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ERDŐS-MACINTYRE TYPE THEOREM'S FOR MLTIPLE DIRICHLET SERIES: EXCEPTIONAL SETS AND OPEN PROBLEMS

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In the paper, we formulate some open problems related to the best description of the values of the exceptional sets in Wiman's inequality for entire functions and in the Erdős-Macintyre type theorems for entire multiple Dirichlet series. At the same time, we clarify the statement of one Ĭ.V. Ostrovskii problem on Wiman's inequality. We also prove three propositions and one theorem. On the one hand, in a rather special case, these results give the best possible description of the values of the exceptional set in the Erdős-Macintyre-type theorem. On the second hand, they indicate the possible structure of the best possible description in the general case.

1. Introduction. For an entire function of the form

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n \tag{1}$$

it is well known (for example, see [1–6]) that by the classical theorem of A. Wiman and G. Valiron for each non-constant entire function of form (1) and for every $\varepsilon>0$ there exists a set $E=E_f(\varepsilon)\subset[1;+\infty)$ of finite logarithmic measure (i.e. $\int_E d\ln r<+\infty$) such that the following classical (*Wiman's inequality*)

$$M_f(r)\le \mu_f(r)\ln^{1/2+\varepsilon}\mu_f(r).$$

holds for $r\in(1,+\infty)\setminus E_f(\varepsilon)$, where

$$M_f(r):=\max\{|f(z)|\colon |z|=r\},\quad \mu_f(r):=\max\{|a_n|r^n\colon n\ge 0\},\quad r>0.$$

Regarding the statement about the Wiman inequality, in 1995, prof. Ĭ.V. Ostrovskii in a conversation with the third co-author of this article formulated the following problem: *what is the best possible description of the values of an exceptional set E*? This problem was considered in a number of articles (for example, see [4,5,7–14]) in view to many other relations obtained in the Wiman-Valiron theory. In this same conversation, Prof. Ĭ.V. Ostrovskii also formulated the following problem.

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Problem 1 (Ĭ.V. Ostrovskii, 1995)**.** *It is well-known that the exceptional set E in classical Wiman's inequality has the form $E = \bigcup\limits_{n=1}^{+\infty}[a_n, b_n]$. Let $E = \bigcup\limits_{n=1}^{+\infty}[a_n, b_n]$ be a arbitrary set of finite logarithmic measure. Is there an entire function such that the set E is an exceptional set in classical Wiman's inequality for given ε?*

Note ([4,5]), for every $\varepsilon > 0$ there exist an entire function f and a set $E_1 \subset [1, +\infty)$ such that for all $r \in E_1$

$$f(r) \geq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon} \quad \text{and} \quad \int_{E_1} (\ln \mu_f(r))^{1/2+\varepsilon}\, d\ln r = +\infty.$$

On the other hand, it was also established [5] that the estimate

$$\int_E \ln^{1/2} \mu_f(r)\, d\ln r < +\infty$$

of an exceptional set E in Wiman's inequality holds almost surely in some probabilistic sense, i.e. the estimate $\int_E \ln^{1/2} \mu_f(r)\, d\ln r < +\infty$ is almost surely the best possible description of size of an exceptional set. Recall that $\ln r = o(\ln \mu_f(r))$ $(t \to +\infty)$ for each transcendental entire function. In this context, the following question arises. It clarifies the question from Problem 1:

Question 1. *Let E be a set of the form specified in Problem 1 and $\int_E \ln^{1/2} r\, d\ln r < +\infty$. Is there an entire function f for each set E such that the set E is exceptional in Wiman's inequality for given $\varepsilon > 0$ and the function f?*

In this article, we will consider the first problem of Ĭ.V. Ostrovskii regarding to the Erdős-Macintyre type theorems for entire multiple Dirichlet series.

Let us introduce some standard notations (see [15, 16]). Let $D^p(\lambda^p)$ be a class of entire functions F on $\mathbb{C}^p$, $p \geq 1$, represented by absolutely convergent in whole space $\mathbb{C}^p$ Dirichlet series of the form

$$F(z) = \sum_{\|n\|=0}^{+\infty} a_n e^{<z,\lambda_n>} \tag{2}$$

with fixed system of exponents $\lambda^p = \{\lambda_n :\ n \in \mathbb{Z}_+^p\}$, $\lambda_n = (\lambda_{n_1}^{(1)}, \ldots, \lambda_{n_p}^{(p)})$, $n = (n_1, \ldots, n_p)$, such that $0 \leq \lambda_k^{(j)} \uparrow +\infty$ $(k \to +\infty)$ for every $j \in \{1, 2, \ldots, p\}$; here $< a, b >= a_1 b_1 + \ldots + a_p b_p$, $\|a\| = a_1 + \cdots + a_p$ for $a = (a_1, \ldots, a_p) \in \mathbb{C}^p$, $b = (b_1, \ldots, b_p) \in \mathbb{C}^p$. For $F \in D^p(\lambda^p)$ and $x \in \mathbb{R}^p$ we denote $I = \{n \in \mathbb{Z}_+^p :\ a_n \neq 0\}$, $\lambda_I^p = \{\lambda_n :\ n \in I\}$ and also

$$M(x, F) = \sup\{|F(x + iy)| :\ y \in \mathbb{R}^p\},\ m(x, F) = \inf\{|F(x + iy)| :\ y \in \mathbb{R}^p\},$$
$$\mu(x, F) = \max\{|a_n| e^{<x,\lambda_n>} :\ n \in \mathbb{Z}_+^p\}.$$

Denote by

$$\gamma(F) = \left\{x \in \mathbb{R}^p :\ \lim_{t \to +\infty} \frac{1}{t} \ln \mu(tx, F) = +\infty\right\}$$

the cone of the growth of maximal term of series (2). Remark, $\gamma(F) = (0; +\infty)$ in the case $F \in D^1(\lambda^1)$.

For entire Dirichlet series of one complex variable ($p = 1$) the following fact is known ([17], see also [18]). In order that for each function $F \in D^1(\lambda^1)$ the asymptotic relations

$$M(x, F) = (1 + o(1))\ \mu(x, F), \quad M(x, F) = (1 + o(1))\ m(x, F) \tag{3}$$

hold as $x \to +\infty$ ($x \in (0; +\infty) \setminus E$), where $E = E(F)$ is some set of finite Lebesgue measure on $\mathbb{R}$, it is sufficient and necessary that for $j = 1$ the following condition

$$\sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1}^{(j)} - \lambda_k^{(j)}} < +\infty. \tag{4}$$

holds. This theorem is analogous to a similar statement established earlier by P.C. Fenton [20] (see also P. Erdős, A.J. Macintyre [19]) for entire functions represented by lacunary power series of one complex variable.

The papers [15, 16] establish analogues of this statement for the class $D^p(\lambda^p)$, $p \geq 2$. It is proved [15] that the asymptotic relations (3) are true as $|x| \to +\infty$ ($x \in K \setminus E$) for every cone K in $\mathbb{R}^p$ with vertex at the origin O such that $\overline{K} \setminus \{O\} \subset (\gamma(F) \cap H)$. Here $E \subset \mathbb{R}^p$ is some measurable set with the Lebesgue measure meas_p for which holds

$$\mathrm{meas}_p(E \cap B_p(r)) = O(r^{p-1}) \quad (r \to +\infty); \tag{5}$$

and $B_p(r) = \{x \in \mathbb{R}^p : \ |x| \leq r\}$ is the ball of radius $r > 0$, a set H is the subset of $x \in \mathbb{R}^p$ such that the sequence $\{< x, \lambda_n >: \ n \in I\}$ admits the increasing arrangement, i.e., $\beta_j(x) =< x, \lambda_n >$ for some $n = n(j) \in \mathbb{Z}_+^p$, $\beta_{j+1}(x) > \beta_j(x)$ ($j \geq 0$) and

$$\sum_{j=j_1}^{+\infty} 1/r_j < +\infty, \quad r_j := \inf\{\beta_{j+1}(x) - \beta_j(x) : \ x \in H, \ |x| = 1\},$$

for some $j_1 \in \mathbb{Z}_+$. The similar assertion from [16] implies, in particular, the following theorem.

Theorem 1 ([16], Theorem 1)**.** *In order that asymptotic relations (3) to be fulfilled for each function $F \in D^p(\lambda^p)$ as $|x| \to +\infty$ ($x \in \mathbb{R}^p \setminus E$), it is necessary and sufficient that condition (4) is fulfilled for every $j \in \{1, 2, \dots, p\}$; here $E = E(F)$ is some measurable set from $\mathbb{R}^p$ such that*

$$\mathrm{meas}_p(E \cap B_p(r)) = o(r^{p-1}\varphi(r)) \quad (r \to +\infty), \tag{6}$$

where $\varphi(r)$ is an arbitrary positive function such that $\varphi(r) \uparrow +\infty$ ($r \to +\infty$) and the set E depends on the function φ in general case.

Earlier, one of the authors of this note (O. Skaskiv) announced the following theorem on International conference on the complex analysis (Lviv, 2004).

Theorem B. *Let $F \in D^p(\lambda^p)$. Suppose the sequence $\|\lambda_I\| = \{\|\lambda_n\| : \ n \in I\}$ can be ordered by increasing (μ_j), $\mu_{j+1} < \mu_j$ ($j \geq 0$), i.e. $\|\lambda_I\| = \{\mu_j : j \geq 0\}$, for each $n \in I$ there exists unique $j = j(n)$ such that $\|\lambda_n\| = \mu_j$, and vice versa, for each $j \geq 0$ there exists unique $n = n(j) \in I$ such that $\mu_j = \|\lambda_n\|$. If*

$$\sum_{j=0}^{+\infty} \frac{1}{\mu_{j+1} - \mu_j} < +\infty, \tag{7}$$

then relations (3) hold as $|x| \to +\infty$ ($x \in \mathbb{R}^p \setminus E$), where E is some measurable set in $\mathbb{R}^p$ such that

$$\mathrm{meas}_p(E \cap S_p(r)) = o(r^{p-1}\varphi(r)) \quad (r \to +\infty), \tag{8}$$

where $S_p(r)$ is a p-dimensional unbounded right cylinder with axis $l_0 = \{x \in \mathbb{R}^p : \ x_1 = \cdots = x_p\}$ and its base is a $(p-1)$-dimensional ball with center at the point O and radius $r > 0$, φ is an arbitrary function such that $\varphi(r) \nearrow +\infty$ ($r \to +\infty$), and the set E depends on the function φ in general case.

The statement of Theorem B follows from Theorem 2, which will be proved below.

Remark 1. Theorem B weakens the conditions of Theorem 1 from [16], while strengthening the estimate (8) of the exceptional set.

Remark 2. The conditions of Theorem 1 and Theorem B are, generally speaking, independent. Indeed, if we choose (for $p = 2$) the sequence $\lambda_I^2 = \{(k, k^3) : \ k \geq 0\}$, then the condition (7) is fulfilled, and the condition (4) for $j = 1$ no. If you choose $\lambda_I^2 = \{(k^3, m^3) : \ (k, m) \in \mathbb{Z}_+^2\}$, then the condition (4) for $j = 1$ and $j = 2$ is fulfilled, and the condition (7) is not fulfilled.

In [9, p. 164], it is noted that in the case $p = 2$ from the condition $\mathrm{meas}_p(E \cap S(r)) = o(r^{p-1})$ ($r \to +\infty$) it follows that $\int_E \frac{dx_1 dx_2}{|x|^{1+\varepsilon}} < +\infty$ for every $\varepsilon > 1$. In fact, this also follows from the condition (5) (at $p = 2$). It is also not difficult to verify that this follows from the conditions (6) or (8) (due to $p = 2$). Similarly, for an arbitrary $p \geq 2$ with the condition (5) (and also from the conditions (6) and (8)) it follows that for every $\varepsilon > 0$ the condition

$$\int\limits_E \frac{dx_1 \ldots dx_p}{|x|^{(p-1)(1+\varepsilon)}} < +\infty$$

holds.

In [8, p. 137] (see also [13]), for the case $p = 1$ it is proved that for every positive non-decreasing function h such that

$$h(x)/x \to +\infty \ (x \to +\infty),$$

for each sequence λ^1 (in particular, for which at $j = 1$ the conditions (4) are fulfilled), there exist a function $f \in D^1(\lambda^1)$, constant $d_1 > 0$ and a set $E_1 \subset [0; +\infty)$ such that for all $x \in E_1$

$$f(x) > (1 + d_1)\mu(x, f) \ , \qquad M(x, f) > (1 + d_1)m(x, f) \tag{9}$$

and

$$m_h E_1 = \int\limits_{E_1 \cap [0;+\infty)} dh(x) = +\infty.$$

Hence, it follows the following proposition.

Proposition 1. *For any sequence λ^p and each positive differentiable function $h \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $h'(t) \nearrow +\infty$ ($t \to +\infty$), there exist a function $F \in D^p(\lambda^p)$, a constant $d > 0$ and a set $E \subset \mathbb{R}_+^p$ such that*

$$F(x) > (1 + d)\mu(x, F) \tag{10}$$

for all $x \in E$, and the set E satisfies the condition

$$\int\limits_E \frac{h'(|x|)}{|x|^{p-1}} dx_1 \ldots dx_p = +\infty.$$

Proof. Let $f_1 \in D^1(\lambda^1)$ be a function from above-cited statement concerning inequalities (9) from [8, c.137]. We choose arbitrary functions $f_j \in D^1((\lambda_k^{(j)}))$, $j \in \{2, ..., p\}$, such that

$$f_j(t) = \sum_{k=0}^{+\infty} a_k^{(j)} \exp\{t\lambda_k^{(j)}\},$$

$a_k^{(j)} > 0$ $(k > 0)$, where $\lambda^p = (\lambda_n)$, $\lambda_n = (\lambda_{n_1}^{(1)}, ..., \lambda_{n_p}^{(p)})$, $n = (n_1, ..., n_p)$.

Then, by the first inequality in (9) and the Cauchy inequality for all $x = (x_1, ..., x_p) \in \mathbb{R}^p$, $x_1 \in E_1$

$$F(x) = f_1(x_1)f_2(x_2)\dots f_p(x_p) \geq (1 + d_1)\prod_{j=1}^{p} \mu(x_j, f_j) = (1 + d_1)\mu(x, F),$$

because $\mu(x, F) = \prod_{j=1}^{p} \mu(x_j, f_j)$. It remains to note that for the set

$$E = \{x = (x_1, ..., x_p) : \ x_1 \in E_1, x_1 \leq x_2 \leq 2x_1, ..., x_1 \leq x_p \leq 2x_1\}$$

ve have

$$\int_E \frac{h'(|x|)}{|x|^{p-1}} dx = \int_{E_1} \left(\int_{x_1}^{2x_1} \cdots \int_{x_1}^{2x_1} \frac{h'(|x|)}{|x|^{p-1}} dx_2 \dots dx_p \right) dx_1 \geq$$

$$\geq \frac{1}{(4p-3)^{\frac{p-1}{2}}} \int_{E_1} h'(x_1)dx_1 = +\infty.$$

The proof of Proposition 1 is complete. □

In [21], we find the following conjecture.

Conjecture 1 (O. Skaskiv, see [21]). *The best possible description of the size of the exceptional sets in relations* (3) *under the conditions* (4) *is the finiteness of the measure*

$$\tau_p(E) := \int_{E \setminus B_p(1)} \frac{dx_1 \dots dx_p}{|x|^{p-1}} < +\infty. \tag{11}$$

If we could prove that under the conditions (4) the description of exceptional set (11) holds, then from the Proposition 1 it follows non-improvability of the estimate (11) of size of the exceptional set in asymptotic relations (3). Remark, in the case of the Borel relation for the entire multiple Dirichlet series it was established [9] that this description of the exceptional set is the best possible one in some sense.

Due to Proposition 1 and Conjecture 1 the following natural question arises.

Question 2. *What are conditions for the multiple entire Dirichlet series $F \in D^p(\lambda^p)$ such that relation* (3) *holds as $|x| \to +\infty$ outside some set E of finite h–τ_p-measure, i.e.*

$$h\text{–}\tau_p(E) := \int_E \frac{h'(|x|)}{|x|^{p-1}} dx_1 \dots dx_p < +\infty \ ?$$

Remark 3. In the case $p = 1$ (i.e. for entire Dirichlet series of one variable) these conditions were founded in the paper [13].

Proposition 2. *For any sequence λ^p there exist a function $F \in D^p(\lambda^p)$, constant $d > 0$ and a set $E \subset \mathbb{R}^p_+$ such that for all $x \in E$ inequality (10) holds and for all $r \geq R_0$*

$$\frac{1}{r^{p-1}}\text{meas}_p(E \cap B_p(r)) \geq c > 0. \tag{12}$$

Proof. Indeed, since for every sequence $\lambda^1 = (\lambda_k^{(1)})$ and every function $f_1 \in D^1(\lambda^1)$ irreducible to an exponential polynomial there exist a constant $d_1 > 0$ and an unbounded set $E_1 \subset \mathbb{R}_+$ such that for all $x_1 \in E_1$ inequalities (9) hold and $meas_1(E_1 \cap [0; +\infty)) = c_1 > 0$, for any function $F \in D^p(\lambda^p)$ of the form $F(x) = f_1(x_1)F_1(x_2, ..., x_p)$, $F_1 \in D^{p-1}(\lambda^{p-1})$, $\lambda^{p-1} = (\lambda_{n_2}^{(2)}, ..., \lambda_{n_p}^{(p)})$, by Cauchy inequality we get

$$M(x, F) = M(x_1, f_1)M(x_2, ..., x_p, F_1) \geq (1 + d_1)\mu(x_1, f_1)\mu(x_2, ..., x_p, F_1).$$

Since in this case at least $\mu(x, F) \leq \mu(x_1, f_1)\mu(x_2, ..., x_p, F_1)$, for every $x \in E = E_1 \times \mathbb{R}^{p-1}$ we have inequality (10). Given that $\text{meas}_p(E \cap B_p(R)) \asymp \text{meas}_p(E \cap \Pi_R)$ $(R \to +\infty)$ for the cube $\Pi_R = \{x \in \mathbb{R}^p : \ |x_j| \leq R, \ 1 \leq j \leq p\}$ and $\text{meas}_p(E \cap \Pi_R) = cR^{p-1}$, $c > 0$, we obtain inequality (12). □

Therefore, it follows from Proposition 2 that, unlike the case of Borel's relation ([9]), it is not possible to significantly improve the description of the exceptional set in asymptotic relations (3) given in Theorem 1 and Theorem B. Actually,

Proposition 3. *Conjecture 1 is false.*

Indeed, suppose that Conjecture 1 is correct, and the sequence λ^p satisfies the condition (4). Then, on the one hand, $\tau_p(E) < +\infty$ for every Dirichlet series from $D^p(\lambda^p)$, and on the other hand, by Proposition 2 there exists a Dirichlet series $D^p(\lambda^p)$ such that inequality (12) holds. But the condition $\tau_p(E) < +\infty$ implies that $\tau_p(E \setminus B_p(r)) \to 0$ $(r \to +\infty)$, i.e. $\tau_p(E \setminus B_p(r)) < \varepsilon$ for enough large $r \geq 1$ and arbitrary $\varepsilon > 0$. Thus, for all $R \geq r$ we have

$$\frac{1}{R^{p-1}}\big(\text{meas}_p(E \cap B_p(R)) - \text{meas}_p(E \cap B_p(r))\big) \leq \int\limits_{E \cap B_p(R) \setminus B_p(r)} \frac{dx}{|x|^{p-1}} < \varepsilon.$$

Therefore, in view of (12)

$$0 < c \leq \frac{\text{meas}_p(E \cap B_p(r))}{R^{p-1}} + \varepsilon \to \varepsilon \quad (R \to +\infty).$$

If we now put $\varepsilon = c/2$, we get a contradiction.

Question 3. *Therefore, Question 2 will probably need some modification?*

In view of Propositions 2 and 3, we can formulate the following conjecture.

Conjecture 2. *The following estimate of exceptional set E*

$$\text{meas}_p(E \cap B_p(r)) = O(r^{p-1}) \quad (r \to +\infty)$$

is the best possible description one under the conditions of Theorem 1.

Now let H be a set of the points x^0 on the sphere $\{x \in \mathbb{R}^p : \ |x| = 1\}$ such that the sequence $\{< x, \lambda_n >: \ a_n \neq 0\}$ can be arranged by increasing $(\beta_j(x^0))$ such that

$$A(x^0) = \sum_{j=j_1}^{+\infty} \frac{1}{\beta_{j+1}(x^0) - \beta_j(x^0)} < +\infty,$$

and j_1 is the smallest integer s such that $\beta_{j+1}(x^0) - \beta_j(x^0) > 0$ for all $j \geq s$. In addition, let

$$H_1^+ = \{x = tx^0 : \ t \in \mathbb{R}_+, \ x^0 \in H_1\}$$

be a cone with the vertex at the origin $O \in \mathbb{R}^p$ and with the directional set
$$H_1 \subset \{x : \ |x| = 1\}.$$

Theorem 2. *Let $F \in D^p(\lambda^p)$, and a set $H_1 \subset H$ such that $\sup\{A(x^0) : \ x^0 \in H_1\} = A < +\infty$. Then for every $\varepsilon > 0$ there exists a set $E \subset \mathbb{R}^p$ such that $\tau_p(E \cap H_1^+) \leq C(\varepsilon) < +\infty$ and*

$$M(x, F) < (1 + \varepsilon)\mu(x, F)$$

for all $x \in \mathbb{R}^p_+ \cap H_1^+ \setminus E$.

Proof of Theorem 2. Let $x_0 \in H_1$, $(\beta_j(x_0))$ be defined above, $\beta_j := \beta_j(x_0)$.

Denote $\Delta_0 = 0$ and for $j \geq 1$

$$\Delta_j = \Delta_j(x_0) := \sum_{k=0}^{j-1} (\beta_{k+1} - \beta_k) \sum_{m=k+1}^{\infty} \left(\frac{1}{\beta_m - \beta_{m-1}} + \frac{1}{\beta_{m+1} - \beta_m} \right).$$

Consider the function

$$f_q(t) = \sum_{j=0}^{+\infty} \frac{a_{n(j)}}{\alpha_j} \ e^{t\beta_j}, \quad t \in \mathbb{R},$$

where $\alpha_j = e^{q\Delta_j}$, $q > 0$.

Since $\Delta_j \geq 0$, then $f_q \in D(\beta)$ with $\beta = (\beta_j(x_0))$ and $\nu(t, f_q) \nearrow +\infty$ $(t \to +\infty)$.

Denote by J the range of the central index $\nu(t, f_q)$, and let (R_k) be the sequence of the jump points of central index $\nu(t, f_q)$, numbered in such a way that $\nu(t, f_q) = k$ for all $t \in [R_k, R_{k+1})$ and $R_k < R_{k+1}$. Then for all $t \in [R_k, R_{k+1})$ and $j \geq 0$ we have

$$\frac{a_{n(j)}}{\alpha_j} e^{t\beta_j} \leq \frac{a_{n(k)}}{\alpha_k} e^{t\beta_k}.$$

We need the following lemma (see [13, Lemma 1] and in the another form [16, Lemma 1]).

Lemma 1 ([13], Lemma 1). *For all $j \geq 0$ and $k \geq 1$ inequality*

$$\frac{\alpha_j}{\alpha_k} \ e^{\tau_k(\beta_j - \beta_k)} \leq e^{-q|n-k|}, \tag{13}$$

is true, where

$$\tau_k = \tau_k(q) = qt_k + \frac{q}{\beta_k - \beta_{k-1}}, \ t_k = \frac{\Delta_{k-1} - \Delta_k}{\beta_k - \beta_{k-1}}.$$

According to Lemma 1, for $t \in [R_k + \tau_k, R_{k+1} + \tau_k)$ we obtain

$$\frac{a_n(j)e^{t\beta_j}}{a_{n(k)}e^{t\beta_k}} \le \frac{\alpha_j}{\alpha_k}\ e^{\tau_k(\beta_j-\beta_k)} \le e^{-q|j-k|} \qquad (j \ge 0).$$

Therefore,

$$\nu(tx_0, F) = k, \quad \mu(tx_0, F) = a_{n(k)}e^{t\beta_k} \quad (t \in [R_k + \tau_k, R_{k+1} + \tau_k),\ k \in \mathbb{N}) \tag{14}$$

and

$$|F(tx_0 + iy) - a_{\nu(tx_0,F)}e^{(tx_0+iy)\lambda_{\nu(tx_0,F)}}| \le$$
$$\le \sum_{j\ne k(\nu(tx_0,F))} \mu(tx_0, F)e^{-q|j-k(\nu(tx_0,F))|} \le 2\ \frac{e^{-q}}{1-e^{-q}}\mu(tx_0, F)$$

for all $t \in [R_k + \tau_k, R_{k+1} + \tau_k)$ and $k \in J$, where $k(\nu(tx_0, F))$ such that $\beta_{k(\nu(tx_0,F))} = (\lambda_{\nu(tx_0,F)}, x_0)$, i.e. for all

$$t \notin E_1(q) = E_1(q, x_0) \overset{def}{=} \bigcup_{k=0}^{+\infty}[R_{k+1} + \tau_k, R_{k+1} + \tau_{k+1}).$$

Thus, inequality

$$|F(x + iy) - a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}| \le 2\ \frac{e^{-q}}{1-e^{-q}}\mu(x, F) \tag{15}$$

holds for all

$$x = tx_0 \notin \bigcup_{\|x_0\|=1, x_0\in H_1} E_1(q) =$$
$$= \bigcup_{\|x_0\|=1, x_0\in H_1} E_1(q, x_0) \overset{def}{=} \bigcup_{\|x_0\|=1, x_0\in H_1}\bigcup_{k=0}^{+\infty}[R_{k+1} + \tau_k, R_{k+1} + \tau_{k+1}).$$

For every $q > 0$ we have

$$\text{meas}_1(E_1(q)) = \sum_{k=0}^{+\infty}\int_{R_{k+1}+\tau_k}^{R_{k+1}+\tau_{k+1}} dx = \sum_{k=0}^{+\infty}(R_{k+1} + \tau_{k+1} - (R_{k+1} + \tau_k)) \le$$
$$\le 2q\sum_{k=0}^{+\infty}\frac{1}{\beta_{k+1} - \beta_k} = 2qA(x_0) \le 2qA. \tag{16}$$

Then for the set $E = \bigcup_{\|x\|=1, x\in H_1} E_1(q)$ from (16) we obtain

$$\tau_p(E) = \int_E \frac{dx}{|x|^{p-1}} = \int_{\|x_0\|=1, x_0\in H_1} dS(x_0) \int_{E_1(q,x_0)\cap[0,+\infty)} dt \le$$
$$\le \int_{\|x_0\|=1, x_0\in H_1} \text{meas}_1(E_1(q, x_0))dS(x_0) \le 2qAS < +\infty,$$

where S is the surface area of a unit sphere in $\mathbb{R}^p$.

To complete the proof, it is enough to choose $q > 0$ such that $2\ \frac{e^{-q}}{1-e^{-q}} \le \varepsilon$ for a given arbitrary $\varepsilon > 0$. □

Remark 4. *It is easy to see that the condition* $\tau_p(E) < +\infty$ *implies*

$$\mathrm{meas}_p(E \cap S_p(r)) = o(r^{p-1}\varphi(r)), \quad \mathrm{meas}_p(E \cap B_p(r)) = o(r^{p-1}) \quad (r \to +\infty).$$

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