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THE HADAMARD COMPOSITIONS OF DIRICHLET SERIES ABSOLUTELY CONVERGING IN HALF-PLANE

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For Dirichlet series with different finite abscissas of absolute convergence in terms of generalized orders the growth of the Hadamard composition of their derivatives is investigated. A relation between the behavior of the maximal terms of Hadamard composition of derivatives and of the derivative of Hadamard composition is established.

1. Introduction. For power series

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \text{ and } g(z)=\sum_{k=0}^{\infty} g_k z^k$$

with the convergence radii $R[f]$ and $R[g]$ the series

$$(f*g)(z)=\sum_{k=0}^{\infty} f_k g_k z^k$$

is called the *Hadamard composition* of f and g ([1,2]). Properties of this composition obtained by J. Hadamard find the applications ([2,3]) in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in paper [4].

For entire functions f and g, the connection between the growth of the maximal term of the Hadamard composition $f^{(n)}*g^{(n)}$ of derivatives and the maximal term of the derivative $(f*g)^{(n)}$ of the Hadamard composition $f*g$ are studied by M. K. Sen ([5,6]).

Since Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of power series, it is natural to pose the question on similar results for the Hadamard composition of such series.

So, let $\Lambda=(\lambda_k)$ be an increasing to $+\infty$ sequence of nonnegative numbers ($\lambda_0=0$), and $S(\Lambda,A)$ be the class of Dirichlet series

$$F(s)=\sum_{k=0}^{\infty} f_k \exp\{s\lambda_k\}, \quad s=\sigma+it \tag{1}$$

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with the exponents Λ and the abscissa of absolute convergence $\sigma_a[F]=A$. If $F\in S(\Lambda,A_1)$ and $G(s)=\sum_{k=0}^{\infty}g_k\exp\{s\lambda_k\}\in S(\Lambda,A_2)$ the Dirichlet series

$$(F*G)(s)=\sum_{k=0}^{\infty}f_kg_k\exp\{s\lambda_k\} \tag{2}$$

is called ([7]) the *Hadamard composition* of F and G.

For a Dirichlet series (1) with

$$\sigma_a[F]=A[F]:=\varliminf_{k\to+\infty}\frac{-\ln|f_k|}{\lambda_k}=A>-\infty$$

and $\sigma<A$ we put

$$M(\sigma,F)=\sup\{|F(\sigma+it)|:\ t\in\mathbb{R}\},$$

and let

$$\mu(\sigma,F)=\max\{|f_k|\exp\{\sigma\lambda_k\}:\ k\ge0\}$$

be the maximal term,

$$\nu(\sigma,F)=\max\{k:\ |f_k|\exp\{\sigma\lambda_k\}=\mu(\sigma,F)\}$$

be the central index and $\Lambda(\sigma,F)=\lambda_{\nu(\sigma,F)}$. The following statement is proved in [7].

Proposition 1. *Let $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ and $m>n$. If $\sigma_a[F]=\sigma_a[G]=+\infty$ and $\ln k=o(\lambda_k\ln\lambda_k)$ as $k\to\infty$ then*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\varrho_R[f*G]$$

and (if $\varrho_R[f*G]<+\infty$)

$$\varliminf_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\lambda_R[f*G],$$

where $\varrho_R[f]$ and $\lambda_R[f]$ are respectively the R-order and the lower R-order of entire Dirichlet series (1). If $\sigma_a[F]=\sigma_a[G]=0$ and $\ln k=o(\lambda_k/\ln\lambda_k)$ as $k\to\infty$ then

$$\varlimsup_{\sigma\uparrow0}|\sigma|\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\varrho^{(0)}[f*G]$$

and

$$\varliminf_{\sigma\uparrow0}|\sigma|\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\lambda^{(0)}[f*G],$$

where $\varrho^{(0)}[f]$ and $\lambda^{(0)}[f]$ are respectively the order and the lower order of Dirichlet series (1) with $\sigma_a[F]=0$.

Naturally, the question of similar properties of the Hadamard composition of Dirichlet series arises when $\sigma_a[F]\ne\sigma_a[G]$. Here we restrict ourselves to the case when $-\infty<\sigma_a[F],\ \sigma_a[G]<+\infty$ and $\sigma_a[F]\ne\sigma_a[G]$.

2. Convergence and growth. In [7] it is proved that if $\sigma_a[F]>-\infty$ and $\sigma_a[G]>-\infty$ then

$$\sigma_a[F*G]\ge\sigma_a[F]+\sigma_a[G].$$

If $\sigma_a[F] = -\infty$ and $\sigma_a[G] = +\infty$ then in view of [7] $\sigma_a[F * G]$ may be equal to any $c \in [-\infty, +\infty]$. We remark also [7] that the inverse inequality

$$\sigma_a[F * G] \le \sigma_a[F] + \sigma_a[G]$$

in general does not hold. In what follows, we assume that

$$\sigma_a[F * G] = \sigma_a[F] + \sigma_a[G]. \tag{3}$$

Equality (3) holds, if for example $\ln k = o(\lambda_k)$ as $k \to \infty$ and there exists either

$$\lim_{k\to\infty} \frac{-\ln|f_k|}{\lambda_k} = A[F] \quad \text{or} \quad \lim_{k\to\infty} \frac{-\ln|g_k|}{\lambda_k} = A[G],$$

where

$$A[G] := \varliminf_{k\to+\infty} \frac{-\ln|g_k|}{\lambda_k}.$$

Indeed, if $\ln k = o(\lambda_k)$ as $k \to \infty$ then (see [13], [14]) $\sigma_a[F] = A[F]$ and, therefore,

$$\sigma_a[F * G] = A[F * G] = \varliminf_{k\to+\infty} \left(\frac{1}{\lambda_k} \ln \frac{1}{|f_k|} + \frac{1}{\lambda_k} \ln \frac{1}{|g_k|} \right) \le$$
$$\le \varliminf_{k\to+\infty} \frac{1}{\lambda_k} \ln \frac{1}{|f_k|} + \lim_{k\to\infty} \frac{1}{\lambda_k} \ln \frac{1}{|g_k|} = A[F] + A[F] = \sigma_a[F] + \sigma_a[G].$$

We remark also that if for all $k \ge k_0$

$$|f_k| \exp\{A[F]\lambda_k\} \ge 1, \quad |g_k| \exp\{A[G]\lambda_k\} \ge 1 \tag{4}$$

then $A[F * G] \le A[F] + A[G]$ and, thus, (3) holds.

The following statement is proved in [7].

Proposition 2. *The equalities* $\sigma_a[F * G] = \sigma_a[(F * G)^{(n)}] = \sigma_a[F^{(n)} * G^{(n)}]$ *hold for every* $n \in \mathbb{N}$.

Unlike the entire Dirichlet series, for Dirichlet series (1) with $\sigma_a[F] \in (-\infty, +\infty)$ the maximal term can be bounded, and in order that $\mu(\sigma, F) \uparrow +\infty$ as $\sigma \uparrow A[F]$, it is necessary and sufficient that (see also [8–10])

$$\varlimsup_{k\to\infty} (\ln|f_k| + A[F]\lambda_k) = +\infty. \tag{5}$$

Indeed, Proposition 2 [11] (see also [12]) implies, that $A[F] = \sup\{\sigma\colon \mu(\sigma, F) < +\infty\}$, thus $\mu(\sigma, F) < +\infty$ for all $\sigma < A[F]$. By the definition of $\mu(\sigma, F)$ for fixed $\sigma < A[F]$ we have $\mu(\sigma, F) \ge |f_k|e^{\lambda_k \sigma}$ $(k \ge 0)$, hence

$$\lim_{\sigma\uparrow A[F]} \mu(\sigma, F) \ge |f_k|e^{\lambda_k A[F]}.$$

Therefore, from (5) we obtain $\lim\limits_{\sigma\uparrow A[F]} \mu(\sigma, F) = +\infty$.

Suppose now that

$$\lim_{\sigma\uparrow A[F]} \mu(\sigma, F) = +\infty \quad \text{and} \quad \varlimsup_{k\to\infty} (\ln|f_k| + A[F]\lambda_k) < +\infty.$$

Then $\ln|f_k| + A[F]\lambda_k \le K < +\infty$ $(k \ge 0)$ and

$$\ln\mu(\sigma, F) \le \sup\{\ln|f_k| + A[F]\lambda_k :\ k \ge 0\} \le K \text{ for every } \sigma < A[F].$$

Therefore $\lim\limits_{\sigma\uparrow A[F]} \mu(\sigma, F) \le e^K < +\infty$, which is a contradiction. Thus, $\varlimsup\limits_{k\to\infty} (\ln|f_k| + A[F]\lambda_k) = +\infty$.

We will assume everywhere below that

$$A[F] = \sigma_a[F] \quad \text{and} \quad \lim_{\sigma\uparrow A[F]} \mu(\sigma, F) = +\infty,$$

thus relation (5) holds and therefore

$$\varlimsup_{k\to\infty} \frac{\lambda_k}{\ln|f_k| + A[F]\lambda_k} = +\infty. \tag{6}$$

Indeed,

$$\varlimsup_{k\to\infty} \frac{\ln|f_k| + A[F]\lambda_k}{\lambda_k} = -\varliminf_{k\to\infty} \frac{-\ln|f_k|}{\lambda_k} + A[F] = 0,$$

thus

$$\lim_{k\to\infty} \frac{\lambda_k}{\ln|f_k| + A[F]\lambda_k} = \infty.$$

But it follows from relation (5) that there exists a sequence $k_j \to +\infty$ such that $\ln|f_{k_j}| + A[F]\lambda_{k_j} > 0$ $(j \ge 1)$. This implies relation (6).

By L we denote the class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L_{\rm si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function.

If $\sigma_a[F] = A[F] = A \in (-\infty, +\infty)$, $\alpha \in L$ and $\beta \in L$ then the quantities

$$\varrho^{(A)}_{\alpha,\beta}[F] := \varlimsup_{\sigma\uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A-\sigma))}, \quad \lambda^{(A)}_{\alpha,\beta}[F] := \varliminf_{\sigma\uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A-\sigma))}$$

are called ([15]–[16]) the generalized (α, β)-order and the generalized lower (α, β)-order of F respectively.

If in the definitions of $\varrho^{(A)}_{\alpha,\beta}[F]$ and $\lambda^{(A)}_{\alpha,\beta}[F]$ we substitute $\ln\mu(\sigma, F)$ instead of $\ln M(\sigma, F)$ then we obtain quantities, which we denote by $\varrho^{(A)}_{\alpha,\beta}[\ln\mu, F]$ and $\lambda^{(A)}_{\alpha,\beta}[\ln\mu, F]$, respectively. Substituting $\Lambda(\sigma, F)$ instead of $\ln M(\sigma, F)$ by analogy we define $\varrho^{(A)}_{\alpha,\beta}[\Lambda, F]$ and $\lambda^{(A)}_{\alpha,\beta}[\Lambda, F]$.

In papers [15, 16] we find the following lemma.

Lemma 1. *Let $\alpha \in L_{\rm si}$, $\beta \in L_{\rm si}$ and*

$$\frac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right) = (1 + o(1))\alpha(x) \tag{7}$$

as $x_0(c) \le x \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $A[F] \in (-\infty, +\infty)$ and $\ln n(x) = o\left(x/\beta^{-1}(c\alpha(x))\right)$ as $x_0(c) \le x \to +\infty$ for each $c \in (0, +\infty)$, where $n(x) = \sum_{\lambda_k \le x} 1$. Then

$$\varrho^{(A)}_{\alpha,\beta}[F] = \varrho^{(A)}_{\alpha,\beta}[\ln\mu, F] = \varlimsup_{k\to\infty} \frac{\alpha(\lambda_k)}{\beta\left(\dfrac{\lambda_k}{\ln|f_k| + A[F]\lambda_k}\right)}. \tag{8}$$

If, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$ *and*

$$\varkappa_k[F]:=\frac{\ln|f_k|-\ln|f_{k+1}|}{\lambda_{k+1}-\lambda_k}\nearrow A[F]$$

as $k_0\le k\to\infty$ *then*

$$\lambda^{(A)}_{\alpha,\beta}[F]=\lambda^{(A)}_{\alpha,\beta}[\ln\mu,F]=\varliminf_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\left(\dfrac{\lambda_k}{\ln|f_k|+A[F]\lambda_k}\right)}. \tag{9}$$

We need also the following lemmas.

Lemma 2. *If* $\alpha(e^x)\in L_{\rm si}$, $\beta\in L_{\rm si}$ *and* $\sigma_a[F]=A[F]=A\in(-\infty,+\infty)$, *then*

$$\varrho^{(A)}_{\alpha,\beta}[\ln\mu,F]\le\varrho^{(A)}_{\alpha,\beta}[\Lambda,F]\le\varrho^{(A)}_{\alpha,\beta}[\ln\mu,F]+\Delta_{\alpha,\beta},\quad \Delta_{\alpha,\beta}=\varlimsup_{x\to+\infty}\frac{\alpha(x)}{\beta(x)}, \tag{10}$$

and

$$\lambda^{(A)}_{\alpha,\beta}[\ln\mu,F]\le\lambda^{(A)}_{\alpha,\beta}[\Lambda,F]\le\lambda^{(A)}_{\alpha,\beta}[\ln\mu,F]+\Delta_{\alpha,\beta}. \tag{11}$$

Proof. Since ([13, p.182], [14, p.17]) for $\sigma_0\le\sigma<A$

$$\ln\mu(\sigma,F)=\ln\mu(\sigma_0,F)+\int\limits_{\sigma_0}^{\sigma}\Lambda(x,F)dx,$$

we have

$$\ln\mu(\sigma,F)=\ln\mu(\sigma_0,F)+(\sigma-\sigma_0)\Lambda(\sigma,F]\le(A-\sigma_0)\Lambda(\sigma,F]+\ln\mu(\sigma_0,F), \tag{12}$$

whence the left-hand side inequalities of (10) and (11) follow.

On the other hand, since $\ln\mu(\sigma,F)\uparrow+\infty$ as $\sigma\uparrow A$, for any $q>1$ we have

$$\ln\mu\left(\sigma+\frac{A-\sigma}{q},F\right)=\int\limits_{\sigma_0}^{\sigma+(A-\sigma)/q}\Lambda(x,F)dx+\ln\mu(\sigma_0,F)\ge$$
$$\ge\int\limits_{\sigma}^{\sigma+(A-\sigma)/q}\Lambda(x,F)dx\ge\frac{A-\sigma}{q}\Lambda(\sigma,F),\quad \sigma\ge\sigma_0^*,$$

i. e.

$$\Lambda(\sigma,F)\le\frac{q}{A-\sigma}\ln\mu\left(\sigma+\frac{A-\sigma}{q},F\right),\quad \sigma\ge\sigma_0^*. \tag{13}$$

Since $\alpha(e^x)\in L_{\rm si}$, for $q=2$ we obtain

$$\alpha(\Lambda(\sigma,F))\le\alpha\left(\exp\left\{\ln\ln\mu\left(\sigma+\frac{A-\sigma}{2},F\right)+\ln\frac{2}{A-\sigma}\right\}\right)\le$$
$$\le\alpha\left(\exp\left\{2\max\left\{\ln\ln\mu\left(\sigma+\frac{A-\sigma}{2},F\right),\ln\frac{2}{A-\sigma}\right\}\right\}\right)=$$

$$= (1+o(1))\alpha\left(\exp\left\{\max\left\{\ln\ln\mu\left(\sigma+\frac{A-\sigma}{2}, F\right), \ln\frac{2}{A-\sigma}\right\}\right\}\right) =$$
$$= (1+o(1))\max\left\{\alpha\left(\ln\mu\left(\sigma+\frac{A-\sigma}{2}, F\right)\right), \alpha\left(\frac{2}{A-\sigma}\right)\right\} \leq$$
$$\leq (1+o(1))\left(\alpha\left(\ln\mu\left(\sigma+\frac{A-\sigma}{2}, F\right)\right)+\alpha\left(\frac{2}{A-\sigma}\right)\right), \quad \sigma\uparrow A.$$

Thus,

$$(1+o(1))\frac{\alpha(\Lambda(\sigma,F))}{\beta(1/(A-\sigma))} \leq \frac{\alpha(\ln\mu(\sigma+(A-\sigma)/2, F))}{\beta(1/(A-\sigma-(A-\sigma)/2))}\frac{\beta(2/(A-\sigma))}{\beta(1/(A-\sigma))}+\frac{\alpha(2/(A-\sigma))}{\beta(1/(A-\sigma))},$$

whence in view of the condition $\alpha\in L_{\rm si}$ and $\beta\in L_{\rm si}$ the right-hand side inequalities of (10) and (11) follow. □

Lemma 3. *If $\alpha\in L_{\rm si}$, $\beta\in L_{\rm si}$, $\alpha(\ln x)=o(\beta(x))$ as $x\to+\infty$ and $\sigma_a[F]=A[F]=A\in(-\infty,+\infty)$ then $\varrho^{(A)}_{\alpha,\beta}[F']=\varrho^{(A)}_{\alpha,\beta}[F]$ and $\lambda^{(A)}_{\alpha,\beta}[F']==\lambda^{(A)}_{\alpha,\beta}[F]$.*

Proof. Since [7] for $\sigma<A$ and $\delta(\sigma)\in(0,\, A-\sigma)$

$$M(\sigma,F')\leq\frac{M(\sigma+\delta(\sigma),F)}{\delta(\sigma)} \tag{14}$$

and for $\sigma_0<\sigma$

$$M(\sigma,F)-M(\sigma_0,F)\leq(\sigma-\sigma_0)M(\sigma,F'). \tag{15}$$

From (15) it follows that $(1+o(1))M(\sigma,F)\leq(A-\sigma_0)M(\sigma,F')$ as $\sigma\uparrow A$, whence

$$\varrho^{(A)}_{\alpha,\beta}[F]\leq\varrho^{(A)}_{\alpha,\beta}[F'],\quad \lambda^{(A)}_{\alpha,\beta}[F]\leq\lambda^{(A)}_{\alpha,\beta}[F'].$$

On the other hand, since $\alpha\in L_{\rm si}$, choosing $\delta(\sigma)=(A-\sigma)/2$, from (14) as in the proof of Lemma 2 we have

$$\alpha(\ln M(\sigma,F'))\leq\alpha(\ln M(\sigma+(A-\sigma)/2,F))+\ln(2/(A-\sigma)))\leq$$
$$\leq\alpha(2\max\{\ln M(\sigma+(A-\sigma)/2,F)),\ln(2/(A-\sigma))\})\leq$$
$$\leq(1+o(1))(\alpha(\ln M(\sigma+(A-\sigma)/2,F)))+\alpha(\ln(2/(A-\sigma))),\quad\sigma\uparrow A,$$

i. e.

$$(1+o(1))\frac{\alpha(\ln M(\sigma,F'))}{\beta(1/(A-\sigma))}\leq\frac{\alpha(\ln M(\sigma+(A-\sigma)/2,F))}{\beta(1/(A-\sigma-(A-\sigma)/2))}\frac{\beta(2/(A-\sigma)))}{\beta(1/(A-\sigma))}+\frac{\alpha(\ln(2/(A-\sigma))}{\beta(1/(A-\sigma))}$$

as $\sigma\uparrow A$, whence in view of the conditions of the lemma we obtain the inequalities

$$\varrho^{(A)}_{\alpha,\beta}[F']\leq\varrho^{(A)}_{\alpha,\beta}[F],\quad \lambda^{(A)}_{\alpha,\beta}[F']\leq\lambda^{(A)}_{\alpha,\beta}[F].$$

□

Using Lemma 1 we prove the following statement.

Proposition 3. *Let the functions α, β and the sequence (λ_k) satisfy the conditions of Lemma 1. Suppose that $-\infty < A[F],\, A[G] < +\infty$ and inequalities (4) hold. Then*

$$\varrho_{\alpha,\beta}^{(A[F*G])}[F*G] = \max\{\varrho_{\alpha,\beta}^{(A[F])}[F],\, \varrho_{\alpha,\beta}^{(A[G])}[G]\} \tag{16}$$

and if, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\varkappa_k[F] \nearrow A[F]$ and $\varkappa_k[G] \nearrow A[G]$ as $k_0 \le k \to \infty$ then

$$\max\{\lambda_{\alpha,\beta}^{(A[F])}[F],\, \lambda_{\alpha,\beta}^{(A[G])}[G]\} \le \lambda_{\alpha,\beta}^{(A[F*G])}[F*G] \le$$
$$\le \min\{\max\{\lambda_{\alpha,\beta}^{(A[F])}[F],\, \varrho_{\alpha,\beta}^{(A[G])}[G]\},\ \max\{\varrho_{\alpha,\beta}^{(A[F])}[F],\, \lambda_{\alpha,\beta}^{(A[G])}[G]\}\}. \tag{17}$$

Proof. Since $|g_k|\exp\{A[G]\lambda_k\} \ge 1$, we have for $\sigma < \sigma < A[F*G]$

$$\ln\mu(\sigma,\, F*G) = \max\{\ln|f_k g_k| + \sigma\lambda_k :\, k \ge 0\} \ge$$
$$\ge \max\{\ln|f_k| + (\sigma - A[G])\lambda_k :\, k \ge 0\} = \ln\mu(\sigma - A[G],\, F)$$

and, thus,

$$\varrho_{\alpha,\beta}^{(A[F*G])}[\ln\mu, F*G] = \varlimsup_{\sigma\uparrow A[F*G]} \frac{\alpha(\ln\mu(\sigma,\, F*G)}{\beta(1/(A[F*G]-\sigma))} \ge$$
$$\ge \varlimsup_{\sigma\uparrow A[F]+A[G]} \frac{\alpha(\ln\mu(\sigma - A[G],\, F)}{\beta(1/(A[F]-(\sigma - A[G])))} = \varlimsup_{\sigma_1\uparrow A[F]} \frac{\alpha(\ln\mu(\sigma_1,\, F)}{\beta(1/(A[F]-\sigma_1))} = \varrho_{\alpha,\beta}^{(A[F])}[\ln\mu, F].$$

Similarly,

$$\varrho_{\alpha,\beta}^{(A[F*G])}[\ln\mu, F*G] \ge \varrho_{\alpha,\beta}^{(A[G])}[\ln\mu, G], \quad \lambda_{\alpha,\beta}^{(A[F*G])}[\ln\mu, F*G] \ge \lambda_{\alpha,\beta}^{(A[F])}[\ln\mu, F]$$

and

$$\lambda_{\alpha,\beta}^{(A[F*G])}[\ln\mu, F*G] \ge \lambda_{\alpha,\beta}^{(A[G])}[\ln\mu, G].$$

Hence by Lemma 1 we get

$$\varrho_{\alpha,\beta}^{(A[F*G])}[F*G] \ge \max\{\varrho_{\alpha,\beta}^{(A[F])}[F],\, \varrho_{\alpha,\beta}^{(A[G])}[G]\},\ \lambda_{\alpha,\beta}^{(A[F*G])}[F*G] \ge \max\{\lambda_{\alpha,\beta}^{(A[F])}[F],\, \lambda_{\alpha,\beta}^{(A[G])}[G]\}.$$

On the other hand, we can assume that $\varrho_{\alpha,\beta}^{(A[F])}[F] < +\infty$ and $\varrho_{\alpha,\beta}^{(A[G])}[g] < +\infty$. Then in view of Lemma 1

$$\ln|f_k| + A[F]\lambda_k \le \frac{\lambda_k}{\beta^{-1}(\alpha(\lambda_k)/\varrho_1)},\ \ \ln|g_k| + A[G]\lambda_k \le \frac{\lambda_k}{\beta^{-1}(\alpha(\lambda_k)/\varrho_2)}$$

for every $\varrho_1 > \varrho_{\alpha,\beta}^{(A[F])}[F]$, $\varrho_2 > \varrho_{\alpha,\beta}^{(A[G])}[g]$ and all $k \ge k_0$. Hence,

$$\ln|f_k g_k| + A[F*G]\lambda_k = \ln|f_k| + A[F]\lambda_k + \ln|g_k| + A[G]\lambda_k \le \frac{2\lambda_k}{\beta^{-1}(\alpha(\lambda_k)/\max\{\varrho_1, \varrho_2\})}$$

and, thus, by Lemma 1 in view of condition $\beta \in L_{\mathrm{si}}$ we obtain

$$\varrho_{\alpha,\beta}^{(A[F*G])}[F*G] \le \varlimsup_{k\to\infty} \frac{\alpha(\lambda_k)}{\beta((1/2)\beta^{-1}(\alpha(\lambda_k)/\max\{\varrho_1,\ \varrho_2\})} = \max\{\varrho_1,\ \varrho_2\},$$

i. e. by the arbitrariness of ϱ_1 and ϱ_2 we get

$$\varrho_{\alpha,\beta}^{(A[F*G])}[F*G] \le \max\{\varrho_{\alpha,\beta}^{(A[F])}[F],\ \varrho_{\alpha,\beta}^{(A[G])}[G]\}.$$

Equality (16) is proved.

By Lemma 1 also we have

$$\ln|f_{k_j} + A[F]\lambda_{k_j} \le \frac{\lambda_{k_j}}{\beta^{-1}(\alpha(\lambda_{k_j})/\lambda_1)}$$

for every $\lambda_1 > \lambda_{\alpha,\beta}^{A[F]}[F]$ and some sequence $(k_j) \uparrow +\infty$. Therefore, as above we have

$$\lambda_{\alpha,\beta}^{(A[F*G])}[F*G] \le \varliminf_{j\to\infty} \frac{\alpha(\lambda_{k_j})}{\beta(\lambda_{k_j}/(\ln|f_{k_j}| + A[F]\lambda_{k_j} + \ln|g_{k_j} + A[G]\lambda_{k_j}))} \le$$
$$\le \varliminf_{j\to\infty} \frac{\alpha(\lambda_{k_j})}{\beta((1/2)\beta^{-1}(\alpha(\lambda_{k_j})/\max\{\lambda_1,\ \varrho_2\}))} = \max\{\lambda_1,\ \varrho_2\},$$

whence in view of the arbitrariness of λ_1 and ϱ_2 we get

$$\lambda_{\alpha,\beta}^{(A[F*G])}[F*G] \le \max\{\lambda_{\alpha,\beta}^{(A[F])}[F],\ \varrho_{\alpha,\beta}^{(A[G])}[G]\}.$$

Similarly,

$$\lambda_{\alpha,\beta}^{(A[F*G])}[F*G] \le \max\{\varrho_{\alpha,\beta}^{(A[F])}[F],\ \lambda_{\alpha,\beta}^{(A[G])}[G]\},$$

whence (17) follows. □

3. Behaviour of the maximal terms of Hadamard compositions. The following result is main in the paper.

Theorem 1. *Let $\alpha(e^x) \in L_{\rm si}$, $\beta \in L_{\rm si}$, conditions (7) hold, and $\ln k = o(\lambda_k/\beta^{-1}(c\alpha(\lambda_k))$ as $k \to \infty$ for each $c \in (0, +\infty)$. Suppose that $-\infty < A[F], A[G] < +\infty$ and inequalities (4) hold. Then for $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$ and $m > n$*

$$\varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) = \max\{\varrho_{\alpha\beta}^{(A[F])}[F],\ \varrho_{\alpha\beta}^{(A[G])}[G]\} \tag{18}$$

and if, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\varkappa_k[F] \nearrow A[F]$ and $\varkappa_k[G] \nearrow A[G]$ as $k_0 \le k \to \infty$ then

$$\max\{\lambda_{\alpha\beta}^{(A[F])}[F],\ \lambda_{\alpha\beta}^{(A[G])}[G]\} \le \varliminf_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) \le$$
$$\le \min\{\max\{\lambda_{\alpha\beta}^{(A[F])}[F],\ \varrho_{\alpha\beta}^{(A[G])}[G]\},\ \max\{\lambda_{\alpha\beta}^{(A[G])}[G],\ \varrho_{\alpha\beta}^{(A[F])}[F]\}\}. \tag{19}$$

Proof. The following inequalities from [7] play an important role in the proof of Theorem 1

$$\Lambda^{m-n}(\sigma,(F*G)^{(n)}) \le \frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \le \Lambda^{m-n}(\sigma,(F*G)^{(m)}) \tag{20}$$

for $\sigma < A[F*G]$. Since $\alpha(e^x) \in L_{\rm si}$, we have

$$\alpha(\Lambda^{m-n}(\sigma,(F*G)^{(n)})) = \alpha(\exp\{(m-n)\ln\Lambda(\sigma,(F*G)^{(n)})\}) =$$

$$= (1+o(1))\alpha(\exp\{\ln\Lambda(\sigma,(F*G)^{(n)})\}) = (1+o(1))\alpha(\Lambda(\sigma,(F*G)^{(n)})), \quad \sigma\to+\infty,$$

and, therefore, (20) implies

$$\alpha(\Lambda(\sigma,(F*G)^{(n)})) \le (1+o(1))\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) \le \alpha(\Lambda(\sigma,(F*G)^{(m)}))$$

as $\sigma\to+\infty$. Hence it follows that

$$\varrho_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \le \varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) \le \le \varrho_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(m)}] \tag{21}$$

and

$$\lambda_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \le \varliminf_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) \le \le \lambda_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(m)}]. \tag{22}$$

The condition $\frac{x}{\beta^{-1}(c\alpha(x))}\uparrow+\infty$ implies $\alpha(x)=o(\beta(x))$ as $x\to+\infty$, that is $\Delta_{\alpha\beta}=0$. By Lemma 2

$$\varrho_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(n)}] = \varrho_{\alpha\beta}^{(A[F*G])}[\ln\mu,(F*G)^{(n)}]$$

and

$$\lambda_{\alpha\beta}^{(A[F*G])}[\Lambda,(F*G)^{(n)}] = \lambda_{\alpha\beta}^{(A[F*G])}[\ln\mu,(F*G)^{(n)}]$$

for each $n\ge 0$. By Lemma 1

$$\varrho_{\alpha\beta}^{(A[F*G])}[\ln\mu,(F*G)^{(n)}] = \varrho_{\alpha\beta}^{(A[F*G])}[(F*G)^{(n)}]$$

and

$$\lambda_{\alpha\beta}^{(A[F*G])}[\ln\mu,(F*G)^{(n)}] = \lambda_{\alpha\beta}^{(A[F*G])}[(F*G)^{(n)}].$$

Finally, by Lemma 3

$$\varrho_{\alpha\beta}^{(A[F*G])}[(F*G)^{(n)}] = \varrho_{\alpha\beta}^{(A[F*G])}[F*G], \quad \lambda_{\alpha\beta}^{(A[F*G])}[(F*G)^{(n)}] = \lambda_{\alpha\beta}^{(A[F*G])}[F*G]$$

for each $n\ge 1$. Therefore, from (21) and (22) we get

$$\varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) = \varrho_{\alpha\beta}^{(A[F*G])}[F*G] \tag{23}$$

and

$$\varliminf_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right) = \lambda_{\alpha\beta}^{(A[F*G])}[F*G]. \tag{24}$$

Using Proposition 3 we obtain (18) and (19). □

We remark that the conditions $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$, $\varkappa_k[F]\nearrow A[F]$ and $\varkappa_k[G]\nearrow A[G]$ as $k_0\le k\to\infty$ for the proof of equalities (23) and (24) are not used.

Since $(F^{(n)}*G^{(n)})(s) = (F*G)^{(2n)}(s)$, for $m=2n$ Theorem 1 implies the following corollary.

Corollary 1. *Let the functions α, β and the sequence (λ_k) satisfy the conditions of Theorem 1. Suppose that $-\infty < A[F], A[G] < +\infty$ and inequalities (4) hold. Then for $n \in \mathbb{N}$*

$$\varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma, F^{(n)} * G^{(n)})}{\mu(\sigma,(F*G)^{(n)})}\right) = \max\{\varrho_{\alpha\beta}^{(A[F])}[F],\ \varrho_{\alpha\beta}^{(A[G])}[G]\}$$

and if, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\varkappa_k[F] \nearrow A[F]$ and $\varkappa_k[G] \nearrow A[G]$ as $k_0 \le k \to \infty$ then

$$\max\{\lambda_{\alpha\beta}^{(A[F])}[F],\ \lambda_{\alpha\beta}^{(A[G])}[G]\} \le \varliminf_{\sigma\uparrow A[F*G]} \frac{1}{\beta(\frac{1}{A[F*G]-\sigma})}\alpha\left(\frac{\mu(\sigma, F^{(n)} * G^{(n)})}{\mu(\sigma,(F*G)^{(n)})}\right) \le$$
$$\le \min\{\max\{\lambda_{\alpha\beta}^{(A[F])}[F],\ \varrho_{\alpha\beta}^{(A[G])}[G]\},\ \max\{\lambda_{\alpha\beta}^{(A[G])}[G],\ \varrho_{\alpha\beta}^{(A[F])}[F]\}\}.$$

4. Hadamard compositions of finite orders. If $\sigma_a[F] = A[F] = A \in (-\infty, +\infty)$ then the quantities

$$\varrho^{(A)}[F] := \varlimsup_{\sigma\uparrow A} \frac{\ln^+ \ln M(\sigma, F))}{-\ln(A-\sigma)}, \quad \lambda^{(A)}[F] := \varliminf_{\sigma\uparrow A} \frac{\ln^+ \ln M(\sigma, F))}{-\ln(A-\sigma)}$$

are called ([8], [17]) the *order of the growth* and the *lower order of the growth* of F, respectively.

The following lemma was proved in [18].

Lemma 4. *Let $\ln\ln n(x) = o(\ln x))$ as $x \to +\infty$ and $\sigma_a[F] = A[F] = A \in (-\infty, +\infty)$. Then*

$$\varrho^{(A)}[F] = \varrho^{(A)}[\ln\mu, F] = \frac{\alpha^*[F]}{1-\alpha^*[F]}, \quad \alpha^*[F] := \varlimsup_{k\to\infty} \frac{\ln^+(\ln|f_k| + A\lambda_k)}{\ln\lambda_k}. \tag{25}$$

If, moreover, $\ln\lambda_{k+1} \sim \ln\lambda_k$ and $\varkappa_k[F] \nearrow A$ as $k_0 \le k \to \infty$ then

$$\lambda^{(A)}[F] = \lambda^{(A)}[\ln\mu, F] = \frac{\alpha_*[F]}{1-\alpha_*[F]},\ \alpha_*[F] := \varliminf_{k\to\infty} \frac{\ln^+(\ln|f_k| + A\lambda_k)}{\ln\lambda_k}. \tag{26}$$

We need also the following lemma.

Lemma 5. *If $\sigma_a[F] = A[F] = A \in (-\infty, +\infty)$, then*

$$\varrho^{(A)}[\ln\mu, F] \le \varrho^{(A)}[\Lambda, F] \le \varrho^{(A)}[\ln\mu, F] + 1 \tag{27}$$

and

$$\lambda^{(A)}[\ln\mu, F] \le \lambda^{(A)}[\Lambda, F] \le \lambda^{(A)}[\ln\mu, F] + 1. \tag{28}$$

Proof. Inequality (12) implies the left-hand sides of (27) and (28). On the other hand, from (13) we have

$$\frac{\ln\Lambda(\sigma, F)}{\ln(1/(A-\sigma))} \le \frac{\ln(2/(A-\sigma))}{\ln(1/(A-\sigma))} + \frac{\ln\ln\mu(\sigma+(A-\sigma)/2, F)}{\ln(1/(A-\sigma-(A-\sigma)/2))}\frac{\ln(2/(A-\sigma))}{\ln(1/(A-\sigma))},$$

whence the right-hand sides of (27) and (28) follow. □

Using Lemmas 4 we prove the following statement.

Proposition 4. *Let* $-\infty < A[F], A[G] < +\infty$ *and* $\ln\ln n(x) = o(\ln x))$ *as* $x \to +\infty$. *Then*

$$\varrho^{(A[F*G])}[F*G] = \max\{\varrho^{(A[F])}[F], \varrho^{(A[G])}[G]\} \tag{29}$$

and if, moreover, $\ln\lambda_{k+1} \sim \ln\lambda_k$, $\varkappa_k[F] \nearrow A[F]$ *and* $\varkappa_k[G] \nearrow A[G]$ *as* $k_0 \le k \to \infty$ *then*

$$\max\{\lambda^{(A[F])}[F], \lambda^{(A[G])}[G]\} \le \lambda^{(A[F*G])}[F*G] \le$$

$$\le \min\{\max\{\lambda^{(A[F])}[F], \varrho^{(A[G])}[G]\},\ \max\{\varrho^{(A[F])}[F], \lambda^{(A[G])}[G]\}\}. \tag{30}$$

Proof. As above, we have $\ln\mu(\sigma,\, F*G) \ge \ln\mu(\sigma - A[G],\, F)$, whence it follows that

$$\varrho^{(A[F*G])}[\ln\mu, F*G] \ge \varrho^{(A[F])}[\ln\mu, F]$$

and, similarly, $\varrho^{(A[F*G])}[\ln\mu, F*G] \ge \varrho^{(A[G])}[\ln\mu, G]$, i. e. by Lemma 4

$$\varrho^{(A[F*G])}[F*G] \ge \max\{\varrho^{(A[F])}[F],\ \varrho^{(A[G])}[G]\}.$$

Similarly,

$$\lambda^{(A[F*G])}[F*G] \ge \max\{\lambda^{(A[F])}[F],\ \lambda^{(A[G])}[G]\}.$$

On the other hand, if $\varrho^{(A[F])}[F] < +\infty$ and $\varrho^{(A[G])}[G] < +\infty$ then by Lemma 4 $\alpha^*[F] < 1$ and $\alpha^*[G] < 1$. Therefore, $\ln|f_k| \le \lambda_k^{\alpha_1}$ and $\ln|g_k| \le \lambda_k^{\alpha_2}$ for every $\alpha_1 \in (\alpha^*[F],\, 1)$, $\alpha_2 \in (\alpha^*[G],\, 1)$ and all $k \ge k_0$. Hence,

$$\alpha^*[F*G] \le \varlimsup_{k\to\infty} \frac{\ln^+(\lambda_k^{\alpha_1} + \lambda_k^{\alpha_2})}{\ln\lambda_k} \le \max\{\alpha_1, \alpha_2\},$$

that is in view of the arbitrariness of α_1 and α_2 we get $\alpha^*[F*G] \le \max\{\alpha^*[F], \alpha^*[G]\}$. Thus, by Lemma 4

$$\varrho^{(A[F*G])}[F*G] = \frac{\alpha^*[F*G]}{1-\alpha^*[F*G]} \le \frac{\max\{\alpha^*[F], \alpha^*[G]\}}{1-\max\{\alpha^*[F], \alpha^*[G]\}} =$$

$$= \frac{\max\{\varrho^{(A[F])}[F]/(1+\varrho^{(A[F])}[F]), \varrho^{(A[G])}[G]/(1+\varrho^{(A[G])}[G])\}}{1-\max\{\varrho^{(A[F])}[F]/(1+\varrho^{(A[F])}[F]), \varrho^{(A[G])}[G]/(1+\varrho^{(A[G])}[G])\}}.$$

If for example $\max\{\varrho^{(A[F])}[F], \varrho^{(A[G])}[G]\} = \varrho^{(A[F])}[F]$ then

$$\max\left\{\frac{\varrho^{(A[F])}[F]}{1+\varrho^{(A[F]}[F]}, \frac{\varrho^{(A[G])}[G]}{1+\varrho^{(A[G])}[G]}\right\} = \frac{\varrho^{(A[F])}[F]}{1+\varrho^{(A[F])}[F]}$$

and, therefore,

$$\varrho^{(A[F*G])}[F*G] \le \frac{\varrho^{(A[F])}[F]/(1+\varrho^{(A[F])}[F])}{1-\varrho^{(A[F])}[F]/(1+\varrho^{(A[F])}[F])} = \varrho^{(A[F])}[F],$$

i. e. $\varrho^{(A[F*G])}[F*G] \le \max\{\varrho^{(A[F])}[F], \varrho^{(A[G])}[G]\}$.

If $\ln\lambda_{k+1} \sim \ln\lambda_k$, $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow +\infty$ as $k_0 \le k \to \infty$ then by Lemma 4 $\ln|f_{k_j}| \le \lambda_{k_j}^{\alpha_0}$ for every $\alpha_0 \in (\alpha_*[F],\, 1)$ and some sequence $(k_j) \uparrow \infty$. Therefore,

$$\alpha_*[F*G] \le \varliminf_{j\to\infty} \frac{\ln^+(\ln|f_{k_j} g_{k_j}| + A[F^*G]\lambda_{k_j})}{\ln\lambda_{k_j}} \le$$

$$\leq \varliminf_{j\to\infty} \frac{\ln^+(\lambda_{k_j}^{\alpha_0}+\lambda_{k_j}^{\alpha_2})}{\ln\lambda_{k_j}} = \max\{\alpha_0, \alpha_2\}.$$

Hence as above we obtain

$$\lambda^{(A[F*G])}[F*G] \leq \max\{\lambda^{(A[F])}[F], \varrho^{(A[G])}[G]\}.$$

Similarly,

$$\lambda^{(A[F*G])}[F*G] \leq \max\{\lambda^{(A[G])}[G], \varrho^{(A[F])}[F]\},$$

and thus, estimates (30) are true. □

Using Lemma 5 and Proposition 4 we prove the following theorem.

Theorem 2. *Let $-\infty < A[F], A[G] < +\infty$, $\ln\ln n(x) = o(\ln x))$ as $x\to+\infty$ and (4) hold. Then for $n \in \mathbb{Z}_+$, $m\in\mathbb{N}$ and $m>n$*

$$(m-n)\max\{\varrho^{(A[F])}[F],\ \varrho^{(A[G])}[G]\} \leq \varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{-\ln(A[F*G]-\sigma)} \ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \leq$$
$$\leq (m-n)(\max\{\varrho^{(A[F])}[F],\ \varrho^{(A[G])}[G]\}+1). \tag{31}$$

If, moreover, $\lambda_{k+1}\sim\lambda_k$, $\varkappa_k[F]\nearrow A[F]$ and $\varkappa_k[G]\nearrow A[G]$ as $k_0\leq k\to\infty$ then

$$(m-n)\min\{\lambda^{(A[F])}[F],\ \lambda^{(A[G])}[G]\} \leq \varliminf_{\sigma\uparrow A[F*G]} \frac{1}{-\ln(A[F*G]-\sigma)} \ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \leq$$
$$\leq (m-n)(\min\{\max\{\lambda^{(A[F])}[F],\ \varrho^{(A[G])}[G]\},\ \max\{\varrho^{(A[F])}[F],\ \lambda^{(A[G])}[G]\}\}+1). \tag{32}$$

Proof. From (20) we get

$$(m-n)\varrho^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \leq \varlimsup_{\sigma\uparrow A[F*G]} \frac{1}{-\ln(A[F*G]-\sigma)} \ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \leq$$
$$\leq (m-n)\varrho^{(A[F*G])}[\Lambda,(F*G)^{(m)}] \tag{33}$$

and

$$(m-n)\lambda^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \leq \varliminf_{\sigma\uparrow A[F*G]} \frac{1}{-\ln(A[F*G]-\sigma)} \ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \leq$$
$$\leq (m-n)\lambda^{(A[F*G])}[\Lambda,(F*G)^{(m)}]. \tag{34}$$

The functions $\alpha(x)=\beta(x)=\ln^+ x$ satisfy the conditions of Lemma 3. Therefore, $\varrho^{(A[F*G])}[\Lambda,(F*G)^{(n)}]=\varrho^{(A[F*G])}[\Lambda,F*G]$ and $\lambda^{(A[F*G])}[\Lambda,(F*G)^{(n)}]=\lambda^{(A[F*G])}[\Lambda,F*G]$. By Lemmas 4 and 5

$$\varrho^{(A[F*G])}[F*G]=\varrho^{(A[F*G])}[\ln\mu,F*G]\leq\varrho^{(A[F*G])}[\Lambda,F*G]\leq$$
$$\leq\varrho^{(A[F*G])}[\ln\mu,F*G]+1=\varrho^{(A[F*G])}[F*G]+1$$

and

$$\lambda^{(A[F*G])}[F*G]=\lambda^{(A[F*G])}[\ln\mu,F*G]\leq\lambda^{(A[F*G])}[\Lambda,F*G]\leq$$
$$\leq\lambda^{(A[F*G])}[\ln\mu,F*G]+1=\lambda^{(A[F*G])}[F*G]+1.$$

Therefore, using (29) and (30) from (33) and (34) we get (31) and (32). □

5. Hadamard compositions of finite R-orders. If $\sigma_a[F]=A[F]=A\in(-\infty,\,+\infty)$ then the quantities

$$\varrho_R^{(A)}[F]:=\overline{\lim_{\sigma\uparrow A}}(A-\sigma)\ln^+\ln M(\sigma,F)),\quad \lambda^{(A)}[F]:=\underline{\lim_{\sigma\uparrow A}}(A-\sigma)\ln^+\ln M(\sigma,F))$$

are called [19] the R-order and the lower R-order of F accordingly.

Lemma 6. *Let* $\overline{\lim_{x\to+\infty}}\dfrac{\ln\ln n(x)}{\ln x}<1$. *Then*

$$\varrho_R^{(A)}[F]=\varrho_R^{(A)}[\ln\mu,F]=\overline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}\ln^+(|f_k|\exp\{A[F]\lambda_k\}). \tag{35}$$

If, moreover, $\ln\lambda_{k+1}\sim\ln\lambda_k$ *and* $\varkappa_k[F]\nearrow A[F]$ *as* $k_0\le k\to\infty$ *then*

$$\lambda_R^{(A)}[F]=\lambda_R^{(A)}[\ln\mu,F]=\underline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}\ln^+(|f_k|\exp\{A[F]\lambda_k\}). \tag{36}$$

Lemma 7. *If* $\sigma_a[F]=A[F]=A\in(-\infty,\,+\infty)$ *then*

$$\varrho_R^{(A)}[F']=\varrho_R^{(A)}[F],\quad \lambda_R^{(A)}[F']=\lambda_R^{(A)}[F].$$

Proof of Lemma 7. From (15) the inequalities $\varrho_R^{(A)}[F]\le\varrho_R^{(A)}[F']$ and $\lambda_R^{(A)}[F]\le\lambda_R^{(A)}[F']$ follow. On the other hand, choosing $\delta(\sigma)=(A-\sigma)/q$ with $q>1$ from (14) we get

$$\ln^+\ln M(\sigma,F')\le\ln^+\ln M(\sigma+(A-\sigma)/q,F)+\ln^+\ln(q/(A-\sigma))+\ln 2$$

and since $(A-\sigma)(\ln^+\ln(q/(A-\sigma))+\ln 2)\to 0$ as $\sigma\uparrow A$ hence it follows that

$$(A-\sigma)\ln^+\ln M(\sigma,F')+o(1)\le$$

$$\le\frac{1}{1-1/q}\left(A-\sigma-\frac{A-\sigma}{q}\right)\ln^+\ln M\left(\sigma+\frac{A-\sigma}{q},F\right),\quad \sigma\uparrow A.$$

Therefore,

$$\varrho_R^{(A)}[F']\le\tfrac{q}{q-1}\varrho_R^{(A)}[F]\quad\text{and}\quad\lambda_R^{(A)}[F']\le\tfrac{q}{q-1}\lambda_R^{(A)}[F],$$

whence in view of the arbitrariness o q we obtain $\varrho_R^{(A)}[F']\le\varrho_R^{(A)}[F]$ and $\lambda_R^{(A)}[F']\le\lambda_R^{(A)}[F]$. □

Lemma 8. *If* $\sigma_a[F]=A[F]A\in(-\infty,\,+\infty)$ *then*

$$\varrho_R^{(A)}[\ln\mu,F]=\varrho_R^{(A)}[\Lambda,F],\quad \lambda_R^{(A)}[\ln\mu,F]=\lambda_R^{(A)}[\Lambda,F].$$

Proof of Lemma 7. From (12) it follows that $\varrho_R^{(A)}[\ln\mu, F] \le \varrho_R^{(A)}[\Lambda, F]$ and $\lambda_R^{(A)}[\ln\mu, F] \le \lambda_R^{(A)}[\Lambda, F]$.
On the other hand, (13) implies

$$(A-\sigma)\ln\Lambda(\sigma,F) \le (A-\sigma)\ln\frac{q}{A-\sigma} + (A-\sigma)\ln\ln\mu\left(\sigma+\frac{A-\sigma}{q}, F\right) =$$
$$= \frac{A-\sigma}{(1-1/q)(A-\sigma)}\left(A-\sigma-\frac{A-\sigma}{q}\right)\ln\ln\mu\left(\sigma+\frac{A-\sigma}{q}, F\right) + o(1)$$

as $\sigma \uparrow A$. Hence it follows that

$$\varrho_R^{(A)}[\Lambda, F] \le (1-1/q)\varrho_R^{(A)}[\ln\mu, F], \quad \lambda_R^{(A)}[\Lambda, F] \le \le (1-1/q)\lambda_R^{(A)}[\ln\mu, F]$$

for each $q > 1$. Thus,

$$\varrho_R^{(A)}[\ln\mu, F] \ge \varrho_R^{(A)}[\Lambda, F], \quad \lambda_R^{(A)}[\ln\mu, F] \ge \lambda_R^{(A)}[\Lambda, F].$$

□

Lemma 6 implies the following statement.

Proposition 5. *Let* $-\infty < A[F], A[G] < +\infty$, $\overline{\lim\limits_{x\to+\infty}}\dfrac{\ln\ln n(x)}{\ln x} < 1$ *and* (4) *holds. Suppose that* $\ln\lambda_{k+1} \sim \ln\lambda_k$, $\varkappa_k[F] \nearrow A[F]$ *and* $\varkappa_k[G] \nearrow A[G]$ *as* $k_0 \le k \to \infty$. *Then*

$$\max\{\varrho_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G], \varrho_R^{(A[G])}[G] + \lambda_R^{(A[F])}[F]\} \le$$
$$\le \varrho_R^{(A[F*G])}[F*G] \le \varrho_R^{(A[F])}[F] + \varrho_R^{(A[G])}[G] \tag{37}$$

and

$$\lambda_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G] \le \lambda_R^{(A[F*G])}[F*G] \le$$
$$\le \min\{\varrho_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G], \varrho_R^{(A[G])}[G] + \lambda_R^{(A[F])}[F]\} \tag{38}$$

Proof of Proposition 5. In view of (4) and (35)

$$\varrho_R^{(A[F*G])}[F*G] = \overline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}(\ln|f_k| + A[F]\lambda_k + \ln|g_k| + A[G]\lambda_k) \le$$
$$\le \overline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}\ln(|f_k|\exp\{A[F]\lambda_k\}) + \overline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}|g_k|\exp\{A[G]\lambda_k\}) = \varrho_R^{(A[F])}[F] + \varrho_R^{(A[G])}[G]$$

and in view of (36)

$$\varrho_R^{(A[F*G])}[F*G] \ge \overline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}\ln(|f_k|\exp\{A[F]\lambda_k\}) + \underline{\lim_{k\to\infty}}\frac{\ln\lambda_k}{\lambda_k}|g_k|\exp\{A[G]\lambda_k\}) =$$
$$= \varrho_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G],$$

i. e. estimates (37) are true. The proof of (38) is similar. □

Finally, using Lemmas 7, 8 and Proposition 5 we prove the following theorem.

Theorem 3. *Let*

$$\overline{\lim_{x\to+\infty}} \frac{\ln\ln n(x)}{\ln x} < 1$$

and $\lambda_{k+1} \sim \lambda_k$ *as* $k \to \infty$. *Suppose that* $A[F], A[G] \in (-\infty, +\infty)$, *and* (4) *holds,* $\varkappa_k[F] \nearrow A[F]$ *and* $\varkappa_k[G] \nearrow A[G]$ *as* $k_0 \le k \to \infty$. *Then for* $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$ *and* $m > n$

$$(m-n)\max\{\varrho_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G], \varrho_R^{(A[G])}[G] + \lambda_R^{(A[F])}[F]\} \le$$
$$\le \overline{\lim_{\sigma\uparrow A[F*G])}} (A[F*G]-\sigma)\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \le (m-n)(\varrho_R^{(A[F])}[F] + \varrho_R^{(A[G])}[G]) \tag{39}$$

and

$$(m-n)(\lambda_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G]) \le \underline{\lim_{\sigma\uparrow A[F*G]}} (A[F*G]-\sigma)\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \le$$
$$\le (m-n)\min\{\varrho_R^{(A[F])}[F] + \lambda_R^{(A[G])}[G], \varrho_R^{(A[G])}[G] + \lambda_R^{(A[F])}[F]\}. \tag{40}$$

Proof. As above from (20) we get

$$(m-n)\varrho_R^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \le \overline{\lim_{\sigma\uparrow A[F*G]}} (A[F*G]-\sigma)\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \le$$
$$\le (m-n)\varrho_R^{(A[F*G])}[\Lambda,(F*G)^{(m)}] \tag{41}$$

and

$$(m-n)\lambda_R^{(A[F*G])}[\Lambda,(F*G)^{(n)}] \le \underline{\lim_{\sigma\uparrow A[F*G]}} (A[F*G]-\sigma)\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} \le$$
$$\le (m-n)\lambda_R^{(A[F*G])}[\Lambda,(F*G)^{(m)}]. \tag{42}$$

By Lemma 7

$$\varrho_R^{(A[F*G])}[\Lambda,(F*G)^{(n)}] = \varrho_R^{(A[F*G])}[\Lambda, F*G]$$

and

$$\lambda_R^{(A[F*G])}[\Lambda,(F*G)^{(n)}] = \lambda_R^{(A[F*G])}[\Lambda, F*G].$$

By Lemmas 5 and 6

$$\varrho_R^{(A[F*G])}[F*G] = \varrho_R^{(A[F*G])}[\ln\mu, F*G] = \varrho_R^{(A[F*G])}[\Lambda, F*G]$$

and

$$\lambda_R^{(A[F*G])}[F*G] = \lambda_R^{(A[F*G])}[\ln\mu, F*G] = \lambda_R^{(A[F*G])}[\Lambda, F*G].$$

Therefore, using estimates (37) and (38) from (41) and (42) we get (39) and (40). □

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