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NORMALITY AND UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS

R. S. Dyavanal, S. B. Kalakoti. *Normality and uniqueness of homogeneous differential polynomials*, Mat. Stud. **59** (2023), 168–177.

The primary goal of this work is to determine whether the results from [19, 20] still hold true when a differential polynomial is considered in place of a differential monomial. In this perspective, we continue our study to establish the uniqueness theorem for homogeneous differential polynomial of an entire and its higher order derivative sharing two polynomials using normal family theory. We also obtain normality criteria for a family of analytic functions in a domain concerning homogeneous differential polynomial of a transcendental meromorphic function satisfying certain conditions. Meanwhile, as a result of this investigation, we proven three theorems that provide affirmative responses for the purpose of this study. Several examples are offered to demonstrate that the conditions of the theorem are necessary.

1. Introduction. We denote by $\mathcal{E}$ the class of non-constant entire functions, i.e. analytic in the whole complex plane $\mathbb{C}$. The fundamentals of Nevanlinna Theory can be found in [2,4,14]. For f, $g \in \mathcal{E}$ and $a \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, if $f - a$ and $g - a$ have the identical zeros with multiplicities then f and g share *a counting multiplicities* (CM), if the multiplicities are not counted, then f and g share *a ignoring multiplicities* (IM) and if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then f and g *share* ∞ CM.

For $a \in \overline{\mathbb{C}}$ and k be a non-negative integer or infinity, denote $E_k(a, f)$ the set of all a points of f, where every point a of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If $E_k(a, f)$=$E_k(a, g)$ then f, g *share a with weight k*.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively. Let $q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$ and $a \in \overline{\mathbb{C}}$. The counting function $N_{(q}(r, \frac{1}{g-a})$ of g means those a-points of g are counted according to multiplicities whose multiplicities are not less than q, $N_{q)}(r, \frac{1}{g-a})$ denotes the counting function of g whose a-points are counted with proper multiplicity where the multiplicities are less than or equal to q and the corresponding reduced counting function is given by $\overline{N}_{(q}(r, \frac{1}{g-a})$, $\overline{N}_{q)}(r, \frac{1}{g-a})$ where the multiplicities are ignored. We call that a function $\varphi(z) \in \mathcal{E}$ is the *small function* of f if $T(r, \varphi) = S(r, f)$.

The *order* $\rho(f)$ *of the growth* of the function $f \in \mathcal{E}$ we define as
$$\rho(f) = \varlimsup_{r\to+\infty} \frac{\log^+ T(r, f)}{\log r}.$$
We consider a *differential monomial generated by a function* $f \in \mathcal{E}$
$$M_j[f] = f^{n_{0j}}(f')^{n_{1j}}(f'')^{n_{2j}} \cdot \ldots \cdot (f^{(k)})^{n_{kj}},$$

2020 *Mathematics Subject Classification:* 30D35, 30D45.
Keywords: entire function; homogeneous differential polynomial.
doi:10.30970/ms.59.2.168-177

The number $\nu_{M_j} = \sum_{i=0}^{k} n_{ij}$ is called the *degree* and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ the *weight* of the *differential monomial* $M_j[f]$. Let us denote

$$P[f] = \sum_{j=1}^{l} b_j M_j[f]$$

a *differential polynomial generated by* f of *degree* $\nu_p = \max\{\nu_{M_j} : 1 \leq j \leq l\}$ and *weight* $\Gamma_p = \max\{\Gamma_{M_j} : 1 \leq j \leq l\}$, where $T(r, b_j) = S(r, f)$ for $j \in \{1, 2, \ldots, l\}$. $P[f]$ is said to be *homogeneous* if $\nu_p = \underline{\nu}_p = \nu$, where $\underline{\nu}_p = \min\{\nu_{M_j} : 1 \leq j \leq l\}$ and k (the highest order of the derivative of f in $P[f]$) are called the *lower degree* and *lower order* of $P[f]$, respectively.

L. Rubel and C. C. Yang [9] first studied the problem of sharing values between entire functions and their derivatives. They proved the following theorem.

Theorem A ([9]). *Let $f \in \mathcal{E}$ and $a \neq b \in \mathbb{C}$. If f and f' share a and b CM, then $f \equiv f'$.*

In 1979, E. Mues and N. Steinmetz [8] extended the above Theorem to the sharing values IM. Uniqueness result considering the power of an entire function was first obtained jointly by L. Z. Yang and J. L. Zhang [15]. In 2009, J. L. Zhang improved the sharing condition to small function as follows:

Theorem B ([16]). *Let $f \in \mathcal{E}$, $a(z)$ be a small function of f and $n, k \in \mathbb{Z}_+$. If f^n and $(f^n)^{(k)}$ share $a(z)$ CM and $n \geq k+1$ then $f^n \equiv (f^n)^{(k)}$ and f is of the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c is a non-zero complex constant and $\lambda^k = 1$.*

In 2011, Yi F. Lü and H. X. Yi [5] extended the above theorem to sharing a polynomial, then the function needs to be transcendental. What will be the result of Theorem B, if function and its derivative share two different polynomials instead of one? As an answer in 2015, S. Majumder proved the below Theorem C.

Theorem C ([6]). *Let $f \in \mathcal{E}$ be transcendental, n, $k \in \mathbb{Z}_+$ and Q_1, Q_2 be non vanishing polynomials. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$ then$(f^n)^{(k)}\frac{Q_2}{Q_1} \equiv f^n$ and if $Q_1 \equiv Q_2$ then f is of the form $f(z) = ce^{\frac{\lambda}{n}z}$ where c is a non zero complex constant and $\lambda^k = 1$.*

In 2018, P. Sahoo and G. Biswas [11] extended the above uniqueness result for $(f^nP(f)) - Q_1$ and $(f^nP(f))^{(k)} - Q_2$ sharing 0 CM where $P(f) = \sum a_i f^i, i = 0, 1, ..., m$ and $f \in \mathcal{E}$. To improve the above result, in 2019 S. Majumder [7] developed weighted sharing and used the concepts of normal families to prove $P[f] - a_1$ and $(P[f])^{(k)} - a_2$ sharing (0, 1) where $f \in \mathcal{E}$ is transcendental, $a_1 = P_1e^Q$ and $a_2 = P_2e^Q$ such that P_1, P_2, and Q are polynomials.

Recently, M. Tejuswini, N. Shilpa, R. S. Dyavanal and B. Narasimha Rao [20] improved the above result for differential monomial as below.

Theorem D ([20]). *Let f be a transcendental entire function, $\varphi_i(z) = A_i(z)e^{B(z)}$ such that $A_i \not\equiv 0$ for $i = 1, 2$ and $B(z)$ are polynomials. Define $M[f] = f^{n_0}(f')^{n_1}\ (f'')^{n_2}....(f^{(k)})^{n_k}$, where $n_0, n_1, n_2, \ldots, n_k$ are positive integers. If $\rho(f) > 2\max\{\deg(B), 1+\deg(A_2)-\deg(A_1)\}$, $M[f] - \varphi_1$ and $(M[f])^{(k)} - \varphi_2$ share $(0, 1)$ and the multiplicities of zeros of $M[f]$ are not less than $k+1$, then $(M[f])^{(k)} \equiv \frac{\varphi_2}{\varphi_1}M[f]$. Also if $\varphi_1 \equiv \varphi_2$, then $f(z) = a + Ce^{\left(\frac{\mu z}{\gamma}\right)}$, where C is a non-zero complex constant, $\mu^k = 1$ and $\gamma = n_0 + n_1 + n_2 + ... + n_k$.*

Remark 1. *If $B(z)$ is a constant then the Theorem D holds and the condition of $\rho(f) > 2\max\{\deg(B), 1+\deg(A_2)-\deg(A_1)\}$ is not needed.*

Recently, B. Chakraborty, W. Lü proved the Theorem given below.

Theorem E ([19]). *Let f be a transcendental meromorphic function $f(z)$ and φ $(\not\equiv 0,\infty)$ be a small function of f such that φ and f has no common zero. Moreover, we assume that $\frac{1}{\varphi}$ and f has no common zero. If every pole of f has multiplicity at least $l \geq 1$, $q_0 > 1+\frac{1}{l}$ and $q_k \geq 1$, then*

$$T(r,f) \leq \frac{1}{(q_0-1-\frac{1}{l})} N\left(r, \frac{1}{M[f]-\varphi(z)}\right) + S(r,f).$$

To develop the normality criteria for a differential monomial in 2021, B. Chakraborty and W. Lü proved the following result.

Theorem F ([19]). *Let $\mathcal{F}$ be the family of analytic functions on a domain D and $k(\geq 1)$, $q_0(\geq 2)$, $q_i(\geq 0)$ $i=1,2,\ldots,k-1$, $q_k(\geq 1)$ be positive integers. If for each $f \in \mathcal{F}$: i) f has only zeros of multiplicity at least k, ii) $f^{q_0}(f')^{q_1}(f'')^{q_2}....(f^{(k)})^{q_k} \neq 1$, then $\mathcal{F}$ is normal on domain D.*

Note, for example, that W. Bergweiler [10] derive the Ahlfors five islands theorem from a corresponding result of R. Nevanlinna concerning perfectly branched values, a rescaling lemma for non-normal families and an existence theorem for quasiconformal mappings.

In this paper, we have extended above Theorems for homogeneous differential polynomials of a function and proved the following Theorems.

Theorem 1. *Let f be a transcendental entire function. If $\varphi_i(z) = A_i(z)e^{B(z)}$ such that $A_i(z) \not\equiv 0$ for $i = 1$, 2 and $B(z)$ are polynomials. If $P[f] = \sum_{j=1}^{l} b_j M_j[f]$ is a homogeneous differential polynomial whose zeros are of multiplicities not less than $k+1$ and $\rho(f) > 2\max\{\deg(B), 1+\deg(A_2)-\deg(A_1)\}$, $P[f]-\varphi_1$ and $(P[f])^{(k)}-\varphi_2$ share $(0,1)$, then $(P[f])^{(k)} \equiv \frac{\varphi_2}{\varphi_1}P[f]$. Also if $\varphi_2 \equiv \varphi_1$, then $f = a + Ce^{\left(\frac{\lambda z}{\nu}\right)}$, where C is a non-zero complex constant, $\lambda^k = 1$ and $\nu = n_{0j}+n_{1j}+n_{2j}+\ldots+n_{kj}$.*

We give some examples to show that the conditions assumed in the Theorem 1 are necessary.

Example 1. Let $P[f] = f(f')^2 + (f')(f'')^2 - (f')^2(f'')$ where $f(z) = e^z + 1$ and $P[f]$ does not have simple zeros. $P' - A_2$ and $P - A_1$ share 0 CM where $A_1 = e^{2z}$ and $A_2 = 2e^{2z}$. In this case $B(z) = e^{2z}$ and $\deg(B) = 1$ which does not satisfy the condition $\rho(f) > 2\max\{\deg(B), 1+\deg(A_2)-\deg(A_1)\}$ and hence $(P[f])' \not\equiv \frac{A_2}{A_1}P[f]$.

Example 2. Let $f = z^2$ and $P[f] = (f')(f^2) + 2(f)(f')(f'') - (f')^3$. Clearly $P[f]$ deos not have simple zeros. For $A_1 = 11z^4 + z^5$ and $A_2 = 3z^5 + z^4$, $P' - A_2$ and $P - A_1$ share 0 CM but $(P[f])' \not\equiv \frac{A_2}{A_1}P[f]$ as f is not transcendental function.

Example 3. Let $P[f] = f'$ where $f(z) = e^z + z$. It is clear that the zeros of $P[f]$ are simple. $P' - A_2$ and $P - A_1$ share 0 CM where $A_1 = 2$, $A_2 = 1$ and $B(z)$ is a constant. $(P[f])' \not\equiv \frac{A_2}{A_1}P[f]$ as the zeros of $P[f]$ are less than $k+1$.

Theorem 2. *Let f be a transcendental meromorphic function and φ $(\not\equiv 0, \infty)$ be a small function of f such that φ and f has no common zero and $\frac{1}{\varphi}$ and f has no common zero. If every pole of f has multiplicity at least $p \geq 1$, $n > 1 + \frac{1}{p}$, $n = \min\{n_{0j} : 1 \leq j \leq l\}$ and $n_{kj} \geq 1$, then*

$$T(r,f) \leq \frac{1}{(n-1-\frac{1}{p})} N\left(r, \frac{1}{P[f]-\varphi(z)}\right) + S(r,f),$$

where $P[f] = \sum\limits_{j=1}^{l} b_j M_j[f]$ $M_j[f] = f^{n_{0j}}(f^{'})^{n_{1j}}(f^{''})^{n_{2j}} \cdot \ldots \cdot (f^{(k)})^{n_{kj}}$.

Corollary 1. *Let f be a transcendental entire (resp. meromorphic function such that every pole of f has multiplicity at least p (≥ 1)) and φ $(\not\equiv 0, \infty)$ be a small function of f such that φ and f has no common zero and $\frac{1}{\varphi}$ and f has no common zero. If $n_{0j} > 1$ (resp. $1 + \frac{1}{p}$) and $n_{kj} \geq 1$, then $P[f] - \varphi$ has infinitely many zeros.*

Theorem 3. *Let $\mathcal{F}$ be the family of an analytic functions on a domain D and k (≥ 1), n_{0j} (> 1) n_{ij} (≥ 0) $(i \in \{1, 2, \ldots, k-1\})$, n_{kj} (≥ 1) be positive integers. If for each $f \in \mathcal{F}$: i) f has only zeros of multiplicity at least k, ii)*

$$P[f] = \sum_{j=1}^{l} (f)^{n_{0j}}(f^{'})^{n_{1j}}(f^{''})^{n_{2j}} \cdot \ldots \cdot (f^{(k)})^{n_{kj}} \neq 1$$

is a non constant homogeneous differential polynomial in f having only one term with $\mu^* = \min\limits_{1\leq j\leq l} \mu_j^* = \min\limits_{1\leq j\leq l} \sum_{i=1}^{k} i n_{ij}$, then $\mathcal{F}$ is normal on domain D.

2. Lemmas. We need the following Lemmas to prove our Theorems.

Lemma 1 ([14]). *Let f be any entire function of finite order and $k \in \mathbb{Z}_+$ be any positive integer. We have $m(r, \frac{f^{(k)}}{f}) = O(\log r)$ $(r \to +\infty)$.*

Lemma 2 ([4]). *Let f be a transcendental entire function and $0 < \delta < \frac{1}{4}$. If for every z with $|z| = r$ the inequality $|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta}$ holds, then there exists a set $F \subset (1, +\infty)$ of finite logarithmic measure, i.e. $\int_F d\ln t < +\infty$, such that*

$$f^{(m)}(z) = \left(\tfrac{\nu(r,f)}{z}\right)^m (1 + o(1))f(z),$$

holds for all $m \geq 0$ and $r \to +\infty$ $(r \notin F)$, where $\nu(r,f)$ is the central index of the power series $f(z) = \sum_{p=0}^{+\infty} f_p z^p$, i.e. $\nu(r,f) = \max\{p \colon |f_p|r^p = \mu_f(r)\}$, $\mu_f(r) = \max\{|f_p|r^p \colon p \geq 0\}$.

Lemma 3 ([18]). *If g is a non-constant meromorphic function, then*

$$N\left(r, \frac{g^{'}}{g}\right) - N\left(r, \frac{g}{g^{'}}\right) = \overline{N}(r,g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g^{'}}\right).$$

Lemma 4 ([21]). *Let f be a transcendental meromorphic function, $P[f] = \sum_{j=1}^{l} b_j M_j[f]$ $\left(M_j[f] = f^{n_{0j}}(f^{'})^{n_{1j}}(f^{''})^{n_{2j}} \cdot \ldots \cdot (f^{(k)})^{n_{kj}}\right)$ is a differential polynomial of degree ν_p and weight Γ_p. Then $T(r, P[f]) = O(T(r,f))$, $S(r, P[f]) = S(r,f)$.*

We prove the following Lemma.

Lemma 5. *Homogeneous differential polynomial $P[f]$ with $n_{0j} \geq 1$ is non-constant for a transcendental meromorphic function f.*

Proof. Let

$$P[f] = \sum_{j=1}^{l} a_j(z) M_j(f),$$

where $M_j(f) = a_j(z)(f)^{n_{0j}}(f')^{n_{1j}}...((f)^{(k)})^{n_{kj}}$ and its degree $\nu_j = n_{0j} + n_{1j} + n_{2j} + n_{3j} + \ldots + n_{kj}$. Since $P[f]$ is a homogeneous differential polynomial $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \ldots = \nu_l = \nu$. Let $\mu_j = 1n_{1j} + 2n_{2j} + 3n_{3j} + \ldots + kn_{kj}$ and $\mu_* = \max\{\mu_j : 1 \leqslant j \leqslant l\}$, consider

$$\frac{l}{f^{\nu}} = a_1(z)\Big(\frac{f'}{f}\Big)^{n_{11}}\Big(\frac{f''}{f}\Big)^{n_{21}} \cdot \ldots \cdot \Big(\frac{f^{(k)}}{f}\Big)^{n_{k1}} \frac{1}{M_1[f]} + a_2(z)\Big(\frac{f'}{f}\Big)^{n_{12}}\Big(\frac{f''}{f}\Big)^{n_{22}} \times \ldots$$

$$\ldots \times \Big(\frac{f^{(k)}}{f}\Big)^{n_{k2}} \frac{1}{M_2[f]} + \ldots + a_l(z)\Big(\frac{f'}{f}\Big)^{n_{1l}}\Big(\frac{f''}{f}\Big)^{n_{2l}} \cdot \ldots \cdot \Big(\frac{f^{(k)}}{f}\Big)^{n_{kl}} \frac{1}{M_l[f]}.$$

This implies

$$\begin{aligned}
l\nu T(r,f) &\leqslant \sum_{j=1}^{l}\Big\{\sum_{i=1}^{k} n_{ij} N\Big(r,\infty;\frac{f^{(i)}}{f}\Big) + T(r, M_j[f])\Big\} + S(r,f) \leq \\
&\leqslant \sum_{j=1}^{l}\Big\{\sum_{i=1}^{k} i n_{ij}(\overline{N}(r,0,f) + \overline{N}(r,\infty,f)) + T(r, M_j[f])\Big\} + S(r,f) \leq \\
&\leqslant \sum_{j=1}^{l}\Big\{\sum_{i=1}^{k} i n_{ij}(N(r,0,P[f]) + N(r,\infty,P[f]))\Big\} + lT(r,P[f]) + S(r,f) \leq \\
&\leqslant l[(2\mu_* + 1)T(r,P[f])] + S(r,f).
\end{aligned}$$

Hence $\nu T(r,f) \leqslant [(2\mu_* + 1)T(r,P[f])] + S(r,f)$. Since f is transcedental, we get $P[f]$ is a non-constant function. □

Lemma 6 ([23]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc $\triangle$ such that all zeros of functions in $\mathcal{F}$ have multiplicity at least k and α be a real number satisfying $0 \leq \alpha < k$, then $\mathcal{F}$ is not normal in any neighbourhood of $z_0 \in \triangle$ if and only if there exist*

(i) *points $z_n \in \triangle$, $z_n \to z_0$,*

(ii) *positive numbers ρ_n, $\rho_n \to 0$ and*

(iii) *functions $f_n \in \mathcal{F}$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function.*

3. Proof of Theorems.

Proof of Theorem 1. Let $P[f]$ be a non-constant homogeneous differential polynomial and

$$F_1^* = \tfrac{P[f]}{\varphi_1} \quad \text{and} \quad G_1^* = \tfrac{P[f]^{(k)}}{\varphi_2}.$$

Suppose $\rho(f) < \infty$. Since each φ_i is a small function of f, $\rho(\varphi_i) < \rho(f)$ for $i = 1, 2$. Hence

$$\rho(F_1^*) = \rho(P[f]) = \rho(f) < \infty.$$

In a similar way $\rho(P[f]) = \rho(P[f]^{(k)}) < \infty$ and hence $\rho(G_1^*) < \infty$. Next we consider two cases for $B(z)$.

Case 1. If $B(z)$ is a constant. Since F_1^* and G_1^* share $(1, 1)$ except for the zeros of $\varphi_i(z)$ for $i = 1, 2$, we get $\overline{N}(r, 1; F_1^*) = \overline{N}(r, 1; G_1^*) + O(\log r)$. Let

$$\psi = \frac{F_1^{*'}(F_1^* - G_1^*)}{F_1^*(F_1^* - 1)} = \frac{F_1^{*'}}{F_1^* - 1}\left(1 - \frac{A_1 P[f]^{(k)}}{A_2 P[f]}\right). \tag{1}$$

We consider two cases for $\psi(z)$.

Case 1.1. If $\psi \not\equiv 0$, (1) implies that $F_1^* \not\equiv G_1^*$. From Lemma 1, we see that $m(r, \infty, \psi) = O(\log r)$. Let α be a zero of F_1^* of multiplicity $\left(s\sum_{j=0}^{k} n_j - \sum_{j=1}^{k} jn_j\right)$, $s \geq k+1$ such that $\varphi_i(\alpha) \neq 0$ for $i = 1, 2$. Now α is the zero of $P[f]$ with the same multiplicity and the zero of $P[f]^{(k)}$ with the multiplicity $\left(s\sum_{j=0}^{k} n_j - \sum_{j=1}^{k} jn_j\right) - k$.

From (1), we see that

$$\psi(z) = O\left((z-\alpha)^{s\sum\limits_{j=0}^{k} n_j - \left(\sum\limits_{j=1}^{k} jn_j + k + 1\right)}\right)$$

and $\psi(z)$ is analytic at $z = \alpha$. Let α_1 be the common zero of $F_1^* - 1$ and $G_1^* - 1$ and $\varphi_i(\alpha_1) \neq 0$, $i = 1, 2$. Let α_1 be a zero of $F_1^* - 1$ of multiplicity s_1. Since F_1^* and G_1^* share $(1, 1)$ except for the zeros of φ_i, $i = 1, 2$. Implies that α_1 is a zero of $(G_1^*) - 1$ of multiplicity t_1. By Taylor's series expansion in the neighborhood of α_1 for F_1^* and G_1^*, we obtain

$$\begin{gathered}
F_1^*(z) - 1 = a_{s_1}(z-\alpha_1)^{s_1} + a_{s_1+1}(z-\alpha_1)^{s_1+1} + \ldots,\ a_{s_1} \neq 0,\\
G_1^*(z) - 1 = b_{t_1}(z-\alpha_1)^{t_1} + b_{t_1+1}(z-\alpha_1)^{t_1+1} + \ldots,\ b_{t_1} \neq 0,\\
F_1^*(z)' = s_1 a_{s_1}(z-\alpha_1)^{s_1-1} + (s_1+1)a_{s_1+1}(z-\alpha_1)^{s_1} + \ldots,
\end{gathered}$$

$$F_1^*(z) - G_1^*(z) = \begin{cases} a_{s_1}(z-\alpha_1)^{s_1} + \ldots\ , & \text{if } s_1 < t_1,\\ -b_{t_1}(z-\alpha_1)^{t_1} - \ldots\ , & \text{if } s_1 > t_1,\\ (a_{s_1} - b_{t_1})(z-\alpha_1)^{s_1} + \ldots\ , & \text{if } s_1 = t_1. \end{cases} \tag{2}$$

By using (2) in (1), we obtain that

$$\psi(z) = O((z-\alpha_1)^{m-1}), \tag{3}$$

where $m \geq \min\{s_1, t_1\}$. Equation (3) implies that $\psi(z)$ is analytic at α_1. Also by using F_1^* and G_1^*, we see that ψ has no poles. So that $T(r, \psi) = O(\log r)$ and hence ψ is a rational function. By (3), $s_1 \geq 2$ and since F_1^* and G_1^* share $(1, 1)$ except for the zeros of $\varphi_i(z)$, $i = 1, 2$, it implies that $t_1 \geq 2$. So that $\overline{N}_{(2}(r, 1; F_1^*) \leq N(r, 0; \psi) \leq T(r, \psi) + O(1) = O(\log r)$ as $r \to \infty$.

By the hypothesis of sharing condition, we get $\overline{N}_{(2}(r, G_1^*) = O(\log r)$ as $r \to \infty$. So that $F_1^* - 1$ and $G_1^* - 1$ have multiple zeros which are finite. Which implies that

$\overline{N}_{(2}(r,\varphi_1;F_1) = \overline{N}_{(2}(r,\varphi_2;F_1^{(k)}) = O(\log r)$ as $r \to \infty$. Now the multiple zeros of $P[f]-\varphi_1$ and $P[f]^{(k)}-\varphi_2$ are finite and also, $P[f]-\varphi_1$ and $P[f]^{(k)}-\varphi_2$ share $(0,1)$, we get

$$\frac{P[f]^{(k)}-\varphi_2}{P[f]-\varphi_1} = \xi e^{\eta}, \tag{4}$$

where $\xi(\neq 0)$ is a rational function and η is a polynomial. From (4) we have

$$\eta = (\log \xi)\frac{\frac{P[f]^{(k)}}{P[f]} - \frac{\varphi_2}{P[f]}}{1-\frac{\varphi_1}{P[f]}}$$

and by Lemma 2, we get

$$\frac{P[f]^{(k)}(z_r)}{P[f](z_r)} = \left(\frac{\nu(r,P[f])}{z_r}\right)^k (1+o(1)),$$

possibly outside a set of finite logarithmic measure E, where $M(r,\, P[f]) = |P[f](z_r)|$. Since $\rho(P[f]) < \infty$, $\log \nu(r,\, P[f]) = O(\log r)$. Also $P[f]$ is transcendental, $\left.\frac{\varphi_i}{P[f]}\right|_{z_r} \to 0$ as $r \to \infty$, $i = 1,2$. Now

$$|\eta(z_r)| = \left|\left(\log\frac{1}{\xi}\right)\frac{\frac{P[f]^{(k)}}{P[f]} - \frac{\varphi_2}{P[f]}}{1-\frac{\varphi_1}{P[f]}}\right| = O(\log r),$$

for $|z_r| = r \in E$. By this we have η is constant. Which implies that

$$P[f]^{(k)} - \varphi_2 \equiv \xi(P[f]-\varphi_1) \quad \text{or} \quad P[f]^{(k)} \equiv \xi P[f] + \varphi_2 - \varphi_1\xi. \tag{5}$$

Case 1.1.1. If $P[f]$ has infinitely many zeros and $\{z_n\}_{n=1}^{\infty}$ be the zeros of $P[f]$ but not the zeros of φ_i, $i=1,2$. Substituting in (5), we get $\xi(z_n) = \frac{\varphi_2(z_n)}{\varphi_1(z_n)}$ which gives $F_1^* \equiv G_1^*$ which is a contradiction.

Case 1.1.2. If $P[f]$ has finitely many zeros then $P[f]$ can be written as $P[f] = f^n$, $\xi \neq \frac{\varphi_2}{\varphi_1}$ and $\rho(P[f]) < \infty$. So that $P[f] = A_3(z)e^{A_4(z)}$ where A_3 is a non-zero polynomial and A_4 is a non constant polynomial. Now $P[f]^{(k)}(z) = (A_3(z)A_4^{(k)}(z) + A_5(z))e^{A_4(z)})$, where $A_5 = A_4^{(k-1)}A_3' + \overline{A}(A_3'',\, A_4')$ and $\overline{A}(A_3'',\, A_4')$ is a differential polynomial in A_3'' and A_4'. Substituting these functions in (5), we obtain $A_3(z)e^{A_4(z)} = \xi(z)A_3(z)e^{A_4(z)} + \varphi_2(z) - \xi(z)\varphi_1(z)$. Comparing the coefficients, we get $A_3A_4^{(k)} + A_5 = \xi A_3$ and $\varphi_2 - \xi\varphi_1 \equiv 0$ or in other words $\xi = \frac{\varphi_2}{\varphi_1}$ which is a contradiction.

Case 1.2. If $\psi \equiv 0$ then $(F_1^*)' \not\equiv 0$ as $P[f](z)$ is a transcendental entire function. Hence, $F_1^* = G_1^*$ i.e $(P[f])^{(k)} \equiv \frac{\varphi_2}{\varphi_1}P[f]$. In particular if $A_1 \equiv A_2$ then

$$(P[f])^{(k)} \equiv P[f]. \tag{6}$$

Let $n_{11}, n_{12}, \ldots, n_{1n_0}$ each with multiplicity $l_{11}, l_{12}, \ldots, l_{1n_0}$ respectively be the zeros of f such that $l_{11} + l_{12} + \ldots + l_{1n_0} = n_{0j}$. Let $n_{21}, n_{22}, \ldots, n_{2n_1}$ be the zeros of f coming from f' each with multiplicity $l_{21}, l_{22}, \ldots, l_{2n_1}$ such that $l_{21} + l_{22} + \ldots + l_{2n_1} = n_{1j}$. Proceeding in the same way, let $n_{k1}, n_{k2}, \ldots, n_{kn_k}$ be the zeros of f coming from $f^{(k)}$ each with multiplicity $l_{k1}, l_{k2}, \ldots, l_{kn_k}$ such that $l_{k1} + l_{k2} + \ldots + l_{kn_k} = n_{kj}$. Substituting these conditions in (6) becomes

$$[(f-n_{11})^{l_{11}}(f-n_{12})^{l_{12}} \cdot \ldots \cdot (f-n_{1n_0})^{l_{1n_0}}][(f-n_{21})^{l_{21}}(f-n_{22})^{l_{22}} \cdot \ldots \cdot (f-n_{2n_1})^{l_{2n_1}}] \times \ldots$$

$$\ldots \times (f-n_{k1})^{l_{k1}}(f-n_{k2})^{l_{k2}} \cdot \ldots \cdot (f-n_{kn_k})^{l_{kn_k}}] =$$

$$([(f-n_{11})^{l_{11}}(f-n_{12})^{l_{12}} \cdot \ldots \cdot (f-n_{1n_0})^{l_{1n_0}}][(f-n_{21})^{l_{21}}(f-n_{22})^{l_{22}} \cdot \ldots \cdot (f-n_{2n_1})^{l_{2n_1}}] \times \ldots$$

$$\ldots \times [(f-n_{k1})^{l_{k1}}(f-n_{k2})^{l_{k2}} \cdot \ldots \cdot (f-n_{kn_k})^{l_{kn_k}}])^{(k)}. \tag{7}$$

Since f is an entire function, it has only one Picard exceptional value say $'a'$. Therefore (7) can be written as

$$(f-a)^{n_{0j}}(f-a)^{n_{1j}} \cdot \ldots \cdot (f-a)^{n_{kj}} = ((f-a)^{n_{0j}}(f-a)^{n_{1j}} \cdot \ldots \cdot (f-a)^{n_{kj}})^{(k)},$$

$$(f-a)^{n_{0j}+n_{1j}+n_{2j}+\ldots+n_{kj}} = ((f-a)^{n_{0j}+n_{1j}+n_{2j}+\ldots+n_{kj}})^{(k)}, \tag{8}$$

$$(f-a)^{\nu} = ((f-a)^{\nu})^{(k)},$$

where $\nu = n_{0j} + n_{1j} + n_{2j} + \ldots + n_{kj}$. Since $(P[f])^{(k)}$ exists, left hand side of (8) does not vanish that is $(f-a)^{\nu} \not\equiv 0$. Therefore, $f = a + Ce^{(\frac{\lambda z}{\nu})}$, where C is a nonzero complex constant and $\lambda^k = 1$.

Case 2. Suppose $B(z)$ is a polynomial with degree≥ 1. Let $r_1 = 2\max\{\deg(B), 1+\deg(A_2)-\deg(A_1)\} \geq 2$ and $r_2 = \frac{r_1-2}{2}$. Since $\deg(B) \leq \rho(f) < \infty$, it can be written as $2 \leq r_1 < \rho(f)$ hence $0 \leq r_2 < \frac{\rho(f)-2}{2}$. For a small positive quantity ϵ, it can be said that $0 \leq r_2 < r = r_2 + \epsilon < \frac{\rho(f)-2}{2}$. By using Lemma 2.4 in [7] we get $\lim_{r\to\infty} \frac{F^{\#}(w_n)}{|w_n|^{\mu}} = +\infty$ where $F^{\#}(w_n)$ is a spherical derivative of F and by Marty's criterion $F^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Replacing F, H by F_1^*, $P[f]$ respectively in (3.10) of [7] and proceeding likewise. Using Hurwitz Theorem to the multiplicities of zeros of $P[f]$ are not less than $k+1$, we get a contradiction. Hence this case is impossible. Hence, by the case 1.2 of case 1, we get the only possibility as $f = a + Ce^{(\frac{\lambda z}{\nu})}$ and using which proves the Theorem.

The case when f is of infinite order can be dealt in a similar manner as in case 2 of [7]. □

Proof of Theorem 2. Let us define $b = b(z) = \frac{1}{\varphi(z)}$. Now by Lemma 5, we have $b(z)P[f]$ is a non-constant. And also

$$\frac{1}{f^{\nu}} = \frac{bP[f]}{f^{\nu}} - \frac{(bP[f])'(bP[f]-1)}{f^{\nu}(bP[f])'}.$$

Where ν is defined in Lemma 5. By using Lemma 3 and 4, the First Fundamental Theorem and Lemma of logarithmic derivative, we obtain

$$\begin{aligned}
\nu m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{bP[f]}{f^{\nu}}\right) + m\left(r, \frac{(bP[f])'}{f^{\nu}}\right) + m\left(r, \frac{bP[f]-1}{(bP[f])'}\right) + O(1) \leq \\
&\leq 2m\left(r, \frac{bP[f]}{f^{\nu}}\right) + m\left(r, \frac{(bP[f])'}{bP[f]}\right) + m\left(r, \frac{bP[f]-1}{(bP[f])'}\right) + O(1) \leq \\
&\leq T\left(r, \frac{(bP[f])'}{bP[f]-1}\right) - N\left(r, \frac{bP[f]-1}{(bP[f])'}\right) + S(r,f) \leq \\
&\leq \overline{N}(r,\infty;f) + N\left(r, \frac{1}{bP[f]-1}\right) - N(r,0;(bP[f])') + S(r,f) \leq \\
&\leq \frac{1}{p}N(r,\infty;f) + N\left(r, \frac{1}{P[f]-\varphi}\right) - (n-1)N(r,0,f) + S(r,f),
\end{aligned} \tag{9}$$

where $n = \min\{n_{0j} \colon 1 \leq j \leq l\}$. Using the First Fundamental Theorem and (9), we get

$$(\nu - n + 1)m\left(r, \frac{1}{f}\right) + (n-1)T(r,f) \leq N\left(r, \frac{1}{P[f] - \varphi(z)}\right) + \frac{1}{p}N(r,\infty,f) + S(r,f) \quad (10)$$

As $n > 1 + \frac{1}{p}$, then from (10), we get

$$T(r,f) \leq \frac{1}{n-1-\frac{1}{p}} N\left(r, \frac{1}{P[f]-\varphi(z)}\right) + S(r,f).$$

This completes the proof. □

Proof of Theorem 3. As normality is a local property. Let us assume that $D = \triangle$ is a unit disc. If possible, suppose that $\mathcal{F}$ is not normal on $\triangle$, then by Lemma 6, there exist $f_n \in \mathcal{F}$, $z_n \in \triangle$ and positive numbers ρ_n with $\rho_n \to 0$ such that

$$g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$$

locally, uniformly in spherical metric, where we choose $\alpha = \frac{\mu^*}{\nu}$. By using Lemma 6, $g(\xi)$ is a non-constant meromorphic function. By Hurwitz's Theorem we have all zeros of $g(\xi)$ are of multiplicity at least k. Next, we define

$$H_n(\xi) = (g_n(\xi))^{n_{01}}(g_n^{'}(\xi))^{n_{11}} \cdot \ldots \cdot (g_n^{(k)}(\xi))^{n_{k1}} + (g_n(\xi))^{n_{02}}(g_n^{'}(\xi))^{n_{12}} \times \ldots$$

$$\ldots \times (g_n^{(k)}(\xi))^{n_{k2}} + \ldots + (g_n(\xi))^{n_{0l}}(g_n^{'}(\xi))^{n_{1l}} \cdot \ldots \cdot (g_n^{(k)}(\xi))^{n_{kl}},$$

$$H(\xi) = (g(\xi))^{n_{0j}}(g^{'}(\xi))^{n_{1j}} \cdot \ldots \cdot (g^{(k)}(\xi))^{n_{kj}}$$

By the assumption, $H(\xi) \not\equiv 0$. Also

$$H_n(\xi) = \rho_n^{\mu_1 - \alpha\nu}[(f_n(z_n + \rho_n\xi))^{n_{01}}(f_n^{'}(z_n + \rho_n\xi))^{n_{11}} \cdot \ldots \cdot (f_n^{(k)}(z_n + \rho_n\xi))^{n_{k1}}] +$$

$$+\rho_n^{\mu_2 - \alpha\nu}[(f_n(z_n + \rho_n\xi))^{n_{02}}(f_n^{'}(z_n + \rho_n\xi))^{n_{12}} \cdot \ldots \cdot (f_n^{(k)}(z_n + \rho_n\xi))^{n_{k2}}] + \ldots$$

$$\ldots + \rho_n^{\mu_3 - \alpha\nu}[(f_n(z_n + \rho_n\xi))^{n_{0l}}(f_n^{'}(z_n + \rho_n\xi))^{n_{1l}} \cdot \ldots \cdot (f_n^{(k)}(z_n + \rho_n\xi))^{n_{kl}}] \to H(\xi)$$

locally uniformly in spherical metric. As $H_n(\xi) \neq 1$ and by the Hurwitz's Theorem, $H(\xi) \neq 1$. Thus by Corollary 1, $g(\xi)$ must be non-constant rational function, otherwise $H(\xi) - 1$ has infinitely many solution, which is not possible. As $\mathcal{F}$ is a family of analytic functions, so $g_n(\xi)$ is analytic. Since, $g_n(\xi) \to g(\xi)$ locally, uniformly in spherical metric, so $g(\xi)$ is an analytic function. But, since $g(\xi)$ is non-constant, so $g(\xi)$ must be a polynomial, say $g(\xi) = C_0 + C_1\xi + ... + C_p\xi^p$. If $p \geq k$, then $H(\xi)$ becomes a non-constant polynomial, which contradicts that $H(\xi) \neq 1$. Thus $p < k$, which in view of Hurwitz's Theorem, contradicts our assumptions on zeros of $f \in \mathcal{F}$. Thus our assumption is wrong. So $\mathcal{F}$ is normal. This completes the proof. □

Acknowledgement. We thank the referee(s) for reading the manuscript carefully and also thankful to UGC for their financial support under NFST scheme [202122-NFST-KAR-01081] dated:14/10/2022.

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Received 01.02.2023
Revised 03.06.2023