УДК 517.956

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# OPTIMAL CONTROL IN THE BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS WITH DEGENERATION

I. D. Pukal'skii, B. O. Yashan. *Optimal control in the boundary value problem for elliptic equations with degeneration*, Mat. Stud. **59** (2023), 76–85.

The problem of optimal control of the system described by the oblique derivative problem for the elliptic equation of the second order is studied. Cases of internal and boundary management are considered. The quality criterion is given by the sum of volume and surface integrals. The coefficients of the equation and the boundary condition allow power singularities of arbitrary order in any variables at some set of points. Solutions of auxiliary problems with smooth coefficients are studied to solve the given problem. Using a priori estimates, inequalities are established for solving problems and their derivatives in special Hölder spaces. Using the theorems of Archel and Riess, a convergent sequence is distinguished from a compact sequence of solutions to auxiliary problems, the limiting value of which will be the solution to the given problem.

The necessary and sufficient conditions for the existence of the optimal solution of the system described by the boundary value problem for the elliptic equation with degeneracy have been established.

**Introduction.** The theory of optimal control of systems, which is described by partial differential equations, is rich in results and is actively developing nowadays. The popularity of this kind of research is connected with its active use in solving problems of natural science, in particular hydro and gas dynamics, heat physics, diffusion, and theory of biological populations.

The basics of the theory of optimal control of deterministic systems described by equations with partial derivatives were systematically described for the first time in the monograph [1]. A papers [2–5] are devoted to the problems of choosing the optimal control of systems described by parabolic boundary value problems with limited internal, starting and boundary control. Problems for high-order degenerate elliptic equations in a half-space are studied in the paper [10]. Under consideration are the questions of the numerical solution by the finite element method (FEM) of the first boundary value problem for an elliptic equation with degeneration on a part of the boundary in [11]. In the paper [12] a class of degenerate elliptic equations with arbitrary power degeneration are considered. The paper [13] shows the unique solvability of the classical Dirichlet problem in cylindrical domain for threedimensional elliptic equations with degeneration of type and order. In [14] the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations is considered.

This paper considers a boundary value problem with an oblique derivative for an elliptic equation with power singularities of arbitrary order in the coefficients of the equation and

2010 *Mathematics Subject Classification:* 35J25, 35K20.
*Keywords:* degeneration; apriori estimation; boundary condition; optimal control.
doi:10.30970/ms.59.1.76-85

the boundary condition for any variables on some set of points. With the help of a priori estimates and the principle of the maximum, the existence of the unique solution to the given problem was proved and the estimates of its derivatives in Hölder spaces with power-law weight were established. The obtained result was used to establish the necessary and sufficient conditions for the existence of optimal control of the system, which is described by a boundary value problem with internal and boundary control. The quality criteria are given by the sum of volume and surface integrals.

**Problem formulation and main limitations.** Let $D$ a bounded domain in $\mathcal{R}^n$ with a boundary $\partial D$, $\dim D = n$, $\Omega$ be some bounded domain, $\overline{\Omega} \subset \overline{D}$, $\dim \Omega \leq n-1$.

Consider in the domain $D$ the task of finding functions $(u(x, q_1(x), q_2(x)); q_1(x); q_2(x))$ on which the functional

$$I(q_1, q_2) = \int_D F_1(x; u(x, q_1(x), q_2(x)); q_1(x))dx+$$
$$+\int_{\partial D} F_2(x; u(x, q_1(x), q_2(x)); q_2(x))d_x S \tag{1}$$

reaches the minimum in the function class $q \in V = \{q | q_1 \in C^{\alpha}(D),\ q_2 \in C^{1+\alpha}(D),\ \nu_{11}(x) \leq q_1(x) \leq \nu_{12}(t,x),\ \nu_{21}(t,x) \leq q_2(x) \leq \nu_{22}(x)\}$ of which $u(x, q_1(x), q_2(x))$ satisfies at $x \in D \backslash \overline{\Omega}$ the equation with a parameter $\lambda$

$$\Big[\sum_{i,j=1}^{n} A_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} A_i(x)\partial_{x_i} + A_0(x) - \lambda\Big] u(x, q_1(x), q_2(x)) = f(x, q_1(x)), \tag{2}$$

and on the border of the domain $\partial D$ boundary condition

$$\lim_{x \longrightarrow z \in \partial D} \Big[\sum_{k=1}^{n} B_k(x)\partial_{x_k} u + B_0(x)u - \varphi(x, q_2(x))\Big] = 0. \tag{3}$$

The order of peculiarities of the coefficients of the equation (2) and boundary condition (3) at the point $P(t,x) \in D \backslash \overline{\Omega}$ will characterize the functions $s(a,x)$: $s(a,x) = \rho^a(x)$ at $\rho(x) \leq 1$, $s(a,x) = 1$ at $\rho(x) \geq 1$, $a \in (-\infty, \infty)$, $\rho(x) = \inf\limits_{z \in \overline{\Omega}} |x - z|$.

Denote by $\gamma$, $l$, $\beta_i$, $\mu_i$, $i \in \{1, 2, \ldots, n\}$, $\mu_0$, $\delta_0$ real numbers, $\beta_i \in (-\infty, \infty)$, $\mu_i \geq 0$, $\mu_0 \geq 0$, $l \geq 0$, $\delta_0 \geq 0$, $\gamma \geq 0$, let $[l]$ be the integer part of $l$, $\{l\} = l - [l]$, $P_1(x^{(1)})$, $H_r(x^{(2)})$ arbitrary points of the domain $\overline{D}$, $x^{(1)} = (x_1^{(1)}, \ldots, x_r^{(1)}, \ldots, x_n^{(1)})$, $x^{(2)} = (x_1^{(1)}, \ldots, x_{r-1}^{(1)}, x_r^{(2)}, x_{r+1}^{(1)}, \ldots, x_n^{(1)})$, $\beta = (\beta_1, \ldots, \beta_n)$.

We define the functional space in which we study problem (1)–(3).

$C^l(\gamma; \beta; a; D)$ denotes the set of functions $u : x \in \overline{D}$, having continuous partial derivatives in the domain $D \setminus \overline{\Omega}$ of the form $\partial_x^k$, $|k| \leq [l]$, and a finite value of the norm

$$\|u; \gamma; \beta; a; D\|_l = \sum_{|k| \leq [l]} \|u; \gamma; \beta; a; D\|_{|k|} + \langle u; \gamma; \beta; a; D\rangle_l,$$

where, e.g., $\|u; \gamma; \beta; 0; D\|_0 = \sup\{|u(P)| \colon \overline{P \in \overline{D}}\} \equiv \|u; D\|_0$,

$$\langle u; \gamma; \beta; a; D\rangle_l \equiv \sum_{|k|=[l]} \Big[\sum_{r=1}^{n} \sup_{(P_1 H_r) \in \subset D} s(a + [l]\gamma, \widetilde{x}) s(\{l\}(\gamma - \beta_r), \widetilde{x}) \times$$
$$\times |x_r^{(1)} - x_r^{(2)}|^{-\{l\}} |\partial_x^k u(P_1) - \partial_x^k u(H_r)| \prod_{i=1}^{n} s(-k_i \beta_i, x)\Big],$$

$(a,\widetilde{x}) = \min(s(a,x^{(1)}), s(a,x^{(2)}))$, $\partial_x^k = \partial_{x_1}^{k_1}, \dots, \partial_{x_n}^{k_n}$, $|k| = k_1 + \dots + k_n$.

Assume that the initial problems (1)–(3) satisfy the following conditions:

a) for the arbitrary vector $\xi = (\xi_1, \dots, \xi_n)$ the following inequality holds

$$\pi_1 |\xi|^2 \le \sum_{i,j=1}^n A_{ij}(t,x) s(\beta_i, x) s(\beta_j, x) \xi_i \xi_j \le \pi_2 |\xi|^2,$$

where $\pi_1, \pi_2$ are fixed positive constants and $s(\beta_i,x)s(\beta_j,x)A_{i,j}(x) \in C^\alpha(\gamma;\beta;0;D)$, $\beta_i \in (-\infty,\infty)$, $s(\mu_i,x)A_i(x) \in C^\alpha(\gamma;\beta;0;D)$, $\mu_i \ge 0$, $s(\mu_0,x)A_0(x) \in C^\alpha(\gamma;\beta;0;D)$, $\mu_0 \ge 0$, $A_0(x) < \lambda$, $0 < \lambda < \infty$;

b) $B_k(x)s(\beta_k,x) \in C^{1+\alpha}(\gamma;\beta;0;D)$, $B_0(x)s(\delta_0,x) \in C^{1+\alpha}(\gamma;\beta;0;D)$, $\delta_0 \ge 0$, $B_0(x)|_{\partial D} > 0$, vector $\overrightarrow{b}^{(s)} = \{s(\beta_1,x)B_1(x), \dots, s(\beta_n,x)B_n(x)\}$ forms with the direction of the external normal $\overrightarrow{n}$ to $\partial D$ at the point $P(x) \in \partial D$ the angle less than $\frac{\pi}{2}$, $\partial D \in C^{2+\alpha}$, $\alpha \in (0,1)$;

c) $f(x,q_1(x)) \equiv F(x) \in C^\alpha(\gamma;\beta;\mu_0;D)$, $\varphi(x,q_2(x)) \equiv \Phi(x) \in C^{1+\alpha}(\gamma;\beta;\delta_0;D)$, $\gamma = \max\{\max_i \beta_i, \max_i(\mu_i - \beta_i), \delta_0, \frac{\mu_0}{2}\}$;

d) functions $F_1(x;u;q_1)$, $f(x,q_1)$, $F_2(x;u;q_2)$, $\varphi(x,q_2)$ have derivatives of the second order with respect to the variables $(u;q_1;q_2)$, which belong as functions of variables $x$ to the spaces $C^\alpha(D)$, $C^{1+\alpha}(\partial D)$, $\nu_{11} \in C^\alpha(D)$, $\nu_{12} \in C^\alpha(D)$, $\nu_{21} \in C^{1+\alpha}(D)$, $\nu_{22} \in C^{1+\alpha}(D)$, respectively.

The following theorem is valid.

**Theorem 1.** *Let the conditions* a)–c) *be fulfilled for the problem* (2), (3). *Then there is a unique solution to the problem* (2), (3) *from space* $C^{2+\alpha}(\gamma;\beta;0;D)$ *and the inequality holds*

$$\|u;\gamma;\beta;0;D\|_{2+\alpha} \le c(\|f;\gamma;\beta;\mu_0;D\|_\alpha + \|\varphi;\gamma;\beta;\delta_0;D\|_{1+\alpha}). \tag{4}$$

To prove Theorem 1, we first establish the correct solvability of boundary value problems with smooth coefficients. From the set of obtained solutions, we select a convergent sequence, the limiting value of which will be the solution of problem (2), (3).

**Estimation of solutions of boundary value problems with smooth coefficients.** Let $D_m = D \cap \{x \in D \mid s(1,x) \ge m^{-1}\}$, $m \ge 1$, be a sequence of domains that, for $m \longrightarrow \infty$ converges to $D$. In the domain $D$ we consider the problem of finding the function $u_m(x)$, that satisfies the equation

$$\Big[\sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^n a_i(x)\partial_{x_i} + a_0(x) - \lambda\Big] u_m(x) = f_m(x, q_1(x)), \tag{5}$$

and on the boundary of the domain $\partial D$ the boundary condition

$$\lim_{x \longrightarrow z \in \partial D} \Big[\sum_{k=1}^n b_k(x)\partial_{x_k} u_m + b_0(x) u_m - \varphi_m(x, q_2(x))\Big] = 0. \tag{6}$$

Here, the coefficients $a_{ij}$, $a_i$, $a_0$, $b_k$, $b_0$, and functions $f_m$, $\varphi_m$ in the domains $D_m$ coincide with $A_{ij}$, $A_i$, $A_0$, $B_k$, $B_0$, $f$, $\varphi$ respectively, and in the domains $D \backslash D_m$ are continuous extensions of the coefficients $A_{ij}$, $A_i$, $A_0$, $B_k$, $B_0$ and functions $f$, $\varphi$ from the domain $D_m$ into the domain $D \backslash D_m$ while maintaining smoothness and normality [6, p. 82].

Denote by $H^l(\gamma;\beta;a;D)$ the set of functions of the space $C^l(D)$ with the norm $\|u_m;\gamma;\beta;a;D\|_l$ equivalent for each $m$ Hölder norm, which is defined in the same way as $\|u;\gamma;\beta;a;D\|_l$, only instead of the function $s(a,x)$ we take $d(a,x) = \min(s(a,x), m^{-a})$ at $a < 0$.

For the norm $\|u_m;\gamma;\beta;a;D\|_l$ correct interpolation inequalities.

**Lemma 1.** *Let $u_m \in H^{2+\alpha}(\gamma;\beta;0;D)$. Then for arbitrary $\varepsilon$, $0<\varepsilon<1$, there is a constant $c(\varepsilon)$, that the inequalities*

$$\|u_m;\gamma;\beta;0;D\|_2 \le \varepsilon^\alpha \langle u_m;\gamma;\beta;0;D\rangle_{2+\alpha} + c(\varepsilon)\|u_m;D\|_0,$$
$$\|u_m;\gamma;\beta;0;D\|_1 \le \varepsilon\|u_m;\gamma;\beta;0;D\|_2 + \frac{c}{\varepsilon}\|u_m;D\|_0 \tag{7}$$

*hold.*

Inequality (7) is obtained by the lemma proof scheme [7]. When conditions a)–c) are satisfied there is the unique solution to the problem (5), (6) in space $H^{2+\alpha}(\gamma;\beta;0;D)$, ([8], Theorem 2.20, p. 233). Let's estimate the norm $\|u_m;\gamma;\beta;0;D\|_{2+\alpha}$.

The theorem is correct.

**Theorem 2.** *If the conditions a)–c) are satisfied, then for a solution of the problem* (5), (6) *we have*

$$\|u_m;\gamma;\beta;0;D\|_{2+\alpha} \le c(\|f_m;\gamma;\beta;2\gamma;D\|_\alpha + \|\varphi_m;\gamma;\beta;\gamma;D\|_{1+\alpha} + \|u_m;D\|_0), \tag{8}$$

*where the constant $c$ does not depend on $m$.*

*Proof.* Using the definition of the norm and inequality (7), we have

$$\|u_m;\gamma;\beta;0;D\|_{2+\alpha} \le (1+\varepsilon^\alpha)\langle u_m;\gamma;\beta;0;D\rangle_{2+\alpha} + (\varepsilon)\|u_m;D\|_0,$$

where $\varepsilon$ is an arbitrary real number from (0,1). Therefore, it is enough to estimate the seminorm $\langle u_m;\gamma;\beta;0;D\rangle_{2+\alpha}$. The definition of the half norm implies the existence points $P_1(x^{(1)})$ and $H_r(x^{(2)})$ in $\overline{D}$, for which the inequality is correct

$$\frac{1}{2}\|u_m;\gamma;\beta;0;D\|_{2+\alpha} \le E(u_m), \tag{9}$$

$$E(u_m) = \sum_{|k|=2}\Big[\sum_{r=1}^{n} d(2\gamma,\widetilde{x})d(\alpha(\gamma-\beta_r),\widetilde{x})|x_r^{(1)}-x_r^{(2)}|^{-\alpha}\times$$
$$\times|\partial_x^k u_m(P_1)-\partial_x^k u_m(H_r)|\prod_{i=1}^{n} d(-k_i\beta_i,\widetilde{x})\Big].$$

If $|x_r^{(1)}-x_r^{(2)}| \ge \frac{\varepsilon_1 n^{-1}}{4} d(\gamma-\beta_r,\widetilde{x}) \equiv N$, $\varepsilon_1$ is an arbitrary real number, $\varepsilon\in(0,1)$, then
$$E(u_m) \le 2\varepsilon_1^{-\alpha}\|u_m;\gamma;\beta;0;D\|_2.$$

Using interpolation inequalities (7), we find

$$E(u_m) \le \varepsilon^\alpha\langle u_m;\gamma;\beta;0;D\rangle_{2+\alpha} + c(\varepsilon)\|u_m;D\|_0. \tag{10}$$

Let $|x_r^{(1)}-x_r^{(2)}| \le N$. We will assume that $d(\gamma,\widetilde{x}) \equiv d(\gamma,x^{(1)})$. Let $|x_n-\xi_n| \le 2N$, $\xi\in\partial D$ або $|x-\xi| \le 2Nn$. Consider a ball $K(a,P)$ of radius $a$, $a>4Nn$, containing points $P_1$, $H_r$ centered at some point $P\in\partial D$. Using the boundary smoothness of $\partial D$, one can straighten $\partial D\cap K(a,P)$ using a mutually unique transformation $x=\psi(y)$ [6, p. 155], suth that the domain $\Pi = D\cap K(a,P)$ passes into the domain $\Pi_1$, for which $y_n \ge 0$.

Let us put $u_m(x) = v_m(y)$. Let us assume that $P_1$, $H_r$, $E$, $d(\gamma,x^{(1)})$ pass during this transformation into $R_1$, $M_r$, $E_1$, $d_1(\gamma,y^{(1)})$.

Let us denote the coefficients of the equation (5) and boundary conditions (6) in the domain $\Pi_1$ by $r_{ij}(y)$, $r_i(y)$, $r_0(y)$, $l_k(y)$, $l_0(y)$. Then $v_m$ will be the solution to the problem

$$\Big[\sum_{ij=1}^{n} r_{ij}(R_1)\partial_{y_i}\partial_{y_j} - \lambda\Big]v_m(y) = \sum_{ij=1}^{n}\Big[r_{ij}(R_1) - r_{ij}(y)\Big]\partial_{y_i}\partial_{y_j}v_m - \sum_{i=1}^{n} r_i(t,y)\partial_{y_i}v_m - \\ -r_0(y)v_m + f_m(\psi(y), q_1(\psi(y))) \equiv F_m(y), \tag{11}$$

$$\sum_{k=1}^{n} l_k(R_1)\partial_{y_k}v_m|_{y_n=0} = \Big(\sum_{k=1}^{n}[l_k(R_1) - l_k(y)]\partial_{y_k}v_m - \\ -l_0(y)v_m + \varphi_m(\psi(y), q_2(\psi(y)))\Big)\Big|_{y_n=0} \equiv G_m(y)\Big|_{y_n=0}. \tag{12}$$

In the problem (11), (12) let us make a replacement $v_m(y) = \omega_m(z)$, $z_i = d_1(\beta_i, y^{(1)})y_i$, $i \in \{1,\dots,n\}$. Then the function $W_m(z) = \eta(z)\omega(z)$ will be the solution to the problem

$$\Big[\sum_{ij=1}^{n} r_{ij}(R_1)d_1(\beta_i, y^{(1)})d_1(\beta_j, y^{(1)})\partial_{z_i}\partial_{z_j} - \lambda\Big]W_m = \\ = \sum_{ij=1}^{n} r_{ij}(R_1)d_1(\beta_i, y^{(1)})d_1(\beta_j, y^{(1)})[\partial_{z_i}\omega_m\partial_{z_j}\eta + \partial_{z_j}\omega_m\partial_{z_i}\eta] + \\ +\omega_m\sum_{ij=1}^{n} r_{ij}(R_1)d_1(\beta_i, y^{(1)})d_1(\beta_j, y^{(1)})\partial_{z_i}\partial_{z_j}\eta + \eta F_m(\widetilde{z}) \equiv F_m^{(1)}(z), \tag{13}$$

$$\sum_{k=1}^{n} l_k(R_1)d_1(\beta_k, y^{(1)})\partial_{z_k}W_m|_{z_n=0} = \Big[\sum_{k=1}^{n} l_k(R_1)d_1(\beta_k, y^{(1)})\omega_m\partial_{z_k}\eta + \\ +\eta G_m(\widetilde{z})\Big]|_{z_n=0} \equiv G_m^{(1)}(z)|_{z_n=0}, \tag{14}$$

where $\widetilde{z} = (d_1^{-1}(\beta_1, y^{(1)})z_1, \dots, d_1^{-1}(\beta_n, y^{(1)})z_n)$, $z_i^{(1)} = d_1(\beta_i, y^{(1)})y_i^{(1)}$,

$$\eta(z) = \begin{cases} 1, z \in H_{1/2}^{(1)}, 0 \le \eta(z) \le 1, \\ 0, z \notin H_{3/4}^{(1)}, |\partial_z^k\eta| \le c_k d_1^{-1}(|k|\gamma, y^{(1)}), \end{cases}$$

$$H_\delta^{(1)} = \big\{z \colon |z_i - z_i^{(1)}| \le 4\delta n^{-1}d_1(\gamma, y^{(1)}),\ i \in \{1,\dots,n\}\big\}.$$

Coefficients of the equation (13) and boundary conditions (14) limited to constants, do not dependent on $R_1(y^{(1)})$. Therefore, using Theorem 7.3 from [9, p. 77], for arbitrary points $S_1(\xi^{(1)}) \in H_{1/2}^{(1)}$, $S_2(\xi^{(2)}) \in H_{1/2}^{(1)}$ we have

$$|\xi^{(1)} - \xi^{(2)}|^{-\alpha}|\partial_z^2\omega_m(S_1) - \partial_z^2\omega_m(S_2)| \le c(\|F_m^{(1)}\|_{C^\alpha(H_{3/4}^{(1)})} + \\ +\|G_m^{(1)}\|_{C^{1+\alpha}(H_{3/4}^{(1)}\cap(z_n=0))} + \|\omega_n; H_{3/4}^{(1)}\|). \tag{15}$$

Using the properties of the function $\eta(z)$, inequalities (7), we obtain

$$\|F_m^{(1)}\|_{C^\alpha(H_{3/4}^{(1)})} \le cd_1^{-1}((2+\alpha)\gamma, y^{(1)})(\|\omega_m;\gamma;0;0;H_{3/4}^{(1)}\|_2 + \|\omega_m; H_{3/4}^{(1)}\|_0 + \\ +\|F_m;\gamma;0;2\gamma;H_{3/4}^{(1)}\|_\alpha), \tag{16}$$

$$\|G_m^{(1)}\|_{C^{1+\alpha}(H_{3/4}^{(1)}\cap(z_n=0))} \le c_1 d_1^{-1}((2+\alpha)\gamma, y^{(1)})\times$$

$$[\|\omega_m;\gamma,0;0;H^{(1)}_{3/4}\|_2+\|\omega_m;H^{(1)}_{3/4}\|_0+\|G_m;\gamma,0;\gamma;H^{(1)}_{3/4}\|_{1+\alpha}]. \tag{17}$$

From the definition of the space $H^{2+\alpha}(\gamma,\beta;0;D)$ we deduce

$$c_2\|\omega_m;\gamma,0;0;H^{(1)}_{3/4}\|_l\leq\|u_m;\gamma,\beta;0;T_{3/4}\|_l\leq c_3\|\omega_m;\gamma,0;0;H^{(1)}_{3/4}\|_l,$$

$T_\delta=\{x\in\Pi,|x_i-x_i^{(1)}|\leq 4\delta n^{-1}d(\gamma-\beta_i,x^{(1)}),\ \ i\in\{1,\dots,n\}\}$.

Let us substitute (16), (17) in (15) and let us return to the variable $x$. We get

$$\begin{aligned}E(u_m)\leq c(&\|F_m;\gamma,\beta;2\gamma;T_{3/4}\|_\alpha+\|G_m;\gamma,\beta;\gamma;T_{3/4}\|_{1+\alpha}+\\&+\|u_m;\gamma,\beta;0;T_{3/4}\|_2+\|u_m;T_{3/4}\|_0),\end{aligned} \tag{18}$$

To find norms $\|F_m;\gamma,\beta;2\gamma;T_{3/4}\|_\alpha$, $\|G_m;\gamma,\beta;\gamma;T_{3/4}\|_{1+\alpha}$ it is sufficient to evaluate the seminorms of each term of the expressions $F_m$, $G_m$. Using inequalities (7), we obtain

$$\begin{aligned}\|F_m;\gamma,\beta;2\gamma;T_{3/4}\|_\alpha&\leq c_4(\|f_m;\gamma,\beta;2\gamma;T_{3/4}\|_\alpha+\|u_m;T_{3/4}\|_0)+\\&+\varepsilon_2\|u_m;\gamma,\beta;0;T_{3/4}\|_{2+\alpha}\end{aligned} \tag{19}$$

$$\begin{aligned}\|G_m;\gamma,\beta;\gamma;T_{3/4}\|_{1+\alpha}&\leq c_5(\|\varphi_m;\gamma,\beta;\gamma;T_{3/4}\|_{1+\alpha}+\|u_m;T_{3/4}\|_0)+\\&+\varepsilon_3\|u_m;\gamma,\beta;0;T_{3/4}\|_{2+\alpha}\end{aligned} \tag{20}$$

Substituting (19), (20) in (18) we find

$$\begin{aligned}E(u_m)\leq c_6(&\|f_m;\gamma,\beta;2\gamma;T_{3/4}\|_\alpha+\|\varphi_m;\gamma,\beta;\gamma;T_{3/4}\|_{1+\alpha}+\|u_m;T_{3/4}\|_0)+\\&+\varepsilon_4\|u_m;\gamma,\beta;0;T_{3/4}\|_{2+\alpha}.\end{aligned} \tag{21}$$

Consider the case $|x_n-\xi_n|\geq 2N$ or $|x-\xi|\geq 2Nn$, $\xi\in\partial D$. Let $T^{(1)}_\delta=\{x\in D||x_i-x_i^{(1)}|\leq 4\delta N\}$. In the problem (5), (6) let us make the replacement $u_m(x)=V_m(t)$, $t_i=d(\beta_i,x^{(1)})x_i$, $i\in\{1,\dots,n\}$. Then the function $W^{(1)}_m(t)=V_m(t)\eta_1(t)$ will be the solution of the problem

$$\begin{aligned}&\sum_{ij=1}^n a_{ij}(P_1)d(\beta_i,x^{(1)})d(\beta_j,x^{(1)})\partial_{t_i}\partial_{t_j}W^{(1)}_m-\lambda W^{(1)}_m=\\&=\sum_{ij=1}^n a_{ij}(P_1)d(\beta_i,x^{(1)})d(\beta_j,x^{(1)})[\partial_{t_i}V_m\partial_{t_j}\eta_1+\partial_{t_j}V_m\partial_{t_i}\eta_1]+\\&+V_m\sum_{ij=1}^n a_{ij}(P_1)d(\beta_i,x^{(1)})d(\beta_j,x^{(1)})\partial_{t_i}\partial_{t_j}\eta_1+\eta_1F^{(2)}_m(\widetilde{t})\equiv F^{(3)}_m(t),\end{aligned} \tag{22}$$

$$\begin{aligned}\sum_{k=1}^n b_k(P_1)d(\beta_k,x^{(1)})\partial_{t_k}W^{(1)}_m|_{\partial D}=\Big[&\sum_{k=1}^n b_k(P_1)d(\beta_k,x^{(1)})V_m\partial_{t_k}\eta_1+\\&+\eta_1G^{(2)}_m(\widetilde{t})\Big]|_{\partial D}\equiv G^{(3)}_m(t)|_{\partial D},\end{aligned} \tag{23}$$

where

$$F^{(2)}_m(x)=\sum_{ij=1}^n[a_{ij}(P_1)-a_{ij}(x)]\partial_{x_i}\partial_{x_j}u_m-\sum_{i=1}^n a_i(x)\partial_{x_i}u_m-a_0(x)u_m+f_m(x,q_1(x)),$$

$$G_m^{(2)}(x) = \sum_{ij=1}^{n} [b_k(P_1) - b_k(x)]\partial_{x_k} u_m - b_0(x) u_m + \varphi_m(x, q_2(x)),$$

$$\eta_1(t) = \begin{cases} 1, & t \in H_{1/2}^{(2)}, 0 \le \eta_1(t) \le 1, \\ 0, & t \notin H_{3/4}^{(2)}, |\partial_t^k \eta_1| \le c_k d^{-1}(|k|\gamma, x^{(1)}), \end{cases}$$

$$H_\delta^{(2)} = \{t\colon |t_i - x_i^{(1)}| \le 4\delta n^{(-1)} d(\gamma, x^{(1)}), i \in \{1, \dots, n\}\},$$

$$\widetilde{t} = (d^{-1}(\beta_1, x^{(1)})t_1, \dots, d^{-1}(\beta_n, x^{(1)})t_n), \quad t_i^{(1)} = d(\beta_i, x^{(1)}) x_i^{(1)}.$$

The coefficients of equation (22) and boundary condition (23) bounded by constants, independent of $P_1(x^{(1)})$. Therefore, using Theorem 7.3 from [9, p. 77] and evaluations (7.8) from [9, p. 77], for arbitrary points $S_1(\tau^{(1)})$ and $S_2(\tau^{(2)}) \in H_{1/2}^{(2)}$ the inequality is valid

$$|\tau^{(1)} - \tau^{(2)}|^{-\alpha} |\partial_t^2 V_m(\tau^{(1)}) - \partial_t^2 V_m(\tau^{(2)})| \le c(\|F_m^{(3)}(x)\|_{C^\alpha(H_{3/4}^{(2)})} +$$
$$+ \|G_m^{(3)}\|_{C^{1+\alpha}(H_{3/4}^{(2)})} + \|V_m; H_{3/4}^{(2)}\|_0).$$

Given the properties of the function $\eta_1(t)$, definition of the space $H^{2+\alpha}(\gamma;\beta;a;D)$ and repeating the reasoning in obtaining the inequality (21), we find

$$E(u_m) \le c_7(\|f_m;\gamma,\beta;2\gamma;T_{3/4}^{(1)}\|_\alpha + \|\varphi_m;\gamma,\beta;\gamma;T_{3/4}^{(1)}\|_{1+\alpha} + \|u_m;T_{3/4}^{(1)}\|_0) +$$
$$+ \varepsilon_5 \|u_m;\gamma,\beta;0;T_{3/4}^{(1)}\|_{2+\alpha}. \tag{24}$$

Considering inequalities (9), (10), (21), (24) and choosing $\varepsilon$, $\varepsilon_4$, $\varepsilon_5$ small enough, we get an estimate (8). □

Let us find the estimate of the norm $\|u_m; D\|_0$.

**Theorem 3.** *If $u_m$ is a classical solution of the problem (5), (6) in the domain $D$ and the conditions a)–c) are satisfied, then for $u_m(x)$ we have*

$$\|u_m; D\|_0 \le \max\{\|f_m(a_0 - \lambda)^{-1}; D\|_0, \|b_0^{-1}\varphi_m; D\|_0\}. \tag{25}$$

Correctness of estimates (25) is established by analyzing all possible locations of the positive maximum and negative minimum of the function $u_m(x)$.

*Proof of Theorem 1.* Since

$$\|f_m;\gamma;\beta;2\gamma;D\|_\alpha \le c\|f;\gamma;\beta;\mu_0;D\|_\alpha, \quad \|\varphi_m;\gamma;\beta;\gamma;D\|_{1+\alpha} \le c\|\varphi;\gamma;\beta;\delta_0;D\|_{1+\alpha},$$

using estimates (8), (25), we obtain

$$\|u_m;\gamma;\beta;0;D\|_{2+\alpha} \le c(\|f;\gamma;\beta;\mu_0;D\|_\alpha + \|\varphi;\gamma;\beta;\delta_0;D\|_{1+\alpha}). \tag{26}$$

The right-hand side of the inequality (26) does not depend on $m$, and sequences $\{W_m^{(k)}\} = \{d(|k|\gamma;x)\prod_{i=1}^n d(-k_i\beta_i,x)|\partial_x^k u_m(x)|\}$, $|k| \le 2$ uniformly bounded and uniformly continuous. According to Arcel's theorem, there are subsequences $\{W_{m(j)}^{(k)}\}$, uniformly convergent to $W^{(k)}$ at $m(j) \longrightarrow \infty$. Passing to the limit of $m(j) \longrightarrow \infty$ in problems (5), (6), we obtain that $u = W^{(0)}$ is the only solution of the problem (2), (3), $u \in C^{2+\alpha}(\gamma;\beta;0;D)$ and correct estimation (4). □

**The problem of optimal control.** For the solvability of the problem (1)–(3) construct a sequence of problem solutions, the limit value of which will be the solution of the problem (1)–(3).

Consider in the domain $D$ the problem of finding functions $(u_m(x, q_1(x), q_2(x)); q_1(x); q_2(x))$, on which the functional

$$I(q_1, q_2) = \int_D F_1(x; u_m(x, q_1(x), q_2(x)); q_1(x))dx+$$
$$+\int_D F_2(x; u_m(x, q_1(x), q_2(x)); q_2(x))d_xS \tag{27}$$

reaches the minimum value in the function class $q \in V$, where $u_m(x, q_1(x), q_2(x))$ satisfies equation (5) and the boundary condition (6).

We assume that conditions a)–d) are fulfilled, when performing which for any $q \in V$ there is a unique solution to the problem (5), (6) from the space $C^{2+\alpha}(\gamma; \beta; 0; D)$ satisfying (4).

Denote by $(G_m^{(1)}(x, \xi), G_m^{(2)}(x, \xi))$ from [8, p. 234] the Green's function of the problem (5), (6).

$$\lambda_1(\xi) = \int_D \frac{\partial F_1(x; u_m; q_1)}{\partial u_m} G_m^{(1)}(x, \xi)dx + \int_{\partial D} \frac{\partial F_2(x; u_m; q_2)}{\partial u_m} G_m^{(1)}(x, \xi)d_xS,$$
$$\lambda_2(\xi) = \int_D \frac{\partial F_1(x; u_m; q_1)}{\partial u_m} G_m^{(2)}(x, \xi)dx + \int_{\partial D} \frac{\partial F_2(x; u_m; q_2)}{\partial u_m} G_m^{(2)}(x, \xi)d_xS,$$
$$H_1(\xi; u_m, \lambda_1, q_1) \equiv \lambda_1(\xi)f_m(\xi, q_1(\xi)) + F_1(\xi; u_m, q_1),$$
$$H_2(\xi; u_m, \lambda_2, q_2) \equiv \lambda_2(\xi)\varphi_m(\xi, q_2(\xi)) + F_2(\xi; u_m, q_2),$$

$q^{(0)} = (q_1^{(0)}, q_2^{(0)})$ is an optimal management, $u_m(x, q_1^{(0)}, q_2^{(0)})$ is an optimal solution of the problem (5), (6), (27).

The theorem is valid.

**Theorem 4.** *Let the conditions a)–d) be fulfilled. Then*

1) *If* $\partial_{q_k} H_k(\xi; u_m, \lambda_k; q_k) > 0$, $k \in \{1, 2\}$, *then* $q^{(0)} = (\nu_{11}(x), \nu_{21}(x))$ *is an optimal control;*
2) *If* $\partial_{q_k} H_k(\xi; u_m, \lambda_k; q_k) < 0$, $k \in \{1, 2\}$, *then* $q^{(0)} = (\nu_{12}(x), \nu_{22}(x))$ *is an optimal control;*
3) *If* $\partial_{q_1} H_1(\xi; u_m, \lambda_1; q_1) > 0$, $\partial_{q_2} H_2(\xi; u_m, \lambda_2; q_2) < 0$, *then* $q^{(0)} = (\nu_{11}(x), \nu_{22}(x))$ *optimal control;*
4) *If* $\partial_{q_1} H_1(\xi; u_m, \lambda_1; q_1) < 0$, $\partial_{q_2} H_2(\xi; u_m, \lambda_2; q_2) > 0$, *then* $q^{(0)} = (\nu_{12}(x), \nu_{21}(x))$ *is an optimal control.*

*Proof.* Consider case 1). Let $\triangle q = (\triangle q_1, \triangle q_2)$ be an arbitrary increase of management $q = (q_1, q_2)$. By $\triangle u_m = \triangle_{q_1} u_m + \triangle_{q_2} u_m$ we denote the increase of the function $u_m(x, q_1(x), q_2(x))$. Then $\triangle_{q_k} u_m$ in the domain $D$ be the solutions of the corresponding boundary value problems

$$\Big[\sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^n a_i(x)\partial_{x_i} + a_0(x) - \lambda\Big]\triangle_{q_k} u_m = \delta_{k1}\triangle_{q_1} f_m(x, q_1(x)),$$

$$\lim_{x \longrightarrow z \in \partial D} \Big[ \sum_{i=1}^{n} b_i(x) \partial_{x_i} \triangle_{q_k} u_m + b_0(x) \triangle_{q_k} u_m \Big] = \delta_{k_2} \triangle_{q_2} \varphi_m(x, q_2(x)), \tag{28}$$

where $\delta_{i,j}$ is the Kronecker symbol, $k \in \{1, 2\}$.

According to Theorem 2.20 from [8, p. 233] there is the Green's function of the problem (28) and increases $\triangle_{q_k} u_m$ are represented by the formulas

$$\triangle_{q_1} u_m = \int_D G_m^{(1)}(x, \xi) \triangle_{q_1} f_m(\xi; q_1(\xi)) d\xi, \triangle_{q_2} u_m = \int_{\partial D} G_m^{(2)}(x, \xi) \triangle_{q_2} \varphi_m(\xi; q_2(\xi)) d_\xi S. \tag{29}$$

Let us consider the increase of functional

$$\triangle I(q_1, q_2) = \triangle_{q_1} I(q_1, q_2) + \triangle_{q_2} I(q_1, q_2) \tag{30}$$

We use Taylor's formula, then

$$\triangle_{q_k} I = \int_D \Big[ \frac{\partial F_1}{\partial u_m} \triangle_{q_k} u_m + O(|\triangle_{q_k} u_m|^2) + \delta_{k_1} \Big( \frac{\partial F_1}{\partial q_1} \triangle q_1 + O(|\triangle q_1|^2) \Big) \Big] dx +$$
$$+ \int_{\partial D} \Big[ \frac{\partial F_2}{\partial u_m} \triangle_{q_k} u_m + O(|\triangle_{q_k} u_m|^2) + \delta_{k_2} \Big( \frac{\partial F_2}{\partial q_2} \triangle q_2 + O(|\triangle q_2|^2) \Big) \Big] d_x S. \tag{31}$$

Substituting (29), (31) in (30) and, while changing the order of integration, we find

$$\triangle I(q_1, q_2) = \int_D [\partial_{q_1} H_1(\xi; u_m; \lambda_1; q_1) \triangle q_1 + O(|\triangle q_1|^2)] dx +$$
$$+ \int_{\partial D} [\partial_{q_2} H_2(\xi; u_m; \lambda_2; q_2) \triangle q_2 + O(|\triangle q_2|^2)] d_x S.$$

If $q_k = \nu_{k1}(x)$ and $\partial_{q_k} H_k > 0$, then for sufficiently small $\triangle q_k$ we have $\triangle I(q_1, q_2) > 0$, $k \in \{1, 2\}$.

Let $q^{(0)}$ is optimal management, that is $\triangle I > 0$. Let us check the fulfillment of condition 1) of Theorem 4. If $\partial_{q_1} H_1$, $\partial_{q_2} H_2$ are sign variables, that is $\partial_{q_1} H_1 > 0$ on $D^+$, $\partial_{q_2} H_2 > 0$ on $\Gamma$, $\Gamma \subset \partial D$ and $\partial_{q_1} H_1 < 0$ on $D \setminus D^+$, $\partial_{q_2} H_2 < 0$ on $\partial D \setminus \Gamma$, then using the mean value theorem, we have

$$\triangle I(q_1, q_2) = \partial_{q_1} H_1(x^+; u_m^+, \lambda_1^+, q_1^+) \int_{D^+} \triangle q_1 dx -$$
$$- |\partial_{q_1} H_1(x^-; u_m^-, \lambda_m^-, q_1^-)| \int_{D \setminus D^+} \triangle q_1 dx + \partial_{q_2} H_2(x^+; u_m^+, \lambda_2^+, q_2^+) \int_{\Gamma} \triangle q_2 dx -$$
$$- |\partial_{q_2} H_2(x^-; u_m^-, \lambda_m^-, q_2^-)| \int_{\partial D \setminus \Gamma} \triangle q_2 d_x S + \int_D O(|\triangle q_1|^2) dx + \int_{\partial D} O(|\triangle q_2|^2) dx.$$

With $\triangle q_k$ small enough sign $\triangle I$ is determined by the first four terms of the sum. The difference of the first two terms changes sign $\triangle I$ depending on the values $\operatorname{mes} D^+$, $\operatorname{mes} \Gamma$, $\triangle q_k$. At rather small values $\operatorname{mes} D^+$, $\operatorname{mes} \Gamma$ and $\triangle q_k > 0$ we have $\triangle I < 0$ and vice versa $\triangle I > 0$, if the values are small $\operatorname{mes}(D \setminus D^+)$, $\operatorname{mes}(\partial D \setminus \Gamma)$ and $\triangle q_k > 0$. So, functional $I(q_1, q_2)$ does not reach the minimum.

Finding the optimal control $q^{(0)}$ in other cases, which depend on the sign of the values $\partial_{q_k} H_k$ is proved similarly. $\square$

Let the conditions of Theorem 4 not be fulfilled. Then the following theorem is correct.

**Theorem 5.** *Let the conditions a)–d) be fulfilled. In order for the control $q^{(0)} = (q_1^{(0)}, q_2^{(0)})$ to be optimal, it is necessary and sufficient that the conditions are fulfilled:*
*1) functions $H_k(\xi; u_m, \lambda_k, q_k)$ in arguments $q_k$ have in the point $q_k^{(0)}$ minimum values, $k \in \{1, 2\}$;*
*2) for an arbitrary vector $(e_k^{(1)}, e_k^{(2)}) \neq 0$ the inequality holds*

$$\partial^2_{u_m} F_k(x; u_m; q_k)(e_k^{(1)})^2 + 2\partial_{q_k}\partial_{u_m} F_k(x; u_m; q_k)e_k^{(1)} e_k^{(2)} + \partial^2_{q_k} F_k(x; u_m; q_k)(e_k^{(2)})^2 > 0.$$

The proof of the Theorem 5 is conducted using the methodology of work [4]. Passing to the limit in the problem (5), (6), (27) as $m(j) \longrightarrow \infty$ we obtain the optimal solution of the problem (1)–(3).

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*Received 31.01.2023*