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GENERALIZED DERIVATIONS ACTING ON LIE IDEALS IN PRIME RINGS AND BANACH ALGEBRAS

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Let R be a prime ring and L a non-central Lie ideal of R. The purpose of this paper is to describe generalized derivations of R satisfying some algebraic identities locally on L. More precisely, we consider two generalized derivations F_1 and F_2 of a prime ring R satisfying one of the following identities:

1. $F_1(x) \circ y + x \circ F_2(y) = 0$,
2. $[F_1(x), y] + F_2([x, y]) = 0$,

for all x, y in a non-central Lie ideal L of R. Furthermore, as an application, we study continuous generalized derivations satisfying similar algebraic identities with power values on nonvoid open subsets of a prime Banach algebra A. Our topological approach is based on Baire's category theorem and some properties from functional analysis.

1. Introduction. Throughout this paper R denotes an associative ring. We shall denote by $Z(R)$ the center of a ring R. An ideal P of R is a prime ideal if $xRy \subseteq P$ yields $x \in P$ or $y \in P$. In particular, if the zero ideal of R is *prime*, then R is said to be a *prime ring*. For $x, y \in R$, we will write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the *Lie product* and *Jordan product*, respectively. An additive subgroup L of R is said to be a *Lie ideal* of R if $[x, r] \in L$ for all $x \in L$ and $r \in R$. An additive mapping $d\colon R \longrightarrow R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F\colon R \longrightarrow R$ is a *generalized derivation* associated to a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. A *Banach algebra* is a normed algebra whose underlying vector space is a Banach space. The closure of a subset X of a Banach algebra $\mathcal{A}$, denoted by $\overline{X}$, is the intersection of all closed subsets of $\mathcal{A}$ containing X. The interior of a subset X of the Banach algebra $\mathcal{A}$, denoted by $\overset{\circ}{X}$, is the largest open set contained in X. Equivalently, $\overset{\circ}{X}$ is the union of all open subsets of $\mathcal{A}$ contained in X.

During the past few decades, there has been an ongoing interest concerning the relationship between a ring R and the behavior of some special additive mappings defined on R. A popular result due to Posner [17] states that a prime ring admitting a non-zero centralizing derivation is a commutative integral domain. This remarkable theorem of Posner has been influential and it has played a key role in the development of various notions. This result was subsequently refined and extended by a number of algebraists. More specifically, they

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studied the commutativity of rings admitting suitably constrained generalized derivation that satisfies specific identities.

In [13, Theorem 2.7] it is proved that if R is a prime ring of characteristic different from two, admitting two generalized derivations F_1 and F_2 such that $F_1(x)F_2(x)+F_2(x)F_1(x)=0$ for all $x \in R$ then $F_1=0$ or $F_2=0$. An interesting result is demonstrated in [15, Theorem 2] by Hvala, it states that if F_1 and F_2 are two generalized derivations on a prime ring R of characteristic different from two, verifying $[F_1(x), F_2(x)]=0$ for all $x \in R$, then there exists $\mu \in C$ such that $F_1=\mu F_2$. Later Demir et al. in [12] obtained the same classification by only considering the main identity on a non-central Lie ideal of a prime ring R, except possibly when R satisfies the standard identity s_4 of degree 4.

Also some authors extended various results on prime Banach algebras. The authors in [1, Theorem 3.1] showed that if $\mathcal{A}$ is a unital prime Banach algebra, F a non-zero continuous generalized derivation with associated derivation d and G_1, G_2 two nonvoid open subsets of $\mathcal{A}$ satisfying $F((xy)^m)-x^my^m \in Z(\mathcal{A})$ or $F((xy)^m)-y^mx^m \in Z(\mathcal{A})$ for all $(x,y) \in G_1\times G_2$ and $m=m(x,y)>1$, then $\mathcal{A}$ is commutative under the additional assumption that $d(Z(\mathcal{A}))\neq 0$.

Motivated by the above mentioned results, it is natural to seek more refined conclusions by considering generalized derivations that satisfy some specific identities only on a non-central Lie ideal of a prime ring. Moreover, as an application, continuous generalized derivations with power values in Banach algebras are also considered.

2. Main results. We will frequently use the following facts which are crucial for developing the proofs of our main results without explicit mention. The following fact is an immediate consequence of [8, Main Theorem].

Fact 1. *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R and F a generalized derivation of R such that $F(L)\subseteq Z(R)$. Then either $F=0$ or R is embedded in a 2×2 matrix ring over a field.*

Fact 2 ([8])**.** *Let R be a prime ring of characteristic different from 2 and F a generalized derivation of R such that $F(R)\subseteq Z(R)$. Then either $F=0$ or R is commutative.*

Fact 3 ([3], Lemma 2)**.** *Let R be a prime ring of characteristic different from 2, L a Lie ideal of R and $C_R(L)=\{a\in R\colon [a,x]=0\ \forall x\in L\}$. If L is not central then $C_R(L)=Z(R)$.*

Fact 4 ([3], Lemma 3)**.** *Let R be a prime ring of characteristic different from 2, L a Lie ideal of R then $C_R([L,L])=C_R(L)$.*

Fact 5 ([3], Lemma 1)**.** *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R. Then there exists a non-zero two-sided ideal I of R such that $0\neq [I,R]\subseteq L$.*

The following fact is an easy consequence of Fact 5 and [11, Theorem 1].

Fact 6. *Let R be a prime ring of characteristic different from 2, Q_r be the right Martindale quotient ring of R, C be the extended centroid of R, F and G be the non-zero generalized derivations of R and L be a non-central Lie ideal of R. If R is not embedded in $M_2(K)$, the algebra of 2×2 matrices over a field K, and the composition (FG) acts as a generalized derivation on the elements of L, then (FG) is a generalized derivation of R and one of the following holds:*

1. *there exists $\alpha\in C$ such that $F(x)=\alpha x$, for all $x\in R$;*

2. *there exists* $\alpha \in C$ *such that* $G(x) = \alpha x$, *for all* $x \in R$;
3. *there exist* $a, b \in Q_r$ *such that* $F(x) = ax$, $G(x) = bx$, *for all* $x \in R$;
4. *there exist* $a, b \in Q_r$ *such that* $F(x) = xa$, $G(x) = xb$, *for all* $x \in R$;
5. *there exist* $a, b \in Q_r$, $\alpha, \beta \in C$ *such that* $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, *for all* $x \in R$.

Fact 7 ([12], Main Theorem)**.** *Let* R *be a prime ring of characteristic different from* 2, U *be its right Utumi quotient ring,* C *be its extended centroid,* L *be a non-central Lie ideal of* R. *Let* $F\colon R \to R$ *and* $G\colon R \to R$ *be non-zero generalized derivations on* R. *If* $[F(u), G(u)] = 0$ *for all* $u \in L$, *then one of the following holds:*

1. *there exists* $\mu \in C$ *such that for any* $x \in R$, $G(x) = \mu F(x)$;
2. R *satisfies* s_4, *the standard identity of degree* 4 *(which is the same as* R *is embedded in a* 2×2 *matrix ring over a field).*

Fact 8 ([19], Theorem 2)**.** *Let* K *be a commutative ring with unity,* R *be a prime* K*-algebra,* G *be a generalized derivation of* R, I *be a non-zero two-sided ideal of* R, $f(x_1, ..., x_n)$ *be a multilinear polynomial over* K, $n \geq 1$ *be a fixed integer. If* $G\big(f(r_1, ..., r_n)\big)^n = 0$, *for all* $r_1, ..., r_n \in I$, *then either* $f(x_1, ..., x_n)$ *is central valued on* R *or* $G = 0$.

Fact 9 ([9], Theorem 2)**.** *If* I *is a non-zero ideal of the prime ring* R, *then* I, R *and* Q_r *satisfy the same generalized polynomial identities with coefficients in* Q_r.

Lemma 1. *Let* R *be a prime ring of characteristic different from* 2, L *a Lie ideal of* R *and* F, G *two generalized derivations of* R *such that* $F(x)y + yG(x) = 0$ *for all* $x, y \in L$. *Then one of the following holds:* 1. $F = G = 0$; 2. $L \subseteq Z(R)$; 3. R *is embeded in a* 2×2 *matrix ring over a field.*

Proof. Suppose that R is not embedded in a 2×2 matrix ring over a field, $L \nsubseteq Z(R)$ and

$$F(x)y + yG(x) = 0 \text{ for all } x, y \in L. \tag{1}$$

Invoking Fact 5, there exists a non-zero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. Equation (1) gives $F(x)[u, r] + [u, r]G(x) = 0$ for all $r \in R,\ u \in I,\ x \in L$. Simple computations leads to $[u, r][G(x), r] = 0$ for all $r \in R,\ u \in I,\ x \in L$. That is

$$[u, r]I[G(x), r] = 0 \text{ for all } r \in R,\ u \in I,\ x \in L.$$

The primeness of R yields $[G(x), r] = 0$ for all $r \in R,\ x \in L$. That is $G(L) \subseteq Z(R)$, it follows from Fact 1 that $G = 0$. Hence, equation (1) yields $F = 0$. □

Lemma 2. *Let* R *be a prime ring of characteristic different from* 2, L *be a Lie ideal of* R *and* G *be a generalized derivation of* R *such that* $G([x, y]) = 0$ *for all* $x, y \in L$. *Then one of the following holds:* 1. $G = 0$; 2. $L \subseteq Z(R)$.

Proof. Suppose that $L \nsubseteq Z(R)$. The main equality along with Fact 5 for all $r, s \in R,\ u, v \in I$ give $G\big(\big[[u, r], [v, s]\big]\big) = 0$ with I a non-zero two-sided ideal of R. Q_r being a prime C-algebra then Fact 8 and Fact 9 combined yield either $G = 0$ or

$$\big[[u, r], [v, s]\big] \in Z(R) \text{ for all } u, v \in I,\ r, s \in R. \tag{2}$$

Set $U = [I, R] := \text{span}\{[x, y] \mid x \in I,\ y \in R\}$, it is clear that U is a Lie ideal of R. Equation (2) and Fact 4 give $C_R(U) = C_R([U, U]) = R$. Then $[I, R]$ is central. A contradiction, thus $G = 0$. □

In ([2, Theorem 2.1]), the authors investigated commutativity in the rings admitting a generalized derivation F that satisfy $F(x \circ y) = 0$ for all x, y in a nonzero ideal I of a semiprime ring R.

In our situation we consider an identity with two generalized derivations on a non-central Lie ideal of R, at the end, we achieve some classifications.

Theorem 1. *Let R be a prime ring of characteristic different from 2, C be its extended centroid, L be a non-central Lie ideal of R and F_1, F_2 be two generalized derivations of R. If*

$$F_1(x) \circ y + x \circ F_2(y) = 0 \quad \text{for all } x, y \in L, \tag{3}$$

then one of the following holds:

1. *there exists $\lambda \in C$ such that $F_1(x) = \lambda x$ and $F_2(x) = -\lambda x$ for any $x \in R$;*
2. *R is embedded in a 2×2 matrix ring over a field.*

Proof. Suppose that R is not embedded in a 2×2 matrix ring over a field. Firstly we point out that, if $F_1(L) \subseteq Z(R)$, then Fact 1 implies $F_1 = 0$ and relation (3) reduces to $x \circ F_2(y) = 0$ for any $x, y \in L$, then $F_2 = 0$ follows by Lemma 1.

Hence, in the sequel we assume that there exists $u_0 \in L$ such that $F_1(u_0) \notin Z(R)$. Denote $a = F_1(u_0)$. In this sense, relation (3) yields

$$a \circ y + u_0 \circ F_2(y) = 0 \text{ for all } y \in L. \tag{4}$$

Application of Fact 6 implies that one of the following cases occurs:

1. there exists $\mu \in C$ such that $F_2(x) = \mu x$, for all $x \in R$;
2. there exist $\alpha, \beta \in C$ such that $F_2(x) = \alpha x + \beta[u_0, x]$, for all $x \in R$.

In the first case, it follows from relation (4) that $a \circ y + u_0 \circ (\mu y) = 0$ for all $y \in L$. This further implies that $[a, [u, r]] + 2[u, r]a + [\mu u_0, [u, r]] + 2[u, r]u_0 = 0$ for all $r \in R, u \in I$, with I a non-zero two-sided ideal of R. Simple computations yield $[a + \mu u_0, r]I[u, r] = 0$ for all $r \in R, u \in I$. Invoking ([6], Lemma 7.24) one can see that $a + \mu u_0 \in C$. In particular $[F_1(u_0), u_0] = 0$.

Now let us consider $F_2(x) = \alpha x + \beta[u_0, x]$, for all $x \in R$. We may assume $\beta \neq 0$, otherwise $[F_1(u_0), u_0] = 0$ follows by the same argument as above. In this case, relation (4) is $a \circ y + u_0 \circ (\alpha y + \beta[u_0, y]) = 0$ for all $y \in L$. Arguing as above, we get by the end to $[u, r][a + \alpha u_0 - \beta u_0^2, r] = 0$ for all $r \in R, u \in I$.

Invoking again ([6, Lemma 7.24]) we deduce $a + \alpha u_0 - \beta u_0^2 \in C$. Then $[F_1(u_0), u_0] = 0$. Generally, in all cases we get $[F_1(u), u] = 0$ for any $u \in L$. Using Fact 7, there exists $\lambda \in C$ such that $F_1(x) = \lambda x$. Relation (3) becomes $x \circ G(y) = 0$ for all $x, y \in L$ with $G(x) = F_2(x) + \lambda x$. Lemma 1 yields $F_2(x) = -\lambda x$. □

It is proved in [18, Theorem 3.3], that if R is a prime ring of characteristic different from 2, F is a generalized derivation of R satisfying $F([x, y]) = [x, y]$ for all x, y in a square closed Lie ideal U of R, then $U \subseteq Z(R)$.

Our result investigate a more generalized identity considered on a non-central Lie ideal and give the corresponding classification to the involved generalized derivation.

Theorem 2. *Let R be a prime ring of characteristic different from 2, C be its extended centroid, L be a non-central Lie ideal of R and F_1, F_2 be two generalized derivations of R. If*

$$[F_1(x), y] + F_2([x, y]) = 0 \quad \text{for all } x, y \in L, \tag{5}$$

then one of the following holds:

1. *there exists $\lambda \in C$ such that $F_1(x) = \lambda x$ and $F_2(x) = -\lambda x$ for any $x \in R$;*
2. *R is embedded in a 2×2 matrix ring over a field.*

Proof. Assume that R is not embedded in a 2×2 matrix ring over a field. Note that, if $F_1(L) \subseteq Z(R)$, then Fact 1 yields $F_1 = 0$ and relation (5) reduces to $F_2([x, y]) = 0$ for all $x, y \in L$, it follows from Lemma 2 that $F_2 = 0$. Hence, we futher assume that there exists $u_0 \in L$ such that $F_1(u_0) \notin Z(R)$ and denote $a = F_1(u_0)$. As a matter of fact, relation (5) implies

$$[a, y] + F_2([u_0, y]) = 0 \text{ for all } y \in L. \tag{6}$$

Application of Fact 6 implies that one of the following cases holds:

1. there exists $\mu \in C$ such that $F_2(x) = \mu x$, for all $x \in R$;
2. there exist $\alpha, \beta \in C$ such that $F_2(x) = \beta^{-1}\big(u_0 \circ x - \alpha x\big)$, for all $x \in R$.

The first case together with relation (6) yield $[a + \mu u_0, y] = 0$ for all $y \in L$. Which gives $[F_1(u_0), u_0] = 0$. On the other hand, for $F_2(x) = \beta^{-1}\big(u_0 \circ x - \alpha x\big)$. Relation (6) reduces to

$$\big[a, [v, r]\big] + \beta^{-1}\big(u_0 \circ \big[u_0, [v, r]\big] - \alpha\big[u_0, [v, r]\big]\big) = 0 \text{ for all } r \in R, v \in I.$$

with I a non-zero two-sided ideal of R. The direct calculations lead us to

$$[v, r][a - \alpha u_0 - \beta^{-1}u_0^2, v] = 0 \text{ for all } r \in R, v \in I.$$

Then $a - \alpha u_0 - \beta^{-1}u_0^2 \in C$. Thus $[F_1(u_0), u_0] = 0$. Generally, in all cases we get $[F_1(u), u] = 0$ for any $u \in L$. Using Fact 7, there exists $\lambda \in C$ such that $F_1(x) = \lambda x$. Relation (5) becomes $G([x, y]) = 0$ for all $x, y \in L$ with $G(x) = F_2(x) + \lambda x$. Using again Lemma 2 it follows that $F_2(x) = -\lambda x$. □

3. Applications on prime Banach Algebras. Throughout this section, $\mathcal{A}$ denotes a real or complex Banach algebra. To prove our main results we need the following lemma.

Lemma 3 ([4]). *Let $\mathcal{A}$ be a Banach algebra. If $P(t) = \sum_{k=0}^{n} b_k t^k$ is a polynomial in the real variable t with the coefficients in $\mathcal{A}$, and if for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{A}$, then every b_k lies in M.*

Theorem 3. *Let $\mathcal{A}$ be a noncommutative prime Banach algebra, O_1, O_2 be nonempty open subsets on $\mathcal{A}$, F_1, F_2 be continuous generalized derivations of $\mathcal{A}$ and n be a fixed positive integer. Suppose that F_1 and F_2 satisfy one of the following assertions:*

i) $(F_1(x) \circ y)^n + x \circ F_2(y) = 0$ for all $(x, y) \in O_1 \times O_2$;

ii) $[F_1(x), y]^n + F_2([x, y]) = 0$ for all $x, y \in O_1 \times O_2$.

Then $F_1 = F_2 = 0$.

Proof. i) Suppose that

$$(F_1(x) \circ y)^n + x \circ F_2(y) = 0 \text{ for all } (x, y) \in O_1 \times O_2. \tag{7}$$

Let $u \in \mathcal{A}$ and $x \in O_1$. Then $x+tu \in O_1$ for a sufficiently small real t. F_1, F_2 are continuous, one can obviously see that $F_i(ru) = rF_i(u)$ for all $u \in A$, $r \in \mathbb{R}$, $i \in \{1,2\}$. Replacing x by $x+tu$ in equation (7), we get

$$\big(F_1(x) \circ y + (F_1(u) \circ y)t\big)^n + \big(x \circ F_2(y) + (u \circ F_2(y))t\big) = 0. \tag{8}$$

Let $P_{n,m}(u,x,y)$ denotes the sum of all monic monomials with n occurrences of $F_1(x) \circ y$ and m occurrences of $F_1(u) \circ y$. It follows from equation (8) that

$$Q(t) = \sum_{k=0}^{n} P_{n-k,k}(u,x,y)t^k + \big(x \circ F_2(y) + (u \circ F_2(y))t\big) = 0.$$

Setting $Q(t) = \sum_{k=0}^{n} q_k(u,x,y)t^k$; with $q_0(u,x,y) = (F_1(x) \circ y)^n + x \circ F_2(y)$, $q_1(u,x,y) = P_{n-1,1}(u,x,y) + u \circ F_2(y)$ and $q_k(u,x,y) = P_{n-k,k}(u,x,y)$ for all $k \in \{2,...,n\}$. As (0) is a closed linear subspace of $\mathcal{A}$, then Lemma 3 yields $q_k(u,x,y) = 0$ for all $k \in \{0,...,n\}$. In particular $q_n(u,x,y) = 0$, thus

$$(F_1(u) \circ y)^n = 0 \text{ for all } (u,y) \in \mathcal{A} \times O_2.$$

Similarly, one can show that

$$(F_1(u) \circ v)^n = 0 \text{ for all } u,v \in \mathcal{A}. \tag{9}$$

As a consequence of the continuity of F_1, it is clear that $(u,v) \mapsto F_1(u) \circ v$ is bilinear. Invoking Fact 8 and equation (9), we obtain $F_1(u) \circ v \in Z(\mathcal{A})$ for all $u,v \in \mathcal{A}$. ([5, Theorem 2.2]) forces $F_1 = 0$. Equation (7) reduces to $x \circ F_2(y) = 0$ for all $(x,y) \in O_1 \times O_2$. Using the same techniques as above, we get to $u \circ F_2(v) = 0$ for all $u,v \in \mathcal{A}$. Invoking again ([5, Theorem 2.2]) it follows that $F_2 = 0$.
ii) Assume that

$$[F_1(x), y]^n + F_2([x,y]) = 0 \text{ for all } (x,y) \in O_1 \times O_2. \tag{10}$$

Let $u \in \mathcal{A}$ and $x \in O_1$. Then $x+tu \in O_1$ for a sufficiently small real t. Taking $x+tu$ instead of x in equation (10), we get

$$\big([F_1(x), y] + [F_1(u), y]t\big)^n + F_2([x,y]) + (F_2([u,y]))t = 0. \tag{11}$$

Let $P_{n,m}(u,x,y)$ denote the sum of all monic monomials with n occurrences of $[F_1(x), y]$ and m occurrences of $[F_1(u), y]$. Equation (11) becomes

$$Q(t) = \sum_{k=0}^{n} P_{n-k,k}(u,x,y)t^k + F_2([x,y]) + (F_2([u,y]))t = 0.$$

Let us consider

$$Q(t) = \sum_{k=0}^{n} q_k(u,x,y)t^k$$

with $q_0(u,x,y) = ([F_1(x), y])^n + F_2([x,y])$, $q_1(u,x,y) = P_{n-1,1}(u,x,y) + F_2([u,y])$ and $q_k(u,x,y) = P_{n-k,k}(u,x,y)$ for all $k \in \{2,...,n\}$. Invoking Lemma 3 we get $q_k(u,x,y) = 0$ for all $k \in \{0,...,n\}$. In particular $q_n(u,x,y) = 0$, then

$$[F_1(u), v]^n = 0 \text{ for all } u,v \in \mathcal{A}. \tag{12}$$

In view of equation (12), Fact 8 gives $[F_1(u), v] \in Z(\mathcal{A})$ for all $u,v \in \mathcal{A}$. Substituting v by $vF_1(u)$, we obtain

$$[F_1(u), v]F_1(u) \in Z(\mathcal{A}) \text{ for all } u,v \in \mathcal{A}. \tag{13}$$

Using ([7, Remark 4]), we get either $[F_1(u), v] = 0$ or $F_1(u) \in Z(\mathcal{A})$ for any $u, v \in \mathcal{A}$, that is $F_1(u) \in Z(\mathcal{A})$ for all $u \in \mathcal{A}$. The latter relation along with Fact 2 implies $F_1 = 0$, similar approach transforms equation (10) to $F_2([u, v]) = 0$ for all $u, v \in \mathcal{A}$. Invoking Lemma 2 we get $F_2 = 0$. □

Theorem 4. *Let $\mathcal{A}$ be a noncommutative prime Banach algebra, $C_{\mathcal{A}}$ be its extended centroid, O_1, O_2 be nonvoid open subsets on $\mathcal{A}$ and F_1, F_2 be continuous generalized derivations of $\mathcal{A}$. If $F_1(x^r) \circ y^s + x^r \circ F_2(y^s) = 0$ for all $(x, y) \in O_1 \times O_2$ where r, s are non-zero integers depending on the pair of elements x and y, then one of the following holds:*

1. *there exists $\lambda \in C_{\mathcal{A}}$ such that $F_1(x) = \lambda x$ and $F_2(x) = -\lambda x$ for any $x \in \mathcal{A}$;*
2. *$\mathcal{A}$ is embedded in a 2×2 matrix ring over a field.*

Proof. Let us fix $x \in O_1$ and set $K_{r,s} = \{y \in \mathcal{A} \mid F_1(x^r) \circ y^s + x^r \circ F_2(y^s) \neq 0\}$. We claim that each $K_{r,s}$ is open in $\mathcal{A}$ or equivalently its complement $K^c_{r,s}$ is closed. For this, we consider a sequence $(y_k)_{k\geq 1} \subset K^c_{r,s}$ converging to y and prove that $y \in K^c_{r,s}$. As $(y_k)_{k\geq 1} \subset K^c_{r,s}$ then $F_1(x^r) \circ y_k^s + x^r \circ F_2(y_k^s) = 0$ for all $k \geq 1$. Hence

$$\lim_{k\to\infty} F_1(x^r) \circ y_k^s + x^r \circ F_2(y_k^s) = F_1(x^r) \circ \Big(\lim_{k\to\infty} y_k\Big)^s + x^r \circ F_2\Big(\Big(\lim_{k\to\infty} y_k\Big)^s\Big) =$$
$$= F_1(x^r) \circ y^s + x^r \circ F_2(y^s) = 0.$$

Therefore, $y \in K^c_{r,s}$, thus $K_{r,s}$ is open. Suppose now that all the $K_{r,s}$ are dense in $\mathcal{A}$ then the intersection of the $K_{r,s}$ is also dense by Baire category theorem, a contradiction with the fact that $O_2 \neq \varnothing$. Hence, there exist some positive integers p, q depending on x, such that $K_{p,q}$ is not dense. Accordingly, there exists a nonvoid open subset O_3 in $K^c_{p,q}$. Therefore,

$$F_1(x^p) \circ y^q + x^p \circ F_2(y^q) = 0 \text{ for all } y \in O_3. \tag{14}$$

Let us consider $z \in O_3$ and $v \in \mathcal{A}$, $z + tv \in O_3$ for all sufficiently small real t. Replacing y by $z + tv$ in (14), we obtain

$$F_1(x^p) \circ (z + tv)^q + x^p \circ F_2((z + tv)^q) = 0. \tag{15}$$

Let $P_{i,j}(x, u)$ denote the sum of all monic monomials with i occurrences of x and j occurrences of u. As $(z + tv)^q = P_{q,0}(z, u) + P_{q-1,1}(z, u)t + ... + P_{1,q-1}(z, u)t^{q-1} + P_{0,q}(z, u)t^q$, it follows from equation (15) that

$$F_1(x^p) \circ \left(\sum_{i=0}^{q} P_{q-i,i}(z, v)t^i\right) + x^p \circ F_2\left(\sum_{i=0}^{q} P_{q-i,i}(z, v)t^i\right) = 0. \tag{16}$$

It follows from (16) that

$$Q(t) = \sum_{i=0}^{q} \Big(F_1(x^p) \circ (P_{q-i,i}(z, v)) + x^p \circ F_2(P_{q-i,i}(z, v))\Big)t^i = 0.$$

Thus
$Q(t) = \sum_{i=0}^{q} a_i(v, x, z)t^i = 0$ with $a_i(v, x, z) = F_1(x^p) \circ \big(P_{q-i,i}(z, v)\big) + x^p \circ F_2\big(P_{q-i,i}(z, v)\big)$. Using Lemma 3, we get $a_i(v, x, z) = 0$ for all $i \in \{0, ..., q\}$. In particular $a_q(v, x, z) = 0$, that is $F_1(x^p) \circ v^q + x^p \circ F_2(v^q) = 0$. In conclusion, we have proved that for a given $x \in O_1$, there exist some positive integers p and q depending on x, such that $F_1(x^p) \circ v^q + x^p \circ F_2(v^q) = 0$ for

all $v \in \mathcal{A}$. Let us fix $v \in \mathcal{A}$. Using a similar approach, we arrive at $F_1(u^p) \circ v^q + u^p \circ F_2(v^q) = 0$ for all $u, v \in \mathcal{A}$. Now let us consider H_1 and H_2 the additive subgroups generated by $\{a^p \mid a \in \mathcal{A}\}$ and $\{a^q \mid a \in \mathcal{A}\}$ respectively, it follows that

$$F_1(x) \circ y + x \circ F_2(y) = 0 \text{ for all } (x, y) \in H_1 \times H_2. \tag{17}$$

Equation (17) along with [10] yield that either $a^p \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$ or H_1 contains a non-central Lie ideal J_1. If $a^p \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, in particular $[a^p, b^p] = 0$ for all $a, b \in \mathcal{A}$. It follows that $\mathcal{A}$ is commutative from ([20], Theorem 2.3). It is a contradiction.

Now suppose that H_1 contains a non-central Lie ideal J_1, similarly H_2 contains also another non-central Lie ideal J_2. Let $I_k = \{x \in \mathcal{A} \mid [x, \mathcal{A}] \subset J_k\}$ with $k = 1, 2$. It follows from ([14, Lemma 1.4]) that I_1, I_2 are both subrings and Lie ideals of $\mathcal{A}$. Therefore, equation (17) becomes

$$F_1(x) \circ y + x \circ F_2(y) = 0 \text{ for all } (x, y) \in [I_1, \mathcal{A}] \times [I_2, \mathcal{A}]. \tag{18}$$

As $[I_1, \mathcal{A}]$ and $[I_2, \mathcal{A}]$ are dense submodules of $[\mathcal{A}, \mathcal{A}]$ then by ([16, Theorem 2]), $[\mathcal{A}, \mathcal{A}]$ satisfy the same identity as $[I_1, \mathcal{A}]$ and $[I_2, \mathcal{A}]$, thus equation (18) becomes $F_1(x) \circ y + x \circ F_2(y) = 0$ for all $x, y \in [\mathcal{A}, \mathcal{A}]$. Since $[\mathcal{A}, \mathcal{A}]$ is a non-central Lie ideal, applying Theorem 1 we get the required result. □

Using the same above arguments, with suitable modification, application of Theorem 2 yields the following result.

Theorem 5. *Let $\mathcal{A}$ be a noncommutative prime Banach algebra, $C_{\mathcal{A}}$ be its extended centroid, O_1, O_2 be nonvoid open subsets on $\mathcal{A}$ and F_1, F_2 be continuous generalized derivations of $\mathcal{A}$. If $[F_1(x^r), y^s] + F_2([x^r, y^s]) = 0$ for all $(x, y) \in O_1 \times O_2$, where r, s are non-zero integers depending on the pair of elements x and y, then one of the following holds:* 1. *there exists $\lambda \in C_{\mathcal{A}}$ such that $F_1(x) = \lambda x$ and $F_2(x) = -\lambda x$ for any $x \in \mathcal{A}$;* 2. *$\mathcal{A}$ is embedded in a 2×2 matrix ring over a field.*

The following examples show that the primeness hypothesis in Theorems 1 and 2 is not superfluous.

Example 1. The ring $\mathcal{R} = M_2(\mathbb{Z}/6\mathbb{Z}) \times \mathbb{Z}/6\mathbb{Z}$ with operations of coordinatewise addition and multiplication is a non prime ring of characteristic 6. Consider $F_M((A, a)) = 3(MA + AM, 0)$ with $M \in [M_2(\mathbb{Z}/6\mathbb{Z}), M_2(\mathbb{Z}/6\mathbb{Z})]$ with associated derivation d_M defined by $d_M((A, a)) = 3(AM - MA, 0)$. For $L = [M_2(\mathbb{Z}/6\mathbb{Z}), M_2(\mathbb{Z}/6\mathbb{Z})] \times \mathbb{Z}/6\mathbb{Z}$ a Lie ideal of $\mathcal{R}$, $F_1 = F_M$ and $F_2 = 0$, we have $F_1((A, a)) \circ (B, b) = 6(MAB + AMB, 0) = 0 \ \forall \ (A, a), (B, b) \in L$. Nevertheless, none of the assertions of Theorem 1 are satisfied.

Example 2. Let us consider the ring $\mathcal{R} = M_2(\mathbb{R}) \times \mathbb{R}$ with operations of coordinatewise addition and multiplication. It is obvious that $\mathcal{R}$ is a non prime ring. Consider the generalized derivation $G_N((A, a)) = (NA + AN, 0)$ with $N \in [M_2(\mathbb{R}), M_2(\mathbb{R})]$ with associated derivation g_N defined by $g_N((A, a)) = (AN - NA, 0)$. We set $L = [M_2(\mathbb{R}), M_2(\mathbb{R})] \times \mathbb{R}$. This is a Lie ideal of $\mathcal{R}$ along with $G_1 = G_N$ and $G_2 = 0$. Simple computations show that $[G_1((A, a)), (B, b)] + G_2\big([(A, a), (B, b)]\big) = 0 \ \forall (A, a), (B, b) \in L$. However, none of the assertions of Theorem 2 is satisfied.

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