{25}$$

if $k \notin \mathcal{K}$. □

Theorem 2 (Existence of solution)**.** *Let vectors $\vec{a}$ and $\vec{b}$ of the problem* (1)–(3) *satisfy the condition $b_0(D_0-\alpha)R_{10}R_{20}\neq 0$ for all $\alpha\geq 0$. Then the problem* (1)–(3) *can have only finite-dimensional kernel whose dimension is not greater than $4K_3(\varepsilon)-2$ for any $0<\varepsilon<(\nu_{10}-\nu_{20})/2$. If the functions φ_1, φ_2 satisfy the conditions of Theorem* 1 *and belong to the space $\mathbf{E}^{q-2}_{h\nu^*_\varepsilon/2}(\Omega)$, then the solution $u=u_\bullet+u_++u_-$ of the problem* (1)–(3) *exists in the space $\mathbf{E}^{2,q}_\theta(\overline{\mathcal{D}})$, where*

$$u_\bullet=\sum_{k=1-K_3(\varepsilon)}^{K_3(\varepsilon)-1}u_k(t)e^{ikx},\quad u_+=\sum_{k=K_3(\varepsilon)}^{\infty}u_k(t)e^{ikx},\quad u_-=\sum_{k=K_3(\varepsilon)}^{\infty}u_{-k}(t)e^{-ikx},$$

$\theta(t)=\min\{(t-h/2)\nu^*_\varepsilon,(h/2-t)\nu^*_\varepsilon\}$, $\nu^*_\varepsilon=\max\{\varepsilon-\nu_{10},\nu_{20}+\varepsilon\}$, and the inequality

$$\|u_++u_-\|^2_{\mathbf{E}^{2,q}_\theta}\leq 2(1+|b_0|^{-2}+|b_0|^{-4})C_1^2\big(\|\varphi_1\|^2_{\mathbf{E}^{q-2}_{h\nu^*_\varepsilon/2}}+\|\varphi_2\|^2_{\mathbf{E}^{q-2}_{h\nu^*_\varepsilon/2}}\big)\tag{26}$$

holds.

Proof. From the conditions of the present theorem it follows that the solution of the problem (5)–(7) is unique when $|k|\geq K_3(\varepsilon)$. Hence, the kernel of the problem (1)–(3) is generated by the $(2K_3(\varepsilon)-1)$-element set $\{k\in\mathbb{Z}\colon |k|<K_3(\varepsilon)\}$ each element of which is associated with two or one linear independent elements of kernel.

In order to prove the inequality (26), we use the formula

$$|u_k^{(j)}(t)|^2\leq\frac{2C_1^2}{|b_0|^{2j}}(1+k^2)^{j-2}\big(e^{2\nu^*_\varepsilon t|k|}|\hat{\varphi}_{1k}|^2+e^{2\nu^*_\varepsilon(h-t)|k|}|\hat{\varphi}_{2k}|^2\big),\qquad |k|\geq K_3(\varepsilon),$$

which is a consequence of (15), (16). Passing to the norms, we obtain the inequality

$$\|u_++u_-\|^2_{\mathbf{E}^{2,q}_\theta}\leq 2B_0C_1^2\sum_{|k|\geq K_3(\varepsilon)}(1+k^2)^{q-2}\big(e^{2(\theta(t)+\nu^*_\varepsilon t)|k|}|\hat{\varphi}_{1k}|^2+e^{2(\theta(t)+\nu^*_\varepsilon(h-t))|k|}|\hat{\varphi}_{2k}|^2\big)\tag{27}$$

(where $B_0=1+|b_0|^{-2}+|b_0|^{-4}$) which proves (26). □

The condition $b_0(D_0-\alpha)R_{10}R_{20}\neq 0$ of Theorem 2 means that the quadratic trinomial $\lambda^2-2a_0\lambda+b_0$ does not have zero roots; its roots are not located on horizontal line and are not the roots of the quadratic trinomial$(a_{10}\lambda+b_{10})(a_{20}\lambda+b_{20})$.

The following theorems demonstrate the increased smoothness of some parts of the solution.

Theorem 3 (Increase of the solution smoothness)**.** *Let the conditions of Theorem* 2 *hold, and let φ_2 be a trigonometric polynomial. Then a solution of the problem* (1)–(3) *exists and belongs to the space $\mathbf{E}^{2,q}_{(h/2-t)\nu^*_\varepsilon}(\overline{\mathcal{D}})$. If φ_1 is a trigonometric polynomial, then a solution of the problem* (1)–(3) *also exists and belongs to the space $\mathbf{E}^{2,q}_{(t-h/2)\nu^*_\varepsilon}(\overline{\mathcal{D}})$.*

Proof. According to Theorem 1, the solution of the problem (1)–(3) exists. In particular, $u=u_\bullet+u_*+u_{**}$, where $u=u_\bullet$ and u_{**} are polynomials, and u_* does not depend on the function φ_2, i.e., $u_*=\sum_{|k|\geq K_3(\varepsilon)}\frac{\delta_{1k}(t)}{\det\delta_k}\hat{\varphi}_{1k}e^{ikx}$.

The estimate for u_* is a consequence of the formula (27), where $\varphi_2=0$, namely

$$\|u_*\|^2_{\mathbf{E}^{2,q}_{(h/2-t)\nu^*_\varepsilon}}\leq 2B_0C_1^2\sum_{|k|\geq K_3(\varepsilon)}(1+k^2)^{q-2}e^{2((h/2-t)\nu^*_\varepsilon+\nu^*_\varepsilon t)|k|}|\hat{\varphi}_{1k}|^2=$$

$$= 2B_0C_1^2 \sum_{|k|\geq K_3(\varepsilon)} (1+k^2)^{q-2} e^{h\nu_\varepsilon^*|k|} |\hat{\varphi}_{1k}|^2 \leq 2B_0C_1^2 \|\varphi_1\|_{\mathbf{E}^{q-2}_{h\nu_\varepsilon^*/2}}.$$

Similarly, if φ_1 is polynomial, then $u = u_\bullet + u_* + u_{**}$, where $u = u_\bullet$ and u_* are polynomials, and the estimate $\|u_{**}\|_{\mathbf{E}^{2,q}_{(t-h/2)\nu_\varepsilon^*}} \leq 2B_0C_1^2 \|\varphi_2\|_{\mathbf{E}^{q-2}_{h\nu_\varepsilon^*/2}}$ is obtained also from (27), if $\varphi_1 = 0$. ☐

It is obvious that the spaces $\mathbf{E}^{2,q}_{\pm(t-h/2)\nu_\varepsilon^*}(\overline{\mathcal{D}})$ are subspaces of the space $\mathbf{E}^{2,q}_{\theta(t)\nu_\varepsilon^*}(\overline{\mathcal{D}})$, which is used in Theorem 2. Hence the smoothness of the solution increases.

Let us separate the right-hand sides of the conditions (2), (3) into three parts $\varphi_j = \varphi_j^\bullet + \varphi_j^+ + \varphi_j^-$, where

$$\varphi_j^\bullet = \sum_{k=1-K_3(\varepsilon)}^{K_3(\varepsilon)-1} \hat{\varphi}_{jk} e^{ikx}, \qquad \varphi_j^+ = \sum_{k=K_3(\varepsilon)}^{\infty} \hat{\varphi}_{jk} e^{ikx}, \qquad \varphi_j^- = \sum_{k=K_3(\varepsilon)}^{\infty} \hat{\varphi}_{j,-k} e^{-ikx}, \qquad j = 1, 2.$$

Theorem 4 (Increase of the solution smoothness)**.** *Let the conditions of Theorem* 2 *hold, and let* φ_1^- *and* φ_2^- *be polynomials. Then for* $\varphi_1^+ \in \mathbf{E}^{q-2}_{(\varepsilon-\nu_{10})h/2}(\Omega)$, $\varphi_2^+ \in \mathbf{E}^{q-2}_{(\nu_{20}+\varepsilon)h/2}(\Omega)$ *a solution of the problem* (1)–(3) *exists in the space* $\mathbf{E}^{2,q}_\theta(\overline{\mathcal{D}})$, *where*

$$\theta(t) = \min\{(\varepsilon - \nu_{10})(h/2 - t), (\nu_{20} + \varepsilon)(t - h/2)\}.$$

If φ_1^+ *and* φ_2^+ *are polynomials, then for* $\varphi_1^- \in \mathbf{E}^{q-2}_{(\nu_{20}+\varepsilon)h/2}(\Omega)$, $\varphi_2^- \in \mathbf{E}^{q-2}_{(\varepsilon-\nu_{10})h/2}(\Omega)$, *the solution of the problem* (1)–(3) *exists in the space* $\mathbf{E}^{2,q}_\theta(\overline{\mathcal{D}})$, *where*

$$\theta(t) = \min\{(\varepsilon - \nu_{10})(t - h/2), (\nu_{20} + \varepsilon)(h/2 - t)\}.$$

Proof. To estimate u_+ and u_- one can use formulas (15) and (16). Further, the scheme of proof is analogical to the schemes of proofs of Theorems 2 and 3. ☐

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