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ON ADEQUACY OF FULL MATRICES

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This paper deals with the following question: whether a ring of matrices or classes of matrices over an adequate ring or elementary divisor ring inherits the property of adequacy?

The property to being adequate in matrix rings over adequate and commutative elementary divisor rings is studied. Let us denote by $\mathfrak{A}$ and $\mathfrak{E}$ an adequate and elementary divisor domains, respectively. Also $\mathfrak{A}_2$ and $\mathfrak{E}_2$ denote a rings of 2×2 matrices over them. We prove that full nonsingular matrices from $\mathfrak{A}_2$ are adequate in $\mathfrak{A}_2$ and full singular matrices from $\mathfrak{E}_2$ are adequate in the set of full matrices in $\mathfrak{E}_2$.

In memory of our teacher and friend Professor Bohdan Zabavsky

1. Introduction. The notion of an adequate ring was originally defined by Helmer [5]. A ring R is adequate if R is a commutative Bezout domain and for every $a \neq 0$, and b in R we can write $a = cd$, with $(c,\ b) = 1$ and with $(d_i, b) \neq 1$ for every nonunit divisor d_i of d. This concept appeared as a formalization of properties of the entire analytic functions ring.

By definition, every adequate ring is a Prufer domain. So every principal ideal domain is an adequate ring. Example of an adequate ring which is not a principal ideal domain is furnished by the set of entire functions with coefficients in a field [4]. Gillman and Henriksen have shown that a von Neumann regular ring is adequate [4]. Also, it is clear to see that a local ring is adequate.

The adequate rings with zero-divisors in Jacobson radical were studied by Kaplansky [7]. Henriksen [6] appears to be the first person to have given an example to show that being adequate is a stronger property than that of being an elementary divisor ring. In proving this, Henriksen observed that in an adequate domain each nonzero prime ideal is contained in the unique maximal ideal [6]. It is a natural question to ask whether or not the converse holds and this question is explicitly raised in [8]. The negative answer to this question is given in [1]. Furthermore, it is shown that there exists an elementary divisor ring which is not adequate but which does have the property that each nonzero prime ideal is contained in the unique maximal ideal. In [15], it was proved that that a commutative Bezout domain in which each nonzero prime ideal is contained in the unique maximal ideal is an elementary divisor ring. In [16], Bezout rings in which each regular element is adequate were studied. The following results have been obtained.

Theorem 1. *Let R be a Bezout ring of stable range 2. A regular element $a \in R$ (element without zero divisors) is an adequate element if and only if R/aR is a semiregular ring.*

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The definition of the ring stable range will be given below.

Theorem 2. *Let R be a semihereditary Bezout ring in which every regular element is adequate. Then R is an elementary divisor ring.*

Zabavskii introduced a new class of elementary divisor rings that contains adequate rings and rings constructed by Henriksen and called the rings belonging to this class generalized adequate rings [14]. Gatalevych [3] was the first, who studied noncommutative adequate rings and their generalizations. He has proved that the generalized right adequate duo Bezout domain is an elementary divisor domain.

The stable range of a ring is one of the important invariants of the algebraic K-theory.

Definition 1. *Stable range of a ring R* (in notation st.r.(R)) is the smallest positive integer n such that whenever

$$a_1R + a_2R + \ldots + a_{n+1}R = R$$

then there are $b_1, \ldots, b_n \in R$ such that

$$(a_1 + b_1a_{n+1})R + \cdots + (a_n + b_na_{n+1})R = R.$$

If such n does not exist, then the stable range of R is infinity.

We say that ring R has *stable range* 1.5 if for each $a, b \in R$ and $0 \neq c \in R$ satisfying $aR + bR + cR = R$ there exists $r \in R$ with

$$(a + br)R + cR = R.$$

This notion was introduced by the second author in [11] and studied in [2, 10, 12]. By Property 1.18, [10, p. 21], adequate rings have stable range 1.5.

Vaserstein [13] established a relationship between the stable range of a ring and the stable range of the matrices rings over it.

Theorem 3. *For any ring R and any $n \geq 1$*

$$\text{st.r.}(M_n(R)) = 1 - \left[-\tfrac{\text{st.r.}(R)-1}{n}\right].$$

where $*$ denotes the least integer greater than or equal to a real number x.

According to this formula if the stable range of the ring R is equal to 1 or 2, then the stable range of the matrix ring $M_n(R)$ has the same stable range. It turns out that this regularity holds for a rings of stable range 1.5. Using Theorem 2.20, [10], p. 90 we get.

Theorem 4. *A second order matrices ring over an adequate ring have stable range* 1.5.

Theorem 5. *Let R be an adequate Bezout ring and $J(R) \neq 0$ then* st.r.$(R) = 1$.

Proof. Let $bR + cR = R$ and $a \in J(R) \setminus \{0\}$. Then, obviously, $aR + bR + cR = R$. As in the proof of Theorem 1, we can show that

$$aR + (b + cr)R = R$$

for some element $r \in R$. Since $a \in J(R)$, we have $(b + cr)R = R$, so st.r.$(R) = 1$. □

Obviously, we have

Corollary 1. *If R is a Bezout ring and there is an adequate element a in $J(R)\backslash\{0\}$ then* $\text{st.r.}(R) = 1$.

A special role among the second-order matrices over an elementary divisor rings is played by full matrices, i.e., matrices whose elements are relatively prime. Thus, according to the result of Kaplansky, R is elementary divisor ring if and only if all full 2×2 matrices over R have the property of canonical diagonal reduction [7].

In the 2×2 matrix ring over an adequate ring Zabavskii and Petrychkovych [9] selected a class of matrices of stable range 1.

In this article we investigate the adequate properties of some classes and matrices over adequate and elementary divisor rings.

Arrangement. To prevent confusion and to reduce of notation, denote by $\mathfrak{A}$ and $\mathfrak{E}$ an adequate and elementary divisor domains, respectively. Also $\mathfrak{A}_2$ and $\mathfrak{E}_2$ denote a rings of 2×2 matrices over them. I is the identity matrix, and $\text{GL}_2(R)$ denote the group of 2×2 invertible matrices of a ring R.

2. Auxiliary results. If $A = BC$, then B is called a left divisor of A and A is called a right multiple of B. If $A = DA_1$ and $B = DB_1$, then D is called a left common divisor of A and B. In addition, if D is a right multiple of each left common divisor of A and B, then D is called the left greatest common divisor of A and B (*in notation* $(A, B)_l$).

(a, b) and $[a, b]$ denote greatest common divisor and least common multiple of a and b.

Let A, B be matrices from $\mathfrak{E}_2$. There are matrices P_A, Q_A, P_B, Q_B in $\text{GL}_2(\mathfrak{E})$ such that

$$P_A A Q_A = \text{diag}(\alpha_1, \alpha_2) := S_A, \quad \alpha_1|\alpha_2,$$
$$P_B B Q_B = \text{diag}(\beta_1, \beta_2) := S_B, \quad \beta_1|\beta_2,$$

where S_A, S_B are Smith normal forms (S.n.f.) of A, B, respectively. Diagonal elements of S.n.f. are called invariant factors. It follows that these matrices can be written as

$$A = P_A^{-1} S_A Q_A^{-1}, \ B = P_B^{-1} S_B Q_B^{-1}.$$

Theorem 6 ([10], Theorem 2.15)**.** *Let $P_B P_A^{-1} = \|s_{ij}\|$. The Smith normal form of $(A,\ B)_l$ is the matrix*

$$\text{diag}\left((\alpha_1,\ \beta_1),\ \ (\alpha_2,\ \beta_2,\ [\alpha_1,\ \beta_1]s_{21})\right).$$

Corollary 2. *In order that $(A,\ B)_l = I$, i.e., $A\mathfrak{E}_2 + B\mathfrak{E}_2 = \mathfrak{E}_2$ it is necessary and sufficient that*

$$(\alpha_2,\ \beta_2,\ [\alpha_1,\ \beta_1]s_{21}) = 1.$$

Let $\beta_i|\alpha_i$, $i = 1, 2$. Denote by $\mathbf{L}(\text{diag}(\alpha_1, \alpha_2),\ \text{diag}(\beta_1, \beta_2))$ the set of all invertible matrices of the form

$$\left\| \begin{matrix} l_{11} & l_{12} \\ \frac{\beta_2}{(\alpha_1,\beta_2)} l_{21} & l_{22} \end{matrix} \right\|.$$

Theorem 7 ([10], Theorem 4.3)**.** *The matrix B is the left divisor of A i.e., $A = BC$ if and only if $\beta_i|\alpha_i$, $i = 1, 2$, and $P_B P_A^{-1} \in \mathbf{L}(S_A,\ S_B)$.*

Theorem 8 ([10], Theorem 4.4)**.** *The set of all left divisors of A having Smith normal form S_B is*

$$(\mathbf{L}(S_A, S_B)P_B)^{-1} S_B \,\mathrm{GL}_2(\mathfrak{E}).$$

Theorem 9 ([10], Theorem 2.19)**.** *Assume that A, B are full matrices in $\mathfrak{A}_2$, moreover $A\mathfrak{A}_2 + B\mathfrak{A}_2 = \mathfrak{A}_2$. Then there exists a matrix $P \in \mathfrak{A}_2$ such that $AP + B = Q$, where Q is an invertible matrix from $\mathfrak{A}_2$.*

3. Main results.

3.1. Adequacy in the set of full nonsingular matrices over adequate domains.

Definition 2. The element b of the ring R is called *left adequate to an element* $a \in R$ if there exist such elements $s, t \in R$ that $b = s \cdot t$, where $tR + aR = R$, and for an arbitrary nonunit element $s' \in R$ such that $sR \subset s'R \neq R$ the condition $s'R + aR \neq R$ is satisfies.

Note that, by the definition the g.c.d. of full matrices entries is equal to 1. It follows that every full matrix in $\mathfrak{E}_2$ has S.n.f. of the form $\mathrm{diag}(1,\ *)$.

Theorem 10. *Full nonsingular matrices from $\mathfrak{A}_2$ are adequate in $\mathfrak{A}_2$.*

Proof. Let B be the full nonsingular matrix from $\mathfrak{A}_2$. An adequate ring is an elementary divisor ring. So B can be written in the form $B = P_B^{-1}\mathrm{diag}(1,\ \beta_2)Q_B^{-1}$, where $\beta_2 \neq 0$. Suppose that

$$A = P_A^{-1}\,\mathrm{diag}(\alpha_1,\ \alpha_2)Q_A^{-1},\ \alpha_1|\alpha_2,$$

is an arbitrary matrix from $\mathfrak{A}_2$ such that $A\mathfrak{A}_2 + B\mathfrak{A}_2 \neq \mathfrak{A}_2$. Since $(A, B)_l$ is the left divisor of B, therefore its invariant factors are divisors of corresponding invariant factors of B (see Theorem 7). It follows that the first invariant factor of $(A, B)_l$ is equal to 1. Consequently, according to Theorem 6, the Smith normal form of $(A, B)_l$ have the form $\Omega = \mathrm{diag}(1,\ \omega_2)$, where $\omega_2 := (\alpha_2,\ \beta_2,\ \alpha_1 s_{21})$, s_{21} is an element of matrix $P_B P_A^{-1} := \|s_{ij}\|$.

Decompose β_2 into product: $\beta_2 = \psi_2\nu_2$, where

$$(\nu_2,\ \omega_2) = (\nu_2,\ \alpha_2,\ \beta_2,\ \alpha_1 s_{21}) = ((\nu_2,\ \beta_2),\ \alpha_2,\ \alpha_1 s_{21}) = (\nu_2,\ \alpha_2,\ \alpha_1 s_{21}) = 1 \tag{1}$$

and for each nontrivial $\psi_2'|\psi_2$ we have

$$(\psi_2',\ \omega_2) = (\psi_2',\ \alpha_2,\ \beta_2,\ \alpha_1 s_{21}) = ((\psi_2',\ \beta_2),\ \alpha_2,\ \alpha_1 s_{21}) = (\psi_2',\ \alpha_2,\ \alpha_1 s_{21}) \neq 1. \tag{2}$$

Set

$$P_T := \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| P_A.$$

The matrix B can be written as $B = ST$, where

$$S := P_B^{-1}\mathrm{diag}(1,\ \psi_2)P_T,\ \ T := P_T^{-1}\mathrm{diag}(1,\ \nu_2)Q_B^{-1}.$$

By Theorem 6, we get $(A, T)_l \sim \mathrm{diag}(1, \lambda_2)$, where $\lambda_2 := (\nu_2,\ \alpha_2,\ \alpha_1 m_{21})$, m_{21} is an element of the matrix $P_T P_A^{-1} := \|m_{ij}\|$, the symbol " $\sim$ " denotes the equivalence. Since

$$P_T P_A^{-1} = \left(\left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| P_A \right) P_A^{-1} = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|,$$

we have $m_{21} = 1$. By virtue of equality (1) we conclude that

$$\lambda_2 = (\nu_2,\ \alpha_2,\ \alpha_1) = 1 \quad \Rightarrow \quad (A, T)_l = I.$$

It means that $A\mathfrak{A}_2 + T\mathfrak{A}_2 = \mathfrak{A}_2$.

Let us describe all left nontrivial divisors of the matrix S. Potential S.n.f. of such divisors have the form $\mathrm{M} = \mathrm{diag}(1, \mu_2)$, where $\mu_2|\psi_2$. According to Theorem 8, the set of all left divisors of the matrix S having the Smith form M has the form $(\mathbf{L}(\Psi, \mathrm{M})P_B)^{-1}\mathrm{M}\mathrm{GL}_2(\mathfrak{A})$, where $\Psi := \mathrm{diag}(1,\ \psi_2)$. The set $\mathbf{L}(\Psi, \mathrm{M})$ consists of all invertible matrices of the form

$$\mathrm{M} = \left\| \begin{matrix} l_{11} & l_{12} \\ \mu_2 l_{21} & l_{22} \end{matrix} \right\|.$$

Let $N := (LP_B)^{-1}\mathrm{M}V$, where $L \in \mathbf{L}(\Psi, \mathrm{M})$, $V \in \mathrm{GL}_2(\mathfrak{A})$. Consider the matrix

$$(LP_B)P_A^{-1} = L(P_B P_A^{-1}) = \left\| \begin{matrix} l_{11} & l_{12} \\ \mu_2 l_{21} & l_{22} \end{matrix} \right\| \left\| \begin{matrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{matrix} \right\| = \left\| \begin{matrix} * & * \\ \mu_2 l_{21} s_{21} + s_{21} l_{22} & * \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} * & * \\ (\mu_2, s_{21})p_{21} & * \end{matrix} \right\|, p_{21} := \frac{\mu_2}{(\mu_2, s_{21})} l_{21} s_{21} + \frac{s_{21}}{(\mu_2, s_{21})} l_{22}.$$

Then

$$(A, N)_l \sim \mathrm{diag}(1, \sigma_2),$$

where

$$\sigma_2 := (\mu_2,\ \alpha_2,\ \alpha_1(\mu_2, s_{21})p_{21}) = (\mu_2,\ \alpha_2,\ \alpha_1\mu_2 p_{21},\ \alpha_1 s_{21} p_{21}) = (\mu_2,\ \alpha_2,\ \alpha_1 s_{21} p_{21}).$$

The element μ_2 is a nontrivial divisor of ψ_2. Therefore by condition (2), $(\mu_2,\ \alpha_2,\ \alpha_1 s_{21}) \neq 1$. Since

$$(\mu_2,\ \alpha_2,\ \alpha_1 s_{21})|(\mu_2,\ \alpha_2,\ \alpha_1 s_{21} p_{21}) = \sigma_2$$

we get $\sigma_2 \neq 1$. It means that $A\mathfrak{A}_2 + N\mathfrak{A}_2 \neq \mathfrak{A}_2$. □

3.2. Singular full matrices over an elementary divisor domains.

Theorem 11. *Full singular matrices from $\mathfrak{E}_2$ are adequate in the set of full matrices in $\mathfrak{E}_2$.*

Proof. Let B be full singular matrix from $\mathfrak{E}_2$. Therefore B can be written in the form $B = P_B^{-1}\mathrm{diag}(1,\ 0)Q_B^{-1}$. Suppose that

$$A = P_A^{-1}\ \mathrm{diag}(1,\ \alpha_2)Q_A^{-1},\ \alpha_1|\alpha_2$$

is an arbitrary full matrix in $\mathfrak{E}_2$ such that $A\mathfrak{E}_2 + B\mathfrak{E}_2 \neq \mathfrak{E}_2$.

The invariant factors of $(A, B)_l$ are divisors of corresponding invariant factors of B. So the first invariant factor of $(A, B)_l$ is equal to 1. Using Theorem 6 we get that the Smith normal form of $(A, B)_l$ is equal to $\Psi = \mathrm{diag}(1,\ \psi_2)$, where $\psi_2 := (\alpha_2,\ s_{21})$, $P_B P_A^{-1} := \|s_{ij}\|$.

Case 1. Assume that

$$\psi_2 := (\alpha_2,\ s_{21}) = 0 \quad \Rightarrow \quad \alpha_2 = s_{21} = 0.$$

It means that

$$A = P_A^{-1}\ \mathrm{diag}(1,\ 0)Q_A^{-1},\ \ P_B P_A^{-1} = \left\| \begin{matrix} e_1 & s_{12} \\ 0 & e_2 \end{matrix} \right\|,\ e_1, e_2 \in U(\mathfrak{E}).$$

Noting that

$$(P_B P_A^{-1}) \operatorname{diag}(1,\ 0) = \operatorname{diag}(1,\ 0) \operatorname{diag}(e_1,\ e_2)$$

it is easy to check that

$$A = B\left(Q_B \operatorname{diag}(e_1,\ e_2)\, Q_A^{-1}\right).$$

Consequently, the matrices A, B are right associates. It follows that B is left adequate to A. So B is right adequate in the set of full matrices in $\mathfrak{E}_2$.

Case 2. Let ψ_2 be nonunit element of $\mathfrak{E}_2 \setminus \{0\}$. Set

$$P_T := \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| P_A.$$

Then the matrix B can be written as $B = ST$, where

$$S := P_B^{-1} \left\| \begin{matrix} 1 & 0 \\ 0 & \psi_2 \end{matrix} \right\| P_T, \quad T := P_T^{-1} \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\| Q_B^{-1}.$$

Consider the matrix $(A, T)_l$. As above we have

$$(A, T)_l \sim \operatorname{diag}(1,\ \lambda_2), \quad \lambda_2 = (\alpha_2,\ m_{21}),$$

where m_{21} is an element of the matrix $P_T P_A^{-1} := \|m_{ij}\|$. Since

$$P_T P_A^{-1} = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|,$$

we get $m_{21} = 1$. So $\lambda_2 = 1$. Consequently, $A\mathfrak{E}_2 + T\mathfrak{E}_2 = \mathfrak{E}_2$.

Consider all left nontrivial divisors of the matrix S. Potential Smith forms of such divisors have the form $\operatorname{diag}(1,\ \mu_2) := \mathrm{M}$, where μ_2 is a non-trivial divisor of ψ_2. According to Theorem 8 the set of all left divisors of the matrix S having the Smith form M has the form

$$(\mathbf{L}(\Psi, \mathrm{M}) P_B)^{-1} \mathrm{M}\ \mathrm{GL}_2(\mathfrak{E}),$$

where $\mathbf{L}(\Psi, \mathrm{M})$ consists of all invertible matrices of the form

$$\mathrm{M} = \left\| \begin{matrix} l_{11} & l_{12} \\ \mu_2 l_{21} & l_{22} \end{matrix} \right\|.$$

Suppose that $N = (LP_B)^{-1}\mathrm{M}V$, where $L \in \mathbf{L}(\Psi, \mathrm{M})$, $V \in \mathrm{GL}_2(\mathfrak{E})$ be an arbitrary matrix in this set. Repeating the considerations made in the proof of Theorem 5, we obtain $A\mathfrak{E}_2 + N\mathfrak{E}_2 \neq \mathfrak{E}_2$. □

Definition 3. Element b of the set $\mathfrak{N}$ is called *right weak adequate* if for an arbitrary element $a \in \mathfrak{N}$ there exist such elements $s, t \in \mathfrak{N}$ that $b = s \cdot t$, where $t\mathfrak{N} + a\mathfrak{N} = \mathfrak{N}$, and $s\mathfrak{N} + a\mathfrak{N} \neq \mathfrak{N}$.

Theorem 12. *Full singular matrices are weak adequate in* $\mathfrak{E}_2$.

Proof. Let $A := P_A^{-1} \operatorname{diag}(\alpha_1, \alpha_2) Q_A^{-1}$ and $B := P_B^{-1} \operatorname{diag}(1, 0) Q_B^{-1}$ are matrices in $\mathfrak{E}_2$. The Case $\alpha_1 = 1$ is considered in Theorem 7.

Let $\alpha_1 \neq 1$. Suppose that $B = ST$, where $(A, T)_l = I$. Obviously that the matrices S, T are full matrices

$$S = P_S^{-1} \operatorname{diag}(1, \sigma_2) Q_S^{-1}, \quad T = P_T^{-1} \operatorname{diag}(1, \tau_2) Q_T^{-1}.$$

Since $\det B = \det S \det T$, we have

$$\sigma_2\tau_2 = 0. \tag{3}$$

As above $I = (A,T)_l = \mathrm{diag}(1, (\tau_2,\ \alpha_2,\ \alpha_1 p_{21}))$, where p_{21} is element of $P_T = \|p_{ij}\|$. It follows that

$$\left(\tau_2,\ \alpha_1\left(\frac{\alpha_2}{\alpha_1},\ p_{21}\right)\right) = 1 \quad \Rightarrow \quad \tau_2 \neq 0.$$

Using equality (3) we get $\sigma_2 = 0$. Hence the matrix S has the form

$$S = P_S^{-1}\mathrm{diag}(1,0)Q_S^{-1}$$

and can be writen as $S = S_1S_2$, where $S_1 := P_S^{-1}\mathrm{diag}(1, \alpha_2+1)$, $S_2 := \mathrm{diag}(1,0)Q_S^{-1}$. Since $(\det S_1,\ \det A) = (\alpha_2+1,\ e\alpha_1\alpha_2) = 1$, where e is invertible element of elementary divisor domain, we conclude that $(A, S_1)_l = I$. □

3.3. Weak adequacy of zero matrix to the set of full matrices.

Theorem 13. *Zero matrix is weak adequate to the set of full matrices in $\mathfrak{E}_2$.*

Proof. Let $A := P_A^{-1}\mathrm{diag}(1,\alpha_2)Q_A^{-1}$ is matrix in $\mathfrak{E}_2$. Zero matrix can be written as $\mathbf{0} = ST$, where

$$S := P_A^{-1}\left\|\begin{matrix}1 & 0\\ 0 & 0\end{matrix}\right\|\left(\left\|\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right\|P_A\right),\quad T := \left(P_A^{-1}\left\|\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right\|\right)\left\|\begin{matrix}0 & 0\\ 0 & 1\end{matrix}\right\|.$$

Obviously that $P_T = \left\|\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right\|P_A$. So $P_TP_A^{-1} = \left\|\begin{matrix}0 & 1\\ 1 & 0\end{matrix}\right\|$. By Theorem 6 we get

$$(A,T)_l \sim \left\|\begin{matrix}1 & 0\\ 0 & (\alpha_2, 1)\end{matrix}\right\| = I.$$

Concequently, $A\mathfrak{E}_2 + T\mathfrak{E}_2 = \mathfrak{E}_2$.

Since $P_S = P_A$, we have $P_SP_A^{-1} = I$. So

$$(A,S)_l \sim \left\|\begin{matrix}1 & 0\\ 0 & \alpha_2\end{matrix}\right\| \neq I.$$

Therefore $A\mathfrak{E}_2 + S\mathfrak{E}_2 \neq \mathfrak{E}_2$.

□

REFERENCES

1. J.W. Brewer, P.F. Conrad, P.R. Montgomery, *Lattice-ordered groups and a conjecture for adequate domains*, Proc. Amer. Math. Soc, **43** (1974), №1, 31–34, 93–108.
2. V.A. Bovdi, V.P. Shchedryk, *Commutative Bezout domains of stable range 1.5*, Linear Algebra and Appl., **568** (2019), 127–134.
3. A. Gatalevych, *On adequate and generalized adequate duo-rings and elementary divisor duo-rings*, Mat. Stud., **9** (1998), №2, 115–119. (in Ukrainian)
http://matstud.org.http://matstud.org.ua/ojs/index.php/matstud/issue/archive

4. L. Gillman, M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc., **82** (1956), 366–391.
5. O. Helmer, *The elementary divisor for rings without chain condition*, Bull. Amer. Math. Soc., **49** (1943), 235–236.
6. M. Henriksen, *Some remarks about elementary divisor rings*, Michigan Math. J., **3** (1956), 159–163.
7. I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc., **66** (1949), 464–491.
8. M. Larsen, W. Lewis, T. Shores, *Elementary divisor rings and finitely presented modules*, Trans. Amer. Mat. Soc., **187** (1974), 231–248.
9. V.M. Petrychkovych, B.V. Zabavskii, *On the stable range of matrix rings*, Ukr. Math. J., **61** (2009), №11, 1853–1857. https://doi.org/10.1007/s11253-010-0317-7
10. V. Shchedryk, Arithmetic of matrices over rings, Kyiv: Akademperiodyka, 2021, 278 p. https://doi.org/10.15407/akademperiodika.430.278
11. V.P. Shchedrik, *Bezout rings of stable rank 1.5 and the decomposition of a complete linear group into products of its subgroups*, Ukr. Math. J., **69** (2017), №1, 138–147. https://doi.org/10.1007/s11253-017-1352-4
12. V.P. Shchedryk, *Some properties of primitive matrices over Bezout B-domain*, Algebra Discrete Math., **4** (2005), №2, 46–57.
13. L.N. Vaserstein, *The stable rank of rings and dimensionality of topological spaces*, Functional Anal. Appl., **5** (1971), 102–110.
14. B.V. Zabavskii, *Generalized adequate rings*, Ukr. Math. J., **48** (1996), №4, 614–617. https://doi.org/10.1007/BF02390621
15. B.V. Zabavsky, A.I. Gatalevych, *A commutative Bezout $PM*$ domain is an elementary divisor ring*, Algebra Discrete Math., **19** (2015), №2, 295–301.
16. B.V. Zabavsky, A.I. Gatalevych, *Diagonal reduction of matrices over commutative semihereditary Bezout rings*, Communications in Algebra, **47** (2019), №4, 1785–1795.

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