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HIGHER-ORDERS ELLIPTIC-PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF THE NONLINEARITY IN UNBOUNDED DOMAINS WITHOUT CONDITIONS AT INFINITY

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Initial-boundary value problems for parabolic and elliptic-parabolic (that is degenerated parabolic) equations in unbounded domains with respect to the spatial variables were studied by many authors. It is well known that in order to guarantee the uniqueness of the solution of the initial-boundary value problems for linear and some nonlinear parabolic and elliptic-parabolic equations in unbounded domains we need some restrictions on behavior of solution as $|x| \to +\infty$ (for example, growth restriction of solution as $|x| \to +\infty$, or the solution to belong to some functional spaces). Note, that we need some restrictions on the data-in behavior as $|x| \to +\infty$ for the initial-boundary value problems for equations considered above to be solvable.

However, there are nonlinear parabolic equations for which the corresponding initial-boundary value problems are uniquely solvable without any conditions at infinity.

We prove the unique solvability of the initial–boundary value problem without conditions at infinity for some of the higher-orders anisotropic parabolic equations with variable exponents of the nonlinearity. A priori estimate of the weak solutions of this problem was also obtained. As far as we know, the initial-boundary value problem for the higher-orders anisotropic elliptic-parabolic equations with variable exponents of nonlinearity in unbounded domains were not considered before.

Introduction. In this paper we consider initial-boundary value problem (in particular, Cauchy problem) for some elliptic-parabolic equations in unbounded domains with respect to space variables. It is well known that to guarantee the unique solvability of such problems for linear parabolic equations, we have to set some conditions on the behavior of the solution as $|x| \to +\infty$. For the first time the result like that was obtained in [24] for Cauchy problem for the heat equation:

$$u_t - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T], \qquad u|_{t=0} = u_0(x),\ x \in \mathbb{R}^n. \tag{1}$$

The authors proved that the problem (1) has unique classic solution under the additional condition on it behavior at infinity:

$$|u(x,t)| \leqslant Ae^{a|x|^2}, \quad (x,t) \in \mathbb{R}^n \times [0,T], \tag{2}$$

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where a, A are the constants (that are dependent on u). It was also proven that this condition is essential, more precisely, it was proven that the problem (1) with $u_0 \equiv 0$ has non-trivial solutions as $Ae^{a|x|^{2+\varepsilon}}$ grows when $|x| \to +\infty$ for any $\varepsilon > 0$. Note, that the restriction (2) can be interpreted as an analogue of the boundary condition at infinity. Similar results for broad classes of both linear and nonlinear parabolic equations were obtained in [2, 9, 20] and others. Also note, that to guarantee the solvability of initial-boundary value problem for parabolic equations mentioned above we have to impose some conditions on the behavior of input data as $|x| \to +\infty$. In particular, in [24] it was shown that the classic solution of the problem (1), (2) exists if u_0 satisfies the condition:

$$|u_0(x)| \leqslant Be^{b|x|^2}, \quad x \in \mathbb{R}^n,$$

where b, B are some constants.

However, there exist nonlinear parabolic equations, the initial-boundary value problems for which are uniquely solvable without any conditions at infinity. The result like that was obtained for the first time in [11] for the equation

$$u_t - \Delta u + |u|^{p-2}u = 0, \quad (x,t) \in \Omega \times (0,T],$$

where Ω is an unbounded domain in $\mathbb{R}^n$, $p > 2$ is a constant.

Later on the similar results were obtained for other nonlinear parabolic equations, in particular, in [1, 3, 4, 5, 8, 11, 14, 19].

Nonlinear differential equations with variable exponents of nonlinearity appear as mathematical models in various physical processes. In particular, these equations describe electroreological substance flows, image recovering processes, electric current in the conductor with changing temperature field (see [22]). Such equations were extensively studied in [7, 12, 16, 18, 21, 23] and many other researches. The corresponding generalizations of Lebesgue and Sobolev spaces were used in these investigations (see [13, 15]).

In this paper we prove the unique solvability of the initial-boundary value problem for the higher-orders anisotropic elliptic-parabolic equations with variable exponents of nonlinearity in unbounded domains without conditions at infinity. The results obtained here are, in particular, generalizations and complements of the results from [10], where the nondegenerate parabolic equations with the constant exponents of the nonlinearity are considered. As far as we know, the initial–boundary value problem for the higher-orders anisotropic elliptic-parabolic equations with variable exponents of nonlinearity in unbounded domains were not considered before.

The paper consists of the introduction and three chapters. In the first chapter we provide the statement of the problem and the formulation of the main result. In the second section, we present auxiliary statements that are used in the third section to prove the main result.

1. Statement of problem and formulation of main result. Let n, m be natural numbers, M is a subset of the set $\{0, 1, \ldots, m\}$ such that $\{0, m\} \subset M$, and $M_0 := M \setminus \{0\}$. Let us N denote the number of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of the dimension n (ordered tuples of n nonnegative integers), the length $|\alpha| = \alpha_1 + \ldots + \alpha_n$ of which are elements of the set M. Let $\mathbb{R}^n$ be the linear space, with the ordered tuples of real numbers $x = (x_1, \ldots, x_n)$ as its elements and $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ as its norm. Denote $\mathbb{R}^N$ to be the linear space, composed of ordered tuples of N real numbers $\xi = (\xi_{\widehat{0}}, \ldots, \xi_\alpha, \ldots) \equiv (\xi_\alpha : |\alpha| \in M)$, the components of which are numbered with multi-indices of dimension n, having lengths from M and ordered lexicographically (it means that $\alpha = (\alpha_1, \ldots, \alpha_n)$ precedes $\beta = (\beta_1, \ldots, \beta_n)$, if $|\alpha| < |\beta|$

or $|\alpha| = |\beta|$ and $\alpha_k > \beta_k$, where $k := \min\{j|\alpha_j \neq \beta_j\}$). From now on the $\widehat{0} = (0, \ldots, 0)$ will be a multi-index of dimension n composed of zeros. Set $|\xi| := \big(\sum_{|\alpha|\in M} |\xi_\alpha|^2\big)^{1/2}$ for any $\xi \in \mathbb{R}^N$. We denote by δv the ordered set of all derivatives $D^\alpha v \equiv \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} v$ of the function $v(x), x \in G$ (G is any open set in $\mathbb{R}^n$) of orders $|\alpha| \in M$ (the ordering rule is the same as for the components of vectors $\xi \in \mathbb{R}^N$).

Let Ω be an unbounded domain in the space $\mathbb{R}^n$. Assume that the boundary $\Gamma := \partial\Omega$ of the domain Ω is a piece-wise smooth surface and denote by ν a unit external normal to Γ. Let $T > 0$ be some fixed number. Set

$$Q := \Omega \times (0, T), \quad \Sigma := \Gamma \times (0, T).$$

Assume that

(**P**) $p\colon \Omega \to \mathbb{R}$ is a measurable function such that

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 2, \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

(**B**) $b\colon \Omega \to \mathbb{R}$ is a measurable and bounded function, $b(x) \geq 0$ for a.e. $x \in \Omega$, and there is an open set $\Omega_0 \neq \varnothing$ such that $b(x) > 0$ for a.e. $x \in \Omega_0$, and $b(x) = 0$ for a.e. $x \in \Omega \setminus \Omega_0$.

Consider the *problem*: find a function $u\colon \overline{Q} \to \mathbb{R}$ that satisfies (in a certain sense) the equation

$$(b(x)u)_t + \sum_{|\alpha|\in M} (-1)^{|\alpha|} D^\alpha a_\alpha(x, t, \delta u) = \sum_{|\alpha|\in M} (-1)^{|\alpha|} D^\alpha f_\alpha(x, t), \quad (x, t) \in Q, \tag{3}$$

the boundary conditions

$$\frac{\partial^j u}{\partial \nu^j}\Big|_\Sigma = 0, \quad j = \overline{0, m-1}, \tag{4}$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega_0, \tag{5}$$

where $a_\alpha\colon Q \times \mathbb{R}^N \to \mathbb{R},\ f_\alpha\colon Q \to \mathbb{R},\ |\alpha| \in M,\ u_0\colon \Omega \to \mathbb{R}$ are given functions, which satisfy certain conditions that will be discussed later. Further the formulated above *initial-boundary value problem for the equation* (3) *with boundary conditions* (4) *and the initial condition* (5) will be briefly called *the problem* (3)–(5).

An example of the equation (3), which we examine here, is

$$(b(x)u)_t + (-\Delta)^m u + c(x, t)|u|^{p(x)-2}u = f(x, t), \quad (x, t) \in Q, \tag{6}$$

where functions b and p satisfy conditions (**B**) and (**P**) respectively, and $c\colon Q \to \mathbb{R}$ is a measurable, locally bounded function, moreover $\operatorname{ess\,inf}_Q c > 0$.

We will consider the weak solutions of the problem (3)–(5). Let us introduce the necessary notation and make appropriate assumptions about input data of this problem.

First of all let us introduce the *functional spaces* we will need next. Let G be any domain in $\mathbb{R}^n$, $r\colon G \to \mathbb{R}$ be a measurable function such that $r(x) \geq 1$ for a.e. $x \in G$, moreover if $r(x) > 1$ for a.e. $x \in G$, then $r'(x), x \in G$, is a function defined by the equality $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for a.e. $x \in G$. Let $L_{r(\cdot)}(G)$ be the linear space of (classes) measurable functions $v\colon G \to \mathbb{R}$ for which the functional $\rho_{G,r}(v) := \int_G |v(x)|^{r(x)} dx$ takes finite values, with a norm

$\|v\|_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 | \rho_{G,r}(v/\lambda) \le 1\}$. This space is a Banach space and is called *generalized Lebesgue space* or *Lebesgue space with variable exponent* (see, for example, [13]). Note, that if $r(x) = r_0 \equiv \text{const} \ge 1$ for a.e. $x \in G$, then $\|\cdot\|_{L_{r(\cdot)}(G)} = \|\cdot\|_{L_{r_0}(G)}$. It is known that if

$$1 < \operatorname*{ess\,inf}_{x\in G} r(x) \le \operatorname*{ess\,sup}_{x\in G} r(x) < \infty,$$

then the conjugated space to $L_{r(\cdot)}(G)$ can be identified with $L_{r'(\cdot)}(G)$. We define the space $L_{r(\cdot)}(D)$ similarly to $L_{r(\cdot)}(G)$, where $D = G \times (0,T)$, with the functional $\rho_{D,r}(w) := \iint_D |w(x,t)|^{r(x)} dxdt$ instead of $\rho_{G,r}(v)$.

We denote by $Bd(\Omega)$ the set of all possible bounded subdomains of the domain Ω. Let $p\colon \Omega \to \mathbb{R}$ be the function given above (see condition ($\mathbb{P}$)). Denote by $L_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ the linear space of (classes) measurable functions $v\colon \Omega \to \mathbb{R}$, which restrictions to an arbitrary domain $\Omega' \in Bd(\Omega)$ belong to $L_{p(\cdot)}(\Omega')$, with a system of seminorms $\{\|\cdot\|_{L_{p(\cdot)}(\Omega')} | \Omega' \in Bd(\Omega')\}$. This space is a complete linear locally convex space. We introduce a complete linear locally convex space $L_{p(\cdot),\mathrm{loc}}(\overline{Q})$ with the system of seminorms $\{\|\cdot\|_{L_{p(\cdot)}(\Omega'\times(0,T))} | \Omega' \in Bd(\Omega')\}$ similar to $L_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$. Note that the sequence $\{v_l\}_{l=1}^{\infty}$ *weakly* converges to v in $L_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ (respectively, in $L_{p(\cdot),\mathrm{loc}}(\overline{Q})$), if for any domain $\Omega' \in Bd(\Omega)$ the sequence $\{v_l|_{\Omega'}\}_{l=1}^{\infty}$ (respectively, $\{v_l|_{\Omega'\times(0,T)}\}_{l=1}^{\infty}$) weakly converges to $v|_{\Omega'}$ (respectively, to $v|_{\Omega'\times(0,T)}$) in $L_{p(\cdot)}(\Omega')$ (respectively, in $L_{p(\cdot)}(\Omega' \times (0,T))$).

For arbitrary $\Omega' \in Bd(\Omega)$ the $H^m(\Omega') := \{v \in L_2(\Omega') \big|\ D^\alpha v \in L_2(\Omega'),\ |\alpha| \le m\}$ denotes the standard Sobolev space with the norm $\|v\|_{H^m(\Omega')} := (\int_{\Omega'} \sum_{|\alpha|\le m} |D^\alpha v|^2 dx)^{1/2}$. We will also consider the space $H^m_{\mathrm{loc}}(\overline{\Omega}) := \{v \in L_{2,\mathrm{loc}}(\overline{\Omega}) \big| D^\alpha v \in L_{2,\mathrm{loc}}(\overline{\Omega}), |\alpha| \le m\}$, which is a complete linear locally convex space with the system of seminorms $\{\|\cdot\|_{H^m(\Omega')} \big| \Omega' \in Bd(\Omega)\}$. The sequence $\{v_j\}$ converges to v in the space $H^m_{\mathrm{loc}}(\overline{\Omega})$ if the sequence $\{v_j\big|_{\Omega'}\}$ converges to $v\big|_{\Omega'}$ in the spaces $H^m(\Omega')$ for any $\Omega' \in Bd(\Omega)$. Let $C^m_c(\Omega)$ (respectively, $C^m_c(\Omega')$, when $\Omega' \in Bd(\Omega)$) be the linear space consisting of m times continuously differentiable and finite on Ω (respectively, on Ω') functions. Suppose that $C^m_c(\overline{\Omega})$ be the linear space consisting of m times continuously differentiable on $\overline{\Omega}$ functions that have bounded supports, that is, their supports are compact sets in $\overline{\Omega}$. Denote the closure of the space $C^m_c(\Omega)$ (respectively, $C^m_c(\Omega')$, when $\Omega' \in Bd(\Omega)$) in $H^m_{\mathrm{loc}}(\overline{\Omega})$ by $\overset{\circ}{H}{}^m_{\mathrm{loc}}(\overline{\Omega})$ (respectively, $\overset{\circ}{H}{}^m_{\mathrm{loc}}(\Omega')$) and the subspace of the space $\overset{\circ}{H}{}^m_{\mathrm{loc}}(\overline{\Omega})$ consisting of functions with a bounded support by $\overset{\circ}{H}{}^m_{\mathrm{c}}(\Omega)$.

For arbitrary $\Omega' \in Bd(\Omega)$ the $H^{m,0}(\Omega' \times (0,T)) := \{w \in L_2(\Omega' \times (0,T)) \big| D^\alpha w \in L_2(\Omega' \times (0,T)), |\alpha| \le m\}$ is the a standard Sobolev space with the norm

$$\|v\|_{H^{m,0}(\Omega'\times(0,T))} := \left(\int_0^T \int_{\Omega'} \sum_{|\alpha|\le m} |D^\alpha v|^2 dxdt \right)^{1/2}.$$

We will also consider the space $H^{m,0}_{\mathrm{loc}}(\overline{Q}) := \{w \in L_{2,\mathrm{loc}}(\overline{\Omega}) \big|\ D^\alpha w \in L_{2,\mathrm{loc}}(\overline{Q}), |\alpha| \le m\}$, which is completely linear locally convex space with the system of seminorms $\{\|\cdot\|_{H^{m,0}(\Omega'\times(0,T))} \big| \Omega' \in Bd(\Omega).\}$ The sequence $\{w_j\}$ converges to w in the $H^{m,0}_{\mathrm{loc}}(\overline{Q})$ if the sequence $\{w_j\big|_{\Omega'\times(0,T)}\}$ converges to $w\big|_{\Omega'\times(0,T)}$ in the $H^{m,0}(\Omega' \times (0,T))$ for any $\Omega' \in Bd(\Omega)$. Denote by $\overset{\circ}{H}{}^{m,0}_{\mathrm{loc}}(\overline{Q})$ (respectively, $\overset{\circ}{H}{}^{m,0}(\Omega' \times (0,T))$ for any $\Omega' \in Bd(\Omega)$) the subspace of the space $H^{m,0}_{\mathrm{loc}}(\overline{Q})$ (respectively, $H^{m,0}(\Omega' \times (0,T))$) consisting of functions w such that $w(\cdot,t) \in \overset{\circ}{H}{}^m_{\mathrm{loc}}(\overline{\Omega})$ (respectively, $\overset{\circ}{H}{}^m(\Omega')$) for a.e. $t \in (0,T)$.

We define the function $\widetilde{b}\colon \Omega \to \mathbb{R}$ by taking $\widetilde{b}(x) := b(x)$ if $x \in \Omega_0$, and $\widetilde{b}(x) := 1$, if $x \in \Omega \setminus \Omega_0$. For any $\Omega' \in Bd(\Omega)$ we denote by $L^2(b;\Omega')$ the linear seminormed space of functions $w\colon \Omega' \to \mathbb{R}$ such that $w = \widetilde{b}^{-1/2}v$, where $v \in L_2(\Omega')$, with seminorm $\|w\|_{L^2(b;\Omega')} := \big(\int_{\Omega'} b(x)|w(x)|^2dx\big)^{1/2}$, and by $L^2_{\rm loc}(b;\overline{\Omega})$ a linear locally convex seminormed space consisting of measurable functions $w\colon \Omega \to \mathbb{R}$, the restrictions of which to an arbitrary $\Omega' \in Bd(\Omega)$ belongs to $L^2(b;\Omega')$, with the topology given by the system of seminorms $\{\|\cdot\|_{L^2(b;\Omega')} \big| \Omega' \in Bd(\Omega)\}$. It is easy to verify that the space $L^2_{\rm loc}(b;\overline{\Omega})$ is a completion of the space $\overset{\circ}{H}{}^m_{\rm loc}(\overline{\Omega})$, and if $b(x) \ge b_0 = \text{const} > 0$ for a.e. $x \in \Omega$ then $v \in L^2_{\rm loc}(b;\overline{\Omega})$ iff $v \in L^2_{\rm loc}(\overline{\Omega})$.

The space $C\big([0,T];L^2_{\rm loc}(b;\overline{\Omega})\big)$ is a space of functions $w(x,t), (x,t) \in Q$, such that for an arbitrary bounded subdomain Ω' of the domain Ω (that is $\Omega' \in Bd(\Omega)$) their restrictions to $\Omega' \times (0,T)$ belong to the space $C\big([0,T];L^2(b;\Omega')\big)$ with the norm $\|w\|_{C\left([0,T];L^2(b;\Omega')\right)} := \max_{t\in[0,T]} \|w(t)\|_{L^2(b;\Omega')}$. The space $C\big([0,T];L^2_{\rm loc}(b;\overline{\Omega})\big)$ is a complete linear locally convex space with a system of seminorms $\{\|\cdot\|_{C([0,T];L_2(b;\Omega'))} | \Omega' \in Bd(\Omega)\}$.

Now let us introduce the conditions on the input data of the considered problem.

Let us $\mathbb{A}_p$, where p is a function that satisfies the condition $(\mathbf{P})$, be the set of ordered tuples $(a_\alpha) := (a_{\widehat{0}}, \ldots, a_\alpha, \ldots)$ of N real-valued functions defined on $Q \times \mathbb{R}^N$, which are numbered by multi-indices of dimension n having lengths from M and ordered lexicographically, and the components of the set (a_α) satisfy the conditions:

$(\mathbf{A_1})$ for every $\alpha, |\alpha| \in M$, function $a_\alpha(x,t,\xi)$, $(x,t,\xi) \in Q \times \mathbb{R}^N$, is Caratheodory type that is the function $a_\alpha(x,t,\cdot)\colon \mathbb{R}^N \to \mathbb{R}$ is continuous for almost every $(x,t) \in Q$, and function $a_\alpha(\cdot,\cdot,\xi)\colon Q \to \mathbb{R}$ is measurable for every $\xi \in \mathbb{R}^N$, moreover $a_\alpha(x,t,0) = 0$ for a.e. $(x,t) \in Q$;

$(\mathbf{A_2})$ the following inequalities hold for a.e. $(x,t) \in Q$ and any $\xi \in \mathbb{R}^N$:

$$|a_{\widehat{0}}(x,t,\xi)| \le h_{\widehat{0}}(x,t)\Big(\sum_{|\alpha|\in M_0} |\xi_\alpha|^{2/p'(x)} + |\xi_{\widehat{0}}|^{p(x)-1}\Big) + g_{\widehat{0}}(x,t),$$

$$|a_\alpha(x,t,\xi)| \le h_\alpha(x,t)|\xi| + g_\alpha(x,t), \quad |\alpha| \in M_0,$$

where $h_\alpha \in L_{\infty,\rm loc}(\overline{Q})$, $|\alpha| \in M$, $g_{\widehat{0}} \in L_{p'(\cdot),\rm loc}(\overline{Q})$, $g_\alpha \in L_{2,\rm loc}(\overline{Q})$, $|\alpha| \in M_0$;

$(\mathbf{A_3})$ there exist constants $B_1 > 0$ and $B_2 \ge 0$ such that the following inequality holds for each $\alpha, |\alpha| \in M_0$, almost all $(x,t) \in Q$ and any ξ and η from $\mathbb{R}^N$:

$$|a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta)| \le \Big(B_1 \sum_{|\alpha|\in M_0} |\xi_\alpha - \eta_\alpha|^2 + B_2|\xi_{\widehat{0}} - \eta_{\widehat{0}}|^2\Big)^{1/2};$$

$(\mathbf{A_4})$ there exist constants $K_1 > 0$, $K_2 \ge 0$, $K_3 > 0$ such that the following inequality holds for a.e. $(x,t) \in Q$, for any ξ and η from $\mathbb{R}^N$:

$$\sum_{|\alpha|\in M} (a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta))(\xi_\alpha - \eta_\alpha)$$
$$\ge K_1 \sum_{|\alpha|\in M_0} |\xi_\alpha - \eta_\alpha|^2 + K_2|\xi_{\widehat{0}} - \eta_{\widehat{0}}|^2 + K_3|\xi_{\widehat{0}} - \eta_{\widehat{0}}|^{p(x)},$$

moreover, if one of the two following conditions is met: $B_2 > 0$ or $p^+ \ge 2(n+1)/n$, then $K_2 > 0$.

Remark 1. If $M = \{0, m\}$, $p^+ \in (2; 2(n+1)/n)$, then, in particular, the elements of the set $\mathbb{A}_p$ are tuples (a_α), whose components are functions $a_{\widehat{0}}(x,t,\xi) := \widetilde{a}_{\widehat{0}}(x,t)|\xi_{\widehat{0}}|^{p(x)-2}\xi_{\widehat{0}}$, $a_\alpha(x,t,\xi) = \widetilde{a}_\alpha(x,t)\xi_\alpha$, $(x,t,\xi) \in Q \times \mathbb{R}^N$, for every $\alpha, |\alpha| = m$, where $\widetilde{a}_\alpha, |\alpha| \in M$, are measurable bounded positive and zero-separated functions. By the way, we will get equation (6) for one of these tuples (a_α).

Let $\mathbb{F}_{p,\mathrm{loc}}(\overline{Q})$ be the set of the ordered arrays of N real valued functions $(f_\alpha := (f_{\widehat{0}}, \ldots, f_\alpha, \ldots)$ defined on Q, which are numbered in the same way as the components of the elements of the set $\mathbb{A}_p$, and satisfy the condition:
(**F**) $f_{\widehat{0}} \in L_{p'(\cdot),\mathrm{loc}}(\overline{Q})$, $f_\alpha \in L_{2,\mathrm{loc}}(\overline{Q})$, $|\alpha| \in M_0$.
Denote

$$\mathbb{U}_{p,\mathrm{loc}}(\overline{Q}) := \overset{\circ}{H}{}^{m,0}_{\mathrm{loc}}(\overline{Q}) \cap L_{p(\cdot),\mathrm{loc}}(\overline{Q}) \cap C\big([0,T]; L^2_{\mathrm{loc}}(b;\overline{\Omega})\big).$$

Let us say that the sequence $\{v_k\}_{k=1}^{\infty}$ converges to v in $\mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ if for each domain $\Omega' \in Bd(\Omega)$ the sequence $\{v_k\big|_{\Omega'\times(0,T)}\}_{k=1}^{\infty}$ converges to $v|_{\Omega'\times(0,T)}$ in $H^{m,0}(\Omega' \times (0,T)) \cap L_{p(\cdot)}(\Omega' \times (0,T)) \cap C\big([0,T]; L^2(b;\Omega')\big)$.

Definition 1. Let $(a_\alpha) \in \mathbb{A}_p$, $(f_\alpha) \in \mathbb{F}_{p,\mathrm{loc}}(\overline{Q})$ and $u_0 \in L^2_{\mathrm{loc}}(b;\overline{\Omega})$. A function $u \in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ is called a *weak solution* to the problem (3)–(5), if it satisfies the initial condition (5) (as element of $C([0,T]; L^2_{\mathrm{loc}}(b;\overline{\Omega})))$ and the integral equality

$$\iint_Q \Big[-bu\psi\varphi' + \sum_{|\alpha|\in M} a_\alpha(x,t,\delta u) D^\alpha \psi\varphi\Big] dxdt = \iint_Q \sum_{|\alpha|\in M} f_\alpha D^\alpha \psi\varphi dxdt \tag{7}$$

for any $\psi \in \overset{\circ}{H}{}^m_c(\Omega) \cap L_{p(\cdot)}(\Omega)$, $\varphi \in C^1_c(0,T)$.

We consider the existence and uniqueness of the weak solution to the problems (3)–(5). In order to formulate and prove the corresponding result, we use the following notation: for arbitrary $R > 0$

$$\Omega_R \quad \text{is a connected component of the set } \ \Omega \cap \{x \in \mathbb{R}^n \big| |x| < R\}, \quad \Gamma_R := \partial\Omega_R,$$

$$Q_R := \Omega_R \times (0,T), \quad \Sigma_R := \Gamma_R \times (0,T).$$

Theorem 1. *Let p satisfy the condition* (**P**), $(a_\alpha) \in \mathbb{A}_p$, $(f_\alpha) \in \mathbb{F}_{p,\mathrm{loc}}(\overline{Q})$ *i* $u_0 \in L^2_{loc}(b;\overline{\Omega})$. *Then the problem* (3)–(5) *has unique weak solution and it satisfies the following estimate for any R, R_0 such that $R \geq 1, 0 < R_0 \leq R/2$:*

$$\max_{t\in[0,T]} \int_{\Omega_{R_0}} b(x)|u(x,t)|^2 dx + \iint_{Q_{R_0}} \Big[\sum_{|\alpha|\in M_0} |D^\alpha u(x,t)|^2 + K_2|u(x,t)|^2 + |u(x,t)|^{p(x)}\Big] dxdt \leq$$

$$\leq C_1\Big\{R^{n-\frac{2q}{q-2}} + \iint_{Q_R} \Big[\sum_{|\alpha|\in M_0} \big|f_\alpha(x,t)\big|^2 + \big|f_{\widehat{0}}(x,t)\big|^{p'(x)}\Big] dxdt + \int_{\Omega_R} b(x)\big|u_0(x)\big|^2 dx\Big\}, \tag{8}$$

where $q = p^+$, if $K_2 = 0$, and $q \in (2; p^-] \cup \{p^+\}$ is arbitrary if $K_2 > 0$, and C_1 is a positive constant depending only on $b, B_1, B_2, K_1, K_2, K_3, p^-, p^+, n, m, q$.

2. Auxiliary statements. In this section, we present the statements that will be used in the next section to prove the main result.

Proposition 1 ([13, 15]). *The following inequalities hold for any $R > 0$ and arbitrary $v \in L_{p(\cdot)}(Q_R)$:*

$$\min\big\{\big(\rho_{p,R}(v)\big)^{1/p^-}, \big(\rho_{p,R}(v)\big)^{1/p^+}\big\} \le \|v\|_{L_{p(\cdot)}(Q_R)} \le \max\big\{\big(\rho_{p,R}(v)\big)^{1/p^-}, \big(\rho_{p,R}(v)\big)^{1/p^+}\big\},$$

$$\min\big\{\big(\|v\|_{L_{p(\cdot)}(Q_R)}\big)^{p^-}, \big(\|v\|_{L_{p(\cdot)}(Q_R)}\big)^{p^+}\big\} \le \rho_{p,R}(v) \le \max\big\{\big(\|v\|_{L_{p(\cdot)}(Q_R)}\big)^{p^-}, \big(\|v\|_{L_{p(\cdot)}(Q_R)}\big)^{p^+}\big\},$$

where

$$\rho_{p,R}(v) := \iint\limits_{Q_R} |v(x,t)|^{p(x)} dxdt, \quad \|v\|_{L_{p(\cdot)}(Q_R)} := \inf\{\lambda > 0 | \rho_{p,R}(v/\lambda) \le 1\}.$$

Lemma 1. *Let the number $R_* \ge 1$ and functions $v \in \overset{\circ}{H}{}^{m,0}(Q_{R_*}) \cap L_{p(\cdot)}(Q_{R_*})$, $g_{\hat{0}} \in L_{p'(\cdot)}(Q_{R_*})$, $g_\alpha \in L_2(Q_{R_*})$, $|\alpha| \in M_0$, be such that*

$$\iint\limits_{Q_{R_*}} \big[-bv\psi\varphi' + \sum_{|\alpha|\in M} g_\alpha D^\alpha \psi\varphi\big] dxdt = 0 \tag{9}$$

for any $\psi \in \overset{\circ}{H}{}^m(\Omega_{R_*}) \cap L_{p(\cdot)}(\Omega_{R_*})$, $\varphi \in C_c^1(0,T)$.

Then $v \in C\big([0,T]; L_2(b;\Omega_R)\big)$ $\forall\ R \in (0,R_*)$ and for arbitrary functions $\theta \in C^1([0,T])$, $w \in C^m(\overline{\Omega})$, $\mathrm{supp}\, w \subset \overline{\Omega}_R$ for some $R \in (0,R_*)$, and any numbers t_1, t_2, $0 \le t_1 < t_2 \le T$, the following equality holds

$$\theta(t_2)\int\limits_\Omega b(x)|v(x,t_2)|^2 w(x)dx - \theta(t_1)\int\limits_\Omega b(x)|v(x,t_1)|^2 w(x)dx -$$
$$-\int\limits_{t_1}^{t_2}\int\limits_\Omega b(x)|v(x,t)|^2 w(x)\theta'(t)dxdt + 2\int\limits_{t_1}^{t_2}\int\limits_\Omega \sum_{|\alpha|\in M} g_\alpha D^\alpha(vw)\theta dxdt = 0. \tag{10}$$

Proof of Lemma 1. We will use the scheme of the proof of Lemma 1 in [6] and Lemma 1 in [7].

First of all replace the variables in the identity (9) such way

$$\mu = 2(t - T/2), \qquad t \in [0,T], \quad \mu \in [-T,T], \tag{11}$$

(then $t = (s+T)/2$) and put

$$\Omega_* := \Omega_{R_*}, \quad G_* := \Omega_{R_*} \times (-T,T), \quad \widetilde{v}(x,\mu) := \begin{cases} v(x,(\mu+T)/2), & \text{if } \ \mu \in (-T,T); \\ 0, & \text{if } \ \mu \notin (-T,T), \end{cases} \tag{12}$$

$$\widetilde{g}_\alpha(x,\mu) := \begin{cases} g_\alpha(x,(\mu+T)/2), & \text{if } \ \mu \in (-T,T); \\ 0, & \text{if } \ \mu \notin (-T,T), \end{cases} \quad x \in \Omega_*. \tag{13}$$

As a result, we get the identity:

$$\iint\limits_{G_*} \big[-b(x)\widetilde{v}(x,\mu)\psi(x)\widetilde{\varphi}'(\mu) + 2^{-1}\sum_{|\alpha|\in M} \widetilde{g}_\alpha(x,\mu) D^\alpha\psi(x)\widetilde{\varphi}(\mu)\big] dxd\mu = 0 \tag{14}$$

for any $\psi \in \overset{\circ}{H}{}^m(\Omega_*) \cap L_{p(\cdot)}(\Omega_*)$, $\widetilde{\varphi} \in C_c^1(-T,T)$.

Let k be any natural number. Let us replace the variables in (14)

$$s = \lambda_k \mu, \quad \mu \in [-T,T], \quad s \in [-\lambda_k T, \lambda_k T] = [-T-2/k, T+k/2],$$

where $\lambda_k := 1 + \frac{2}{kT} > 1$ (then $\mu = s/\lambda_k$). As a result, we get equality

$$\int\limits_{-T-2/k}^{T+2/k} \int\limits_{\Omega_*} \Big[-b(x)\widetilde{v}(x,s/\lambda_k)\psi(x)\widehat{\varphi}'(s) + (2\lambda_k)^{-1} \sum_{|\alpha|\in M} \widetilde{g}_\alpha(x,s/\lambda_k) D^\alpha\psi(x)\widehat{\varphi}(s)\Big] dx ds = 0 \quad (15)$$

for any $\psi \in \overset{\circ}{H}{}^m(\Omega_*) \cap L_{p(\cdot)}(\Omega_*)$, $\widehat{\varphi} \in C_c^1(-T-2/k, T+2/k)$.

Let $\omega_1(t) = Ce^{t^2/(t^2-1)}$, if $|t| < 1$, and $\omega_1(t) = 0$, if $|t| \geq 1$, where $C > 0$ is a constant such that $\int_{-\infty}^{+\infty} \omega_1(t)dt = 1$. It is obvious that $\omega_1 \in C_c^\infty(\mathbb{R})$ i $\omega_1(t) \geq 0$ for all $t \in \mathbb{R}$. For an arbitrary $\rho > 0$, we put $\omega_\rho(t) = \rho^{-1}\omega_1(t/\rho)$, $t \in \mathbb{R}$. It is known that the functions $\omega_\rho, \rho > 0$, are called *mollifiers*. It is easy to verify the correctness of the following chain of equalities

$$\int\limits_{-\infty}^{+\infty} \widetilde{v}(x,s/\lambda_k)\frac{d}{ds}\omega_{1/k}(\tau-s)ds = -\int\limits_{-\infty}^{+\infty} \widetilde{v}(x,s/\lambda_k)\frac{d}{d\tau}\omega_{1/k}(\tau-s)ds =$$

$$= -\frac{d}{d\tau}\int\limits_{-\infty}^{+\infty} \widetilde{v}(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds, \quad \tau \in \mathbb{R}. \quad (16)$$

Let us take in (15) $\widehat{\varphi}(s) = \omega_{1/k}(\tau-s)$ in (15), $s \in (-T-k/2, T+k/2)$, where $\tau \in [-T,T]$. As a result, taking (16) into account, we get

$$\int\limits_{\Omega_*} \Big[b(x)\big(v_k(x,\tau)\big)_\tau \psi(x) + \sum_{|\alpha|\in M} g_{\alpha,k}(x,\tau)D^\alpha\psi(x)\Big] dx = 0, \quad \tau \in [-T,T], \quad (17)$$

where

$$v_k(x,\tau) := \int\limits_{-\infty}^{+\infty} \widetilde{v}(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds = \int\limits_{|s-\tau|<1/k} \widetilde{v}(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds,$$

$$g_{\alpha,k}(x,\tau) := (2\lambda_k)^{-1}\int\limits_{-\infty}^{+\infty} \widetilde{g}_\alpha(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds = (2\lambda_k)^{-1}\int\limits_{|s-\tau|<1/k} \widetilde{g}_\alpha(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds,$$

$(x,\tau) \in \Omega \times [-T,T]$, $|\alpha| \in M$, and $\psi \in \overset{\circ}{H}{}^m(\Omega_*) \cap L_{p(\cdot)}(\Omega_*)$ is an arbitrary function.

Let us show that

$$v_k \underset{k\to\infty}{\longrightarrow} \widetilde{v} \quad \text{in} \quad L_{p(\cdot)}(G_*). \quad (18)$$

At the same time we will use the notation for convenience: $p_1 := p^-, p_2 := p^+$. As the space $C(\overline{G_*})$ is dense in $L_{p(\cdot)}(G_*)$ (see, e.g., [13]), then there exists a sequence of elements $\{\widetilde{v}_l\}_{l=1}^\infty$ of the space $C(\overline{G_*})$ such that $\|\widetilde{v} - \widetilde{v}_l\|_{L_{p(\cdot)}(G_*)} \to 0$ as $l \to +\infty$. Taking this into account and using Hölder's inequality, we have the following:

$$\iint\limits_{G_*} |v_k(x,\tau)-\widetilde{v}(x,\tau)|^{p(x)}dxd\tau =$$

$$= \iint\limits_{G_*}\Big| \int\limits_{|s-\tau|<1/k} \widetilde{v}(x,s/\lambda_k)\omega_{1/k}(\tau-s)ds - \widetilde{v}(x,\tau)\Big|^{p(x)} dxd\tau =$$

$$= \iint\limits_{G_*}\Big| \int\limits_{|s-\tau|<1/k} \big[\widetilde{v}(x,s/\lambda_k)-\widetilde{v}_l(x,s/\lambda_k)\big]\omega_{1/k}(\tau-s)ds +$$

$$+ \int\limits_{|s-\tau|<1/k} \big[\widetilde{v}_l(x,s/\lambda_k)-\widetilde{v}_l(x,\tau/\lambda_k)\big]\omega_{1/k}(\tau-s)ds +$$

$$+\big[\widetilde{v}_l(x,\tau/\lambda_k)-\widetilde{v}_l(x,\tau)\big]+\big[\widetilde{v}_l(x,\tau)-\widetilde{v}(x,\tau)\big]\Big|^{p(x)} dxd\tau \le$$

$$\le C_2 \iint\limits_{G_*}\Big[\Big| \int\limits_{|s-\tau|<1/k} \big[\widetilde{v}(x,s/\lambda_k)-\widetilde{v}_l(x,s/\lambda_k)\big]\omega_{1/k}(\tau-s)ds\Big|^{p(x)} +$$

$$+\Big| \int\limits_{|s-\tau|<1/k} \big[\widetilde{v}_l(x,s/\lambda_k)-\widetilde{v}_l(x,\tau/\lambda_k)\big]\omega_{1/k}(\tau-s)ds\Big|^{p(x)} +$$

$$+\big|\widetilde{v}_l(x,\tau/\lambda_k)-\widetilde{v}_l(x,\tau)\big|^{p(x)}+\big|\widetilde{v}_l(x,\tau)-\widetilde{v}(x,\tau)\big|^{p(x)}\Big] dxd\tau \le$$

$$\le C_2\Big[\iint\limits_{G_*}\Big(\int\limits_{|s-\tau|<1/k} \big|\widetilde{v}(x,s/\lambda_k)-\widetilde{v}_l(x,s/\lambda_k)\big|^{p(x)}ds\Big)\times$$

$$\times\Big(\int\limits_{|s-\tau|<1/k} \big|\omega_{1/k}(\tau-s)\big|^{p'(x)}ds\Big)^{p(x)-1} dxd\tau +$$

$$+ \max_{i\in\{1,2\}}\Big(\max_{x\in\Omega_*,s,\tau\in[-T-1/k,T+1/k],|s-\tau|\le 1/k} \big|\widetilde{v}_l(x,s/\lambda_k)-\widetilde{v}_l(x,\tau/\lambda_k)\big|\Big)^{p_i}\cdot \mathrm{mes}G_* +$$

$$+ \max_{i\in\{1,2\}}\Big(\max_{x\in\overline{\Omega_*},\tau\in[-T,T]} \big|\widetilde{v}_l(x,\tau/\lambda_k)-\widetilde{v}_l(x,\tau)\big|\Big)^{p_i}\cdot \mathrm{mes}G_* +$$

$$+\iint\limits_{G_*}\big|\widetilde{v}_l(x,\tau)-\widetilde{v}(x,\tau)\big|^{p(x)}dxd\tau\Big] \equiv C_2\big[I_1(k,l)+I_2(k,l)+I_3(k,l)+I_4(l)\big], \qquad (19)$$

where $C_2 := 4^{p^+-1}$. Here we used the corollary of discrete Hölder's inequality

$$\sum_{i=1}^m a_i b_i \le \left(\sum_{i=1}^m |a_i|^q\right)^{1/q}\left(\sum_{i=1}^m |b_i|^{q'}\right)^{1/q'}, \qquad (20)$$

where $m\in\mathbb{N},\ a_i\ge 0, b_i\ge 0, i=\overline{1,m},\ q>1$, when $m=4$ and $q=p(x), b_i=1$, that is inequality

$$\left(\sum_{i=1}^4 a_i\right)^{p(x)} \le 4^{p(x)-1}\sum_{i=1}^4 |a_i|^{p(x)}, \quad a_i\ge 0, i=\overline{1,4}, \quad \text{for a.e. } x\in\Omega_*.$$

Let $\varepsilon>0$ be a sufficiently small real number. We show that for sufficiently large values of $k\in\mathbb{N}$, the right-hand side of the inequality (19) is smaller than ε. Given that $|\omega_{1/k}(t)|\le$

$k \cdot \max\limits_{|z|\le 1} \omega_1(z) = C_3 k, \quad t \in \mathbb{R}$, where $C_3 := \max\limits_{|z|\le 1} \omega_1(z) > 0$ (C_3 is a constant which does not depend on k), thus

$$\Big(\int\limits_{|s-\tau|<1/k} \big|\omega_{1/k}(\tau - s)\big|^{p'(x)} ds\Big)^{p(x)-1} \le \Big(\int\limits_{|z|<\frac{1}{k}} \big|\omega_{1/k}(z)\big|^{p'(x)} dz\Big)^{p(x)-1} \le$$

$$\le \Big(\int\limits_{|z|<\frac{1}{k}} \big|C_3 k\big|^{p'(x)} dz\Big)^{p(x)-1} \le (C_3 k)^{p(x)} \Big(2k^{-1}\Big)^{p(x)-1} = (2C_3)^{p(x)} k/2 \le C_4 k,$$

where $C_4 := \max\limits_{i\in\{1,2\}} (2C_3)^{p^i}/2$. Using the Cauchy-Buniakovsky-Schwarz inequality we have the following estimates:

$$I_1(k,l) := \iint\limits_{G_*} \Big(\int\limits_{|s-\tau|<1/k} \big|\widetilde{v}(x, s/\lambda_k) - \widetilde{v}_l(x, s/\lambda_k)\big|^{p(x)} ds\Big) \times$$

$$\times \Big(\int\limits_{|s-\tau|<1/k} \big|\omega_{1/k}(\tau - s)\big|^{p'(x)} ds\Big)^{p(x)-1} dx d\tau \le$$

$$\le C_4 k \iint\limits_{G_*} \Big(\int\limits_{|s-\tau|<1/k} \big|\widetilde{v}(x, s/\lambda_k) - \widetilde{v}_l(x, s/\lambda_k)\big|^{p(x)} ds\Big) dx d\tau =$$

$$= \Big[\ s + \tau = z, s = z + \tau, ds = dz\ \Big] =$$

$$= C_4 k \iint\limits_{G_*} \Big(\int\limits_{|z|<\frac{1}{k}} \big|\widetilde{v}\big(x, (z+\tau)/\lambda_k\big) - \widetilde{v}_l\big(x, (z+\tau)/\lambda_k\big)\big|^{p(x)} dz\Big) dx d\tau =$$

$$= C_4 k \int\limits_{|z|<\frac{1}{k}} dz \int\limits_{\Omega_*} dx \int\limits_{-T}^{T} \big|\widetilde{v}\big(x, (z+\tau)/\lambda_k\big) - \widetilde{v}_l\big(x, (z+\tau)/\lambda_k\big)\big|^{p(x)} d\tau =$$

$$= \Big[\ (z+\tau)/\lambda_k = t, \quad \tau = \lambda_k t - z, \quad d\tau = \lambda_k dt\ \Big] \le$$

$$\le 2C_4\lambda_k \int\limits_{\Omega_*} dx \int\limits_{-T}^{T} \big|\widetilde{v}(x,t) - \widetilde{v}_l(x,t)\big|^{p(x)} dt = 2C_4\lambda_k \iint\limits_{G_*} \big|\widetilde{v}(x,t) - \widetilde{v}_l(x,t)\big|^{p(x)} dx dt$$

(note that C_4 is a constant that does not depend on k and l). From this and taking into account the fact that $\|\widetilde{v} - \widetilde{v}_l\|_{L_{p(\cdot)}(G_*)} \to 0$ as $l \to +\infty$, and sequence $\{\lambda_k\}$ is bounded it follows based on the statement of Proposition 1 the existence of $l_0 \in \mathbb{N}$ such that

$$I_1(k, l_0) \le 2C_4\lambda_k \iint\limits_{G_*} \big|\widetilde{v}(x,t) - \widetilde{v}_{l_0}(x,t)\big|^{p(x)} dx dt < \frac{\varepsilon}{4C_2}, \quad k \in \mathbb{N}. \tag{21}$$

Then

$$I_4(l_0) = \iint\limits_{G_*} \big|\widetilde{v}_{l_0}(x,t) - \widetilde{v}(x,t)\big|^{p(x)} dx d\tau < \frac{\varepsilon}{4C_2}. \tag{22}$$

As $\widetilde{v}_{l_0} \in C(\overline{G_*})$ and $\overline{G_*}$ is a compact, and therefore the function $\widetilde{v}_{l_0}$ is uniformly continuous on $\overline{G_*}$, then there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ the following estimation takes place:

$$I_2(k, l_0) := \max_{i\in\{1,2\}} \Big(\max_{x\in\Omega_*, s,\tau\in[-T-1/k, T+1/k], |\tau-s|\le\frac{1}{k}} \big|\widetilde{v}_{l_0}(x, s/\lambda_k) - \widetilde{v}_{l_0}(x, \tau/\lambda_k)\big|\Big)^{p_i} \mathrm{mes} G_* < \frac{\varepsilon}{4C_2}, \tag{23}$$

$$I_3(l_0) := \max_{i\in\{1;2\}} \Big(\max_{x\in\overline{\Omega_*}, \tau\in[-T,T]} \big|\widetilde{v}_{l_0}(x, \tau/\lambda_k) - \widetilde{v}_{l_0}(x, \tau)\big|\Big)^{p_i} \mathrm{mes} G_* < \frac{\varepsilon}{4C_2}. \tag{24}$$

From (19) when $l = l_0$, taking into account (21)–(24), we get

$$\iint\limits_{G_*} \big|v_k(x, \tau) - \widetilde{v}(x, \tau)\big|^{p(x)} dxd\tau < \varepsilon \quad \forall k > k_0.$$

Since $\varepsilon > 0$ is an arbitrary number, we get what we need.

Similarly as (18), we can prove that

$$v_k \underset{k\to\infty}{\longrightarrow} \widetilde{v} \text{ in } H^{m,0}(G_*), \quad g_{\widehat{0},k} \underset{k\to\infty}{\longrightarrow} \widetilde{g}_{\widehat{0}}\big|_{G_*} \text{ in } L_{p'(\cdot)}(G_*), \tag{25}$$

$$g_{\alpha,k} \underset{k\to\infty}{\longrightarrow} \widetilde{g}_\alpha \text{ in } L_2(G_*), \quad |\alpha| \in M_0. \tag{26}$$

Let k and l be arbitrary natural numbers. Then from equality (17) we get

$$\int\limits_\Omega \Big[b(x)\big(v_{kl}(x, \tau)\big)_\tau \psi(x) + \sum_{|\alpha|\in M} g_{\alpha,kl}(x, \tau) D^\alpha \psi(x)\Big] dx = 0, \quad \tau \in [-T, T], \tag{27}$$

where $v_{kl}(x, \tau) := v_k(x, \tau) - v_l(x, \tau)$, $g_{\alpha,kl}(x, \tau) := g_{\alpha,k}(x, \tau) - g_{\alpha,l}(x, \tau), (x, \tau) \in G$, $\psi \in \overset{\circ}{H}{}^m(\Omega_*) \cap L_{p(\cdot)}(\Omega_*)$ is an arbitrary function.

Let $w \in C^m_c(\overline{\Omega})$, $\mathrm{supp} w \subset \overline{\Omega_{R_*}}$. For arbitrary $\tau \in [-T, T]$ let us put in (27) $\psi(x) = v_{kl}(x, \tau)w(x), x \in \Omega$. As a result, for $\tau \in [-T, T]$ we have

$$\int\limits_\Omega \Big[b(x)(v_{kl}(x, \tau))_\tau v_{kl}(x, \tau)w(x) + \sum_{|\alpha|\in M} g_{\alpha,kl}(x, \tau) D^\alpha\big(v_{kl}(x, \tau)w(x)\big)\Big] dx = 0. \tag{28}$$

Let $\theta \in C^1\big([-T, T]\big)$. For every $\tau \in [-T, T]$ multiply (28) by $\theta(\tau)$ and integrate obtained equality with respect to τ from τ_1 to τ_2 $(-T \le \tau_1 < \tau_2 \le T)$. Then we get

$$\frac{1}{2}\int\limits_{\tau_1}^{\tau_2} \theta(\tau)w(x)\frac{d}{d\tau}\Big(\int\limits_\Omega b(x)|v_{kl}(x, \tau)|^2 dx\Big)d\tau +$$

$$+\sum_{|\alpha|\in M}\int\limits_{\tau_1}^{\tau_2}\int\limits_\Omega g_{\alpha,kl}(x, \tau)D^\alpha\big(v_{kl}(x, \tau)w(x)\big)\theta(\tau)dxd\tau = 0,$$

whence, using the partial integration formula and multiplying the obtained equality by 2, we obtain

$$\theta(\tau_2)\int\limits_\Omega b(x)|v_{kl}(x, \tau_2)|^2 w(x)dx - \theta(\tau_1)\int\limits_\Omega b(x)|v_{kl}(x, \tau_1)|^2 w(x)dx -$$

$$-\int\limits_{\tau_1}^{\tau_2}\int\limits_\Omega b(x)|v_{kl}(x, \tau)|^2 w(x)\theta'(\tau)dxd\tau +$$

$$+2\sum_{|\alpha|\in M}\int_{\tau_1}^{\tau_2}\int_{\Omega} g_{\alpha,kl}(x,\tau)D^\alpha\big(v_{kl}(x,\tau)w(x)\big)\theta(\tau)dxd\tau=0. \quad (29)$$

Let us take in (29) $\theta(\tau)=1$, if $\tau\in[0,T]$, $\theta(-T)=0$ and $0\le\theta(\tau)\le 1$, $|\theta'(\tau)|\le\frac{2}{T}$, if $\tau\in[-T;0)$, and $w(x)\ge 0, x\in\Omega$, $w(x)=1$, if $x\in\Omega_R$, where $R\in(0,R_*)$ is any fixed real number, and $w(x)=0$ if $x\notin\Omega_{R_1}$, where $R_1=(R+R_*)/2$. Then from (29) (by setting $\tau_1=-T$, $\tau_2=\tau\in[0,T]$) we can easily obtain the inequality

$$\max_{\tau\in[0,T]}\int_{\Omega_R} b(x)|v_{kl}(x,\tau)|^2dx\le 2\iint_{G_{R_*}}\sum_{|\alpha|\in M}\big|g_{\alpha,kl}(x,\tau)D^\alpha\big(v_{kl}(x,\tau)w(x)\big)\big|dxd\tau$$

$$+\frac{2}{T}\iint_{G_{R_*}} b(x)|v_{kl}(x,\tau)|^2w(x)dxd\tau. \quad (30)$$

Let us estimate $\max\limits_{\tau\in[-T;0]}\int_{\Omega_R} b(x)|v_{kl}(x,\tau)|^2dx$ similarly and as a result we get an estimate of the value $\max\limits_{\tau\in[-T;T]}\int_{\Omega_R} b(x)|v_{kl}(x,\tau)|^2dx$ as the right side of the inequality (30). On the basis of (18), (25) and (26) the right-hand side of the inequality (30) goes to zero as $k,l\to+\infty$, then the left-hand side does too. So, the sequence $\big\{b^{1/2}v_k\big|_{G_R}\big\}_{k=1}^{\infty}$ of elements of the Banach space $C\big([-T,T];L_2(\Omega_R)\big)$ (from now on $G_R:=\Omega_R\times(-T,T)$) is fundamental in this space, and therefore convergent in it. But since $b^{1/2}v_k\big|_{G_R}\underset{k\to\infty}{\longrightarrow} b^{1/2}\widetilde{v}\big|_{G_R}$ in $L_2(G_R)$, $v_k\big|_{G_R}\underset{k\to\infty}{\longrightarrow}\widetilde{v}\big|_{G_R}$ in $C\big([-T,T];L_2(b;\Omega_R)\big)$, and therefore it is clear that we have $\widetilde{v}\big|_{G_R}\in C\big([-T,T];L_2(b;\Omega_R)\big)$, where $R\in(0,R_*)$ is an arbitrary number. Hence, taking into account (11)–(13), we get that $v\in C\big([0,T];L_2(b;\Omega_R)\big)\ \forall\ R\in(0,R_*)$.

Now let us put in (17) $\psi(x)=v_k(x,\tau)w(x)\theta(\tau)$, $x\in\Omega$, for any $\tau\in[-T,T]$, where $w\in C_c^m(\overline{\Omega})$, $\mathrm{supp}w\subset\overline{\Omega_{R_*}}$, $\theta\in C^1\big([-T,T]\big)$. Using the same reasoning as when we got (29), we obtain a similar equality (29) with k instead of kl, $\tau_1=t_1$, $\tau_2=t_2$. Having passed to the limit in this equality as $k\to\infty$ and taking into account (11), we get (10). □

Remark 2. If $v\big|_{\Omega_{R_*}\times(0,T)}\in L_2\big(0,T;\overset{\circ}{H}{}^m(\Omega_{R_*})\big)$ and the conditions of the Lemma 1 are fulfilled, then $v\in C\big([0,T];L_2(\Omega_{R_*})\big)$ and the equality (10) holds with $w\equiv 1$. It easily follows from the proof of Lemma 1.

Lemma 2. *Let $R_*\ge 1$, $(a_\alpha)\in\mathbb{A}_p$ and for every $l\in\{1,2\}$ functions $(f_{\alpha,l})\in\mathbb{F}_{p,\mathrm{loc}}(\overline{Q})$, $u_{0,l}\in L^2_{loc}(b;\overline{\Omega})$ and $u_l\in\mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ satisfy the initial condition*

$$u_l(x,0)=u_{0,l}(x),\quad x\in\Omega_0\cap\Omega_{R_*}, \quad (31)$$

and integral equality

$$\iint_{Q_{R_*}}\Big[-bu_l\psi\varphi'+\sum_{|\alpha|\in M}a_\alpha(x,t,\delta u_l)D^\alpha\psi\varphi\Big]dxdt=\iint_{Q_{R_*}}\sum_{|\alpha|\in M}f_{\alpha,l}D^\alpha\psi\varphi dxdt \quad (32)$$

for any $\psi\in\overset{\circ}{H}{}^m(\Omega_{R_*})\cap L_{p(\cdot)}(\Omega_{R_*})$, $\varphi\in C_c^1(0,T)$.

Then any numbers R, R_0 such that $R \ge 1$, $0 < 2R_0 \le R \le R_*$, satisfy the inequality

$$\max_{t\in[0,T]} \int\limits_{\Omega_{R_0}} b(x)\big|u_1(x,t) - u_2(x,t)\big|^2 dx + \iiint\limits_{Q_{R_0}} \Big[\sum_{|\alpha|\in M_0} \big|D^\alpha u_1(x,t) - D^\alpha u_2(x,t)\big|^2 +$$

$$+K_2\big|u_1(x,t) - u_2(x,t)\big|^2 + \big|u_1(x,t) - u_2(x,t)\big|^{p(x)}\Big] dxdt \le$$

$$\le C_1\Big\{ R^{n-\frac{2q}{q-2}} + \iiint\limits_{Q_R} \Big[\sum_{|\alpha|\in M_0} \big|f_{\alpha,1}(x,t) - f_{\alpha,2}(x,t)\big|^2 + \big|f_{\widehat{0},1}(x,t) - f_{\widehat{0},2}(x,t)\big|^{p'(x)}\Big] dxdt +$$

$$+ \int\limits_{\Omega_R} b(x)\big|u_{0,1}(x) - u_{0,2}(x)\big|^2 dx \Big\}, \qquad (33)$$

where q and C_1 are the same as in Theorem 1.

Proof of Lemma 2. Set $v := u_1 - u_2$. From the integral identities obtained from (32), we get, respectively, for $l = 1$ and $l = 2$

$$\iiint\limits_{Q_{R_*}} \Big[-bv\psi\varphi' + \sum_{|\alpha|\in M} \big(a_\alpha(x,t,\delta u_1) - a_\alpha(x,t,\delta u_2)\big) D^\alpha\psi\varphi\Big] dxdt =$$

$$= \iiint\limits_{Q_{R_*}} \sum_{|\alpha|\in M} (f_{\alpha,1} - f_{\alpha,2}) D^\alpha\psi\varphi dxdt$$

for any $\psi \in \overset{\circ}{H}{}^m_c(\Omega_{R_*}) \cap L_{p(\cdot)}(\Omega_{R_*})$, $\varphi \in C^1_c(0,T)$. Hence, on the basis of Lemma 1, we get

$$\theta(t_2) \int\limits_{\Omega_{R_*}} b(x)|v(x,t_2)|^2 w(x)dx - \theta(t_1) \int\limits_{\Omega_{R_*}} b(x)|v(x,t_1)|^2 w(x)dx -$$

$$- \int\limits_{t_1}^{t_2} \int\limits_{\Omega_{R_*}} b(x)|v(x,t)|^2 w(x)\theta'(t)dxdt +$$

$$+2 \int\limits_{t_1}^{t_2} \int\limits_{\Omega_{R_*}} \sum_{|\alpha|\in M} \big(a_\alpha(x,t,\delta u_1) - a_\alpha(x,t,\delta u_2)\big) D^\alpha(vw)\theta dxdt =$$

$$= 2 \int\limits_{t_1}^{t_2} \int\limits_{\Omega_{R_*}} \sum_{|\alpha|\in M} (f_{\alpha,1} - f_{\alpha,2}) D^\alpha(vw)\theta dxdt, \qquad (34)$$

where $\theta \in C^1\big([0,T]\big)$, $w \in C^m(\overline{\Omega})$, $\text{supp} w \subset \overline{\Omega}_R$ for some $R \in (0, R_*)$, $t_1, t_2 \in [0,T]$ are arbitrary.

Let R_0 and R be any numbers such that $0 < 2R_0 < R \le R_*$, $R \ge 1$. Set

$$\zeta(x) := \begin{cases} (R^2 - |x|^2)/R, & \text{if } |x| \le R; \\ 0, & \text{if } |x| > R. \end{cases}$$

Let us take $t_1 = 0$, $t_2 = \tau \in (0,T]$, $\theta(t) = 1$, $t \in [0,T]$, $w(x) = \zeta^s(x), x \in \Omega$, in (34), where $s > m$ is a sufficiently large number (it is obvious that if $s > m$ we have $\zeta^s \in C^m_c(\overline{\Omega})$,

$\mathrm{supp}\zeta^s \subset \overline{\Omega_R}$). As a result, we get the equality

$$\int\limits_{\Omega_R} b(x)|v(x,\tau)|^2\zeta^s(x)dx + 2\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M}\big(a_\alpha(x,t,\delta u_1)-a_\alpha(x,t,\delta u_2)\big)D^\alpha(v\zeta^s)dxdt =$$

$$= 2\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M}(f_{\alpha,1}-f_{\alpha,2})D^\alpha(v\zeta^s)dxdt + \int\limits_{\Omega_R} b(x)\big|u_{0,1}(x)-u_{0,2}(x)\big|^2\zeta^s(x)dx, \qquad (35)$$

where $Q_R^\tau := \Omega_R \times (0,\tau)$.

Now we note the following. Let $\widetilde{v} \in \overset{\circ}{H}{}^m_{\mathrm{loc}}(\Omega)$, $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \in M_0$, $g_\alpha \in L_{2,\mathrm{loc}}(\Omega)$. It is obvious that

$$\int\limits_{\Omega} g_\alpha D^\alpha(\widetilde{v}\zeta^s)dx = \int\limits_{\Omega} g_\alpha D^\alpha\widetilde{v}\zeta^s dx + \int\limits_{\Omega} g_\alpha\big(D^\alpha(\widetilde{v}\zeta^s) - D^\alpha\widetilde{v}\zeta^s\big)dx. \qquad (36)$$

Taken into account Lemma 3.1 from [1], we have

$$\int\limits_{\Omega} g_\alpha\big(D^\alpha(\widetilde{v}\zeta^s) - D^\alpha\widetilde{v}\zeta^s\big)dx \le \varepsilon\int\limits_{\Omega}|g_\alpha|^2\zeta^s dx + \varepsilon\int\limits_{\Omega}\sum_{|\beta|=|\alpha|}\big|D^\beta\widetilde{v}\big|^2\zeta^s dx$$

$$+C_\alpha(\varepsilon)\int\limits_{\Omega}|\widetilde{v}|^2\zeta^{s-2|\alpha|}dx, \qquad (37)$$

where $\varepsilon > 0$ is an arbitrary number, $C_\alpha(\varepsilon) > 0$ is a constant which does not depend on R.

So, from (35) based on (36), (37) we will get

$$\int\limits_{\Omega_R} b(x)|v(x,\tau)|^2\zeta^s(x)dx + 2\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M}\big(a_\alpha(x,t,\delta u_1)-a_\alpha(x,t,\delta u_2)\big)D^\alpha v\zeta^s dxdt$$

$$\le \varepsilon_1\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M_0}\big|a_\alpha(x,t,\delta u_1)-a_\alpha(x,t,\delta u_2)\big|^2\zeta^s dxdt$$

$$+\varepsilon_1\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M_0}\big|f_{\alpha,1}-f_{\alpha,2}\big|^2\,\zeta^s dxdt + \varepsilon_1\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M_0}|D^\alpha v|^2\zeta^s dxdt$$

$$+C_5(\varepsilon_1)\iint\limits_{Q_R^\tau}|v|^2\sum_{i\in M_0}\zeta^{s-2i}dxdt$$

$$+2\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M}|f_{\alpha,1}-f_{\alpha,2}||D^\alpha v|\zeta^s dxdt + \int\limits_{\Omega_R} b(x)\big|u_{0,1}(x)-u_{0,2}(x)\big|^2\zeta^s(x)dx, \qquad (38)$$

where $\varepsilon_1 > 0$ is an arbitrary number, $C_5(\varepsilon_1) > 0$ is a constant which doesn't depend on R.

Let us evaluate the terms of the inequality (38). Using the condition ($\mathbf{A_4}$) and taking into account that $v := u_1 - u_2$, we get

$$\iint\limits_{Q_R^\tau}\sum_{|\alpha|\in M}\big(a_\alpha(x,t,\delta u_1)-a_\alpha(x,t,\delta u_2)\big)\big(D^\alpha u_1 - D^\alpha u_2\big)\zeta^s dxdt \ge$$

$$\geq \iint\limits_{Q_R^\tau} \Big[K_1 \sum_{|\alpha|\in M_0} |D^\alpha v|^2 + K_2|v|^2 + K_3|v|^{p(x)}\Big]\zeta^s dxdt. \tag{39}$$

Using the condition $(\mathbf{A_3})$, we obtain

$$\iint\limits_{Q_R^\tau} \sum_{|\alpha|\in M_0} \big|a_\alpha(x,t,\delta u_1) - a_\alpha(x,t,\delta u_2)\big|^2 \zeta^s dxdt \leq$$

$$\leq (N-1) \iint\limits_{Q_R^\tau} \Big(B_1 \sum_{|\alpha|\in M_0} |D^\alpha v|^2 + B_2|v|^2\Big)\zeta^s dxdt. \tag{40}$$

Further we use Young inequality

$$|ab| \leq \varepsilon|a|^\gamma + \varepsilon^{-\frac{1}{\gamma-1}}|b|^{\gamma'}, \tag{41}$$

where $a,b\in\mathbb{R}$, $\varepsilon>0$, $\gamma>1,\gamma'=\frac{\gamma}{\gamma-1}$. Note that since $\varepsilon^{-\frac{1}{\gamma-1}} = (\varepsilon^{-1})^{\frac{1}{\gamma-1}}$ for $\gamma>1$, the function $(1,+\infty)\ni\gamma\to\varepsilon^{-\frac{1}{\gamma-1}}$ is descending if $0<\varepsilon<1$.

By applying inequality (41) with $\gamma=2$, we get

$$\iint\limits_{Q_R^\tau} \sum_{|\alpha|\in M} |f_{\alpha,1} - f_{\alpha,2}||D^\alpha v|\zeta^s dxdt \leq \iint\limits_{Q_R^\tau} |f_{\widehat{0},1} - f_{\widehat{0},2}||v|\zeta^s dxdt +$$

$$+\varepsilon_2 \iint\limits_{Q_R^\tau} \sum_{|\alpha|\in M_0} |D^\alpha v|^2\zeta^s dxdt + \frac{1}{\varepsilon_2}\iint\limits_{Q_R^\tau} \sum_{|\alpha|\in M_0} |f_{\alpha,1} - f_{\alpha,2}|^2\zeta^s dxdt, \tag{42}$$

where $\varepsilon_2>0$ is an arbitrary number.

Let $(x,t)\in Q_R$ be any point such that $v(x,t)$, $p(x)$ are defined and $p^-\leq p(x)\leq p^+$. Let us put in inequality (41) $a=|v(x,t)|^2\zeta^{\frac{s}{\gamma}}(x)$, $b=\zeta^{\frac{s}{\gamma'}-2i}(x)$, $\gamma=\frac{p(x)}{2}$,$\gamma'=\frac{p(x)}{p(x)-2}$, $\varepsilon=\varepsilon_3\in(0,1)$, $i\in M_0$. As a result, we get the following inequality:

$$|v(x,t)|^2\zeta^{s-2i}(x) \leq \varepsilon_3|v(x,t)|^{p(x)}\zeta^s(x) + \varepsilon_3^{-\frac{2}{p(x)-2}}\zeta^{s-\frac{2ip(x)}{p(x)-2}}(x) \leq$$

$$\leq \varepsilon_3|v(x,t)|^{p(x)}\zeta^s(x) + \varepsilon_3^{-\frac{2}{p^--2}}\zeta^{s-\frac{2ip(x)}{p(x)-2}}(x)$$

for almost every $(x,t)\in Q_R$. Let us integrate it, assuming that $s>\frac{2mp(x)}{p(x)-2}$ for of almost all $(x,t)\in Q_R$. As a result, we get

$$\iint\limits_{Q_R^\tau} |v(x,t)|^2\zeta^{s-2i}(x)dxdt \leq \varepsilon_3 \iint\limits_{Q_R^\tau} |v(x,t)|^{p(x)}\zeta^s(x)dxdt + \varepsilon_3^{-\frac{2}{p^--2}} \iint\limits_{Q_R^\tau} \zeta^{s-\frac{2ip(x)}{p(x)-2}}(x)dxdt, \tag{43}$$

where $\varepsilon_3\in(0,1)$, $i\in M_0$, $s>\frac{2mp^-}{p^--2}$.

Let $q\in(2,p^-]$. It is obvious that $L_{p(\cdot)}(Q_R)\subset L_q(Q_R)$. Similarly to the previous one, we get

$$\iint\limits_{Q_R^\tau} |v(x,t)|^2\zeta^{s-2i}(x)dxdt \leq \varepsilon_4 \iint\limits_{Q_R^\tau} |v(x,t)|^q\zeta^s(x)dxdt + \varepsilon_4^{-\frac{2}{q-2}} \iint\limits_{Q_R^\tau} \zeta^{s-\frac{2iq}{q-2}}(x)dxdt, \tag{44}$$

where $\varepsilon_4>0$ is arbitrary, $i\in M_0$, $s>2mq/(q-2)$.

We will also use the following estimate based on (41):

$$\iint\limits_{Q_R^\tau} \left|f_{\widehat{0},1}(x,t)-f_{\widehat{0},2}(x,t)\right||v(x,t)|\zeta^s(x)dxdt \le \varepsilon_5 \iint\limits_{Q_R^\tau} |v(x,t)|^{p(x)}\zeta^s(x)dxdt+$$

$$+\varepsilon_5^{-\frac{1}{p^- -1}} \iint\limits_{Q_R^\tau} \left|f_{\widehat{0},1}(x,t)-f_{\widehat{0},2}(x,t)\right|^{p'(x)}\zeta^s(x)dxdt, \tag{45}$$

where $\varepsilon_5 \in (0,1)$ is an arbitrary constant.

From (38) on the basis of (39)–(43), (45) for sufficiently small values of ε_1, ..., ε_5, we obtain

$$\int\limits_{\Omega_R} b(x)|v(x,\tau)|^2\zeta^s dx + \iint\limits_{Q_R^\tau} \Big[K_1 \sum_{|\alpha|\in M_0} |D^\alpha v(x,t)|^2 + (2K_2-\sigma)|v(x,t)|^2+$$

$$+K_3|v(x,t)|^{p(x)}\Big]\zeta^s(x)dxdt \le C_6 \sum_{i\in M_0} \iint\limits_{Q_R} \zeta^{s-\frac{2ip(x)}{p(x)-2}}(x)dxdt+$$

$$+C_7 \iint\limits_{Q_R} \Big(\sum_{|\alpha|\in M_0} \left|f_{\alpha,1}(x,t)-f_{\alpha,2}(x,t)\right|^2 + \left|f_{\widehat{0},1}(x,t)-f_{\widehat{0},2}(x,t)\right|^{p'(x)}\Big)\zeta^s(x)dxdt+$$

$$+\int\limits_{\Omega_R} b(x)\left|u_{0,1}(x)-u_{0,2}(x)\right|^2\zeta^s dx, \tag{46}$$

where $s > \frac{2mp^-}{p^- -2}$ is an arbitrary constant; C_6, C_7 are positive constants depending only on p^-, p^+, $m, n, B_1, B_2, K_1, K_2, K_3, s$; $\sigma = 0$, if $B_2 = 0$, and $\sigma = K_2$, if $B_2 > 0$, and therefore, based on our assumption, $K_2 > 0$.

Note that

$$\frac{2mp^-}{p^- -2} \ge \frac{2mp(x)}{p(x)-2} \ge \frac{2ip(x)}{p(x)-2} \ge \frac{2ip^+}{p^+ -2} \ge \frac{2p^+}{p^+ -2}$$

for almost every $(x,t) \in Q$, $i \in M_0$. It is easy to see that $0 \le \zeta(x) \le R$ when $x \in \mathbb{R}^n$ and $\zeta(x) \ge R - R_0$ when $|x| \le R_0$. Considering what we mentioned above and, in particular, the fact that $0 < 2R_0 \le R \le R_*, R \ge 1$, we get the following from (46)

$$\max_{t\in[0,T]} \int\limits_{\Omega_{R_0}} b(x)|v(x,t)|^2 dx + \iint\limits_{Q_{R_0}} \Big[K_1 \sum_{|\alpha|\in M_0} |D^\alpha v(x,t)|^2 + (2K_2-\sigma)|v(x,t)|^2+$$

$$+K_3|v(x,t)|^{p(x)}\Big]dxdt \le C_8 R^{n-\frac{2p^+}{p^+ -2}}+$$

$$+C_9 \iint\limits_{Q_R} \Big(\sum_{|\alpha|\in M_0} \left|f_{\alpha,1}(x,t)-f_{\alpha,2}(x,t)\right|^2 + \left|f_{\widehat{0},1}(x,t)-f_{\widehat{0},2}(x,t)\right|^{p'(x)}\Big)dxdt+$$

$$+\int\limits_{\Omega_R} b(x)\left|u_{0,1}(x)-u_{0,2}(x)\right|^2 dx, \tag{47}$$

where C_8, C_9 are positive constants depending only on $p^-, p^+, m, n, K_1, K_2, K_3, B_1$ and B_2.

From (47) we easily obtain the inequality (33) with $q = p^+$. Note that we have assumed until now that $K_2 \ge 0$.

Let $K_2 > 0$. Take any $q \in (2, p^-]$. Directly make sure that for an arbitrary point $(x,t) \in Q$ such that $v(x,t)$ and $p(x)$ are defined and $p^- \le p(x) \le p^+$, the next inequality is correct

$$K_2|v(x,t)|^2 + K_3|v(x,t)|^{p(x)} \ge K_4|v(x,t)|^q, \tag{48}$$

where $K_4 = \min\{K_2, K_3\}$. From (38) based on (39)–(42), (44), (45) and (48) and reasoning similarly to the above, we will get (33) with $q \in (2, p^-]$. □

Corollary 1. *Let $R_* \ge 1$, $(a_\alpha) \in \mathbb{A}_p$, $(f_\alpha) \in \mathbb{F}_{p,\mathrm{loc}}(\overline{Q})$, $u_0 \in L_{2,loc}(\overline{\Omega})$. Suppose that the function $w \in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ satisfies the initial condition $w(x,0) = u_0(x)$, $x \in \Omega_0 \cap \Omega_{R_*}$, and integral equality*

$$\iint\limits_{Q_{R_*}} \Big[-bw\psi\varphi' + \sum_{|\alpha|\in M} a_\alpha(x,t,\delta w)D^\alpha\psi\varphi\Big]dxdt = \iint\limits_{Q_{R_*}} \sum_{|\alpha|\in M} f_\alpha D^\alpha\psi\varphi dxdt \tag{49}$$

for any $\psi \in \overset{\circ}{H}{}^m(\Omega_{R_*}) \cap L_{p(\cdot)}(\Omega_{R_*})$, $\varphi \in C_c^1(0,T)$.

Then the inequality, that differs from the inequality (8) *only in w instead of u, holds for any numbers R_0,R such that $R \ge 1$, $0 < 2R_0 \le R \le R_*$.*

3. Proof of main result.

Proof of Theorem 1. Let k be any natural number. Let u_k be a function from the space $\overset{\circ}{H}{}^{m,0}(Q_k) \cap L_{p(\cdot)}(Q_k) \cap C\big([0,T]; L_2(b;\Omega_k)\big)$, which satisfies the initial condition $u(x,0) = u_0(x)$, $x \in \Omega_0 \cap \Omega_k$, and the integral equality

$$\iint\limits_{Q_k} \Big[-bu_k\psi\varphi' + \sum_{|\alpha|\in M} a_\alpha(x,t,\delta u_k)D^\alpha\psi\varphi\Big]dxdt = \iint\limits_{Q_k} \sum_{|\alpha|\in M} f_\alpha D^\alpha\psi\varphi dxdt \tag{50}$$

for any $\psi \in \overset{\circ}{H}{}^m(\Omega_k) \cap L_{p(\cdot)}(\Omega_k)$, $\varphi \in C_c^1(0,T)$.

The proof of the existence of the function u_k is based on the Faedo-Galerkin method. The uniqueness of this function is easy to prove by considering Remark 2 and using the condition $(\mathbf{A_4})$. We will extend the function u_k to $\overline{Q}$ by zero for each $k \in \mathbb{N}$, and keep the same notation u_k to this extension. Let us show that the sequence $\{u_k\}_{k=1}^\infty$ contains a subsequence that converges to the weak solution of the problem (3)–(5).

Let k and l be arbitrary natural numbers such that $1 < k < l$, and R_0, R be any real numbers such that $0 < 2R_0 \le R \le k-1$, $R \ge 1$. Suppose that q be a real number that satisfies the corresponding conditions from the statement of the Theorem 1 and the condition $n - 2q/(q-2) < 0$.

Let $\varepsilon > 0$ be any however small number. Let us fix an arbitrarily chosen value $R_0 > 0$ and choose the value $R \ge \max\{1; 2R_0\}$ to be so large that

$$C_1 R^{n-2q/(q-2)} < \varepsilon, \tag{51}$$

where q, C_1 are the constants from formulation of Theorem 1 and the condition $n - 2q/(q-2) < 0$ holds (clearly that C_1 does not depend on R_0 and R). Based on Lemma 2, with $R_* = R+1$, for any natural numbers $k \ge R+1$ and $l > k$ we get

$$\max_{t\in[0,T]} \int\limits_{\Omega_{R_0}} b(x)|u_k(x,t) - u_l(x,t)|^2 dx + \iint\limits_{Q_{R_0}} \Big[\sum_{|\alpha|\in M_0} \big|D^\alpha u_k(x,t) - D^\alpha u_l(x,t)\big|^2 +$$
$$+ |u_k(x,t) - u_l(x,t)|^{p(x)}\Big]dxdt \le C_1 R^{n-2q/(q-2)}. \tag{52}$$

From (51), (52) it follows that the left side of the inequality (52) is smaller than ε. This means that $\{u_k\big|_{Q_{R_0}}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\overset{\circ}{H}{}^{m,0}(Q_{R_0})\cap L_{p(\cdot)}(Q_{R_0})\cap C\big([0,T];L_2(b;\Omega_{R_0})\big)$. Since $R_0>0$ is an arbitrary number, it yields the existence of a function $u\in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ such that

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad \mathbb{U}_{p,\mathrm{loc}}(\overline{Q}). \tag{53}$$

Now note that, based on the condition $(\mathbf{A_3})$, we have

$$\iint\limits_{Q_{R_0}} \sum_{|\alpha|\in M_0} \big|a_\alpha(x,t,\delta u_k)-a_\alpha(x,t,\delta u)\big|^2 dxdt \le$$
$$\le (N-1)\iint\limits_{Q_{R_0}} \Big[B_1 \sum_{|\beta|\in M_0} \big|D^\beta(u_k-u)\big|^2 + B_2|u_k-u|^2\Big]dxdt, \quad R_0>0. \tag{54}$$

Since R_0 is arbitrary, it follows from (53) and (54) that

$$a_\alpha\big(\circ,\diamond,\delta u_k(\circ,\diamond)\big)\underset{k\to\infty}{\longrightarrow} a_\alpha\big(\circ,\diamond,\delta u(\circ,\diamond)\big) \quad \text{in} \quad L_{2,\mathrm{loc}}(\overline{Q}), \quad |\alpha|\in M_0. \tag{55}$$

Now we will show that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of the sequence $\{u_k\}_{k=1}^{\infty}$ such that

$$a_{\widehat{0}}\big(\circ,\diamond,\delta u_{k_j}(\circ,\diamond)\big)\underset{j\to\infty}{\longrightarrow} a_{\widehat{0}}\big(\circ,\diamond,\delta u(\circ,\diamond)\big) \quad \text{weakly in} \quad L_{p'(\cdot),\mathrm{loc}}(\overline{Q}). \tag{56}$$

Let $R_0>0$ be any number. From Corollary 1 for any $k>2R_0$ $(k\in\mathbb{N})$ we have estimate:

$$\max_{t\in[0,T]}\int\limits_{\Omega_{R_0}} b(x)\big|u_k(x,t)\big|^2dx + \iint\limits_{Q_{R_0}}\Big[\sum_{|\alpha|\in M_0}\big|D^\alpha u_k(x,t)\big|^2 + |u_k(x,t)|^{p(x)}\Big]dxdt \le C_{10}(R_0), \tag{57}$$

where $C_{10}(R_0)>0$ is a constant that does not depend on k.

Based on the condition $(\mathbf{A_2})$ and inequality (20), taking into account (57), we have

$$\iint\limits_{Q_{R_0}} \big|a_{\widehat{0}}\big(x,t,\delta u_k(x,t)\big)\big|^{p'(x)}dxdt \le$$
$$\le \iint\limits_{Q_{R_0}} \big|h_{\widehat{0}}(x,t)\big(\sum_{|\alpha|\in M_0}|D^\alpha u_k(x,t)|^{2/p'(x)} + |u_k(x,t)|^{p(x)-1}\big) + g_{\widehat{0}}(x,t)\big|^{p'(x)}dxdt \le$$
$$\le \iint\limits_{Q_{R_0}} \big(N\big|h_{\widehat{0}}(x,t)\big|^{p(x)}+1\big)^{\frac{p'(x)}{p(x)}}\big(\sum_{|\alpha|\in M_0}|D^\alpha u_k(x,t)|^2 + |u_k(x,t)|^{p(x)} +$$
$$+|g_{\widehat{0}}(x,t)|^{p'(x)}\big)dxdt \le C_{11}(R_0), \tag{58}$$

where $C_{11}(R_0)>0$ is a constant that does not depend on k, but may depend on R_0.

On the basis of (53), (58) and the condition $(\mathbf{A_1})$, taking into account the reflexivity of the space $L_{p'(\cdot)}(Q_{R_0})$, we can conclude that there exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of a sequence $\{u_k\}_{k=1}^{\infty}$ and function $\chi_{\widehat{0}}\in L_{p'(\cdot),\mathrm{loc}}(\overline{Q})$ such that

$$u_{k_j}\underset{j\to\infty}{\longrightarrow} u, \quad a_{\widehat{0}}\big(\circ,\diamond,\delta u_{k_j}(\circ,\diamond,)\big)\underset{j\to\infty}{\longrightarrow} a_{\widehat{0}}\big(\circ,\diamond,\delta u(\circ,\diamond,)\big) \quad \text{almost everywhere on} \quad Q, \tag{59}$$

$$a_{\widehat{0}}\big(\circ,\diamond,\delta u_{k_j}(\circ,\diamond)\big)\underset{j\to\infty}{\longrightarrow}\chi_{\widehat{0}}(\circ,\diamond)\quad\text{weakly in}\quad L_{p'(\cdot),\mathrm{loc}}(\overline{Q}). \tag{60}$$

From (59) and (60) (see [17]) we obtain

$$\chi_{\widehat{0}}(\circ,\diamond)=a_{\widehat{0}}\big(\circ,\diamond,\delta u(\circ,\diamond)\big). \tag{61}$$

Let $\psi\in \overset{\circ}{H}{}^m_{\mathrm{c}}(\Omega)\cap L_{p(\cdot)}(\Omega)$. For every $j\geq j_0$, where $j_0\in\mathbb{N}$ is such that $\mathrm{supp}\psi\subset\overline{\Omega_{k_{j_0}}}$, given the definition of u_{k_j}, we have

$$\iint\limits_{Q}\Big[-bu_{k_j}\psi\varphi'+\sum_{|\alpha|\in M}a_\alpha(x,t,\delta u_{k_j})D^\alpha\psi\varphi\Big]dxdt=\iint\limits_{Q}\sum_{|\alpha|\in M}f_\alpha D^\alpha\psi\varphi dxdt. \tag{62}$$

Consider in (62) the limit as $j\to+\infty$, taking into account (53), (55), (60), (61). As a result, we get (7) for a given function ψ. Since ψ is an arbitrary function, we have proved that u is a weak solution of the problem (3)–(5).

Let us prove *the uniqueness of the weak solution* of the researched problem. Assume the contrary. Let u_1,u_2 be (different) weak solutions of the problem (3)–(5). Lemma 2 (R_* is an arbitrary number) yields

$$\iint\limits_{Q_{R_0}}\big|u_1(x,t)-u_2(x,t)\big|^{p(x)}dxdt\leq C_1R^{n-2q/(q-2)}, \tag{63}$$

where R_0,R are arbitrary constants such that $0<2R_0\leq R$, $R\geq 1$, and $q>0$ is such that $n-2q/(q-2)<0$ (the constant $C_1>0$ does not depend on R_0 and R). Fix $R_0>0$ and consider the limit of (63) as $R\to+\infty$. As a result, we get that $u_1=u_2$ almost everywhere on Q_{R_0}. Since $R_0>0$ is an arbitrary number, we have that $u_1=u_2$ almost everywhere on Q. Thus, we have proved the correctness of the problem (3)–(5). □

Conclusions. We have considered one class of higher orders anisotropic elliptic-parabolic equations, defined in unbounded domains with respect to the spacial variables and such that initial-boundary problem for them are uniquely solvable without any restrictions on the behavior of the solution and the growth of the input data at infinity. The studied equations have variable exponents of nonlinearity and, accordingly, their solutions are taken from the generalized Lebesgue and Sobolev spaces. In our opinion, the class of equations studied here can be extended while preserving its basic properties.

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