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SHARP BOUNDS OF LOGARITHMIC COEFFICIENT PROBLEMS FOR FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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The logarithmic coefficients play an important role for different estimates in the theory of univalent functions. Due to the significance of the recent studies about the logarithmic coefficients, the problem of obtaining the sharp bounds for the second Hankel determinant of these coefficients, that is $H_{2,1}(F_f/2)$ was paid attention. We recall that if f and F are two analytic functions in $\mathbb{D}$, the function f is subordinate to F, written $f(z) \prec F(z)$, if there exists an analytic function ω in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{D}$. It is well-known that if F is univalent in $\mathbb{D}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{D}$ if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. In the current study, we obtain the sharp bounds of the second Hankel determinant of the logarithmic coefficients for families $\mathcal{S}_s^*(\psi)$ and $\mathcal{C}_s(\psi)$ where were defined by the concept subordination and ψ is considered univalent in $\mathbb{D}$ with positive real part in $\mathbb{D}$ and satisfies the condition $\psi(0) = 1$. Note that $f \in \mathcal{S}_s^*(\psi)$ if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \psi(z), \quad z \in \mathbb{D}$$

and $f \in \mathcal{C}_s(\psi)$ if

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \psi(z), \quad z \in \mathbb{D}.$$

It is worthwhile mentioning that the given bounds in this paper extend and develop some related recent results in the literature. In addition, the results given in these theorems can be used for determining the upper bound of $|H_{2,1}(F_f/2)|$ for other popular families.

1. Introduction. Let $\mathcal{A}$ be the family of analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon\ |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{D} \tag{1}$$

and let $\mathcal{S}$ be the family of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{D}$ if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. K. Sakaguchi [14] defined and studied the family $\mathcal{S}_s^*$ of all functions that are starlike with respect to symmetric points. He proved that $f \in \mathcal{S}_s^*$

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if $\operatorname{Re}\big(zf'(z)/(f(z)-f(-z))\big) > 0$ for $z \in \mathbb{D}$. Recently, V. Ravichandran [13] introduced the following classes $\mathcal{S}_s^*(\psi)$ and $\mathcal{C}_s(\psi)$ where ψ is considered univalent in $\mathbb{D}$ with positive real part in $\mathbb{D}$ and satisfies the condition $\psi(0) = 1$.

A function $f \in \mathcal{A}$ of the form (1) is in the family $\mathcal{S}_s^*(\psi)$ if

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \psi(z), \quad z \in \mathbb{D}$$

and is in the family $\mathcal{C}_s(\psi)$ if

$$\frac{2(zf'(z))'}{f'(z)+f'(-z)} \prec \psi(z), \quad z \in \mathbb{D}.$$

It is obvious these are families of close-to-convex functions, so they are univalent in $\mathbb{D}$. Taking $\psi(z) = (1+z)/(1+z)$, the above categories reduce to the categories $\mathcal{S}_s^*$ and $\mathcal{C}_s$ (convex functions with respect to symmetric points, see [3]). Note that the odd functions in $\mathcal{S}^*(\psi)$ $(\mathcal{C}(\psi))$ are in the family $\mathcal{S}_s^*(\psi)$ $(\mathcal{C}_s(\psi))$ [13] where $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ are known as W.C. Ma and D. Minda categories [10].

For $q, n \in \mathbb{N} := \{1, 2, 3, \dots\}$, the *q-th Hankel determinant* of a function $f \in \mathcal{A}$ with the form (1) is defined by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (\text{with } a_1 := 1)\,.$$

There are more outcomes for Hankel determinants of any degree and their applications in [12]. It is clear that the Hankel determinants $H_{2,1}(f)$ and $H_{2,2}(f)$ correspond to the Fekete-Szegö and second Hankel determinant functionals, respectively.

The logarithmic coefficients γ_n of the function $f \in \mathcal{S}$ are defined with the purpose of the following form

$$F_f(z) := \log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \; z \in \mathbb{D}, \quad \text{where} \quad \log 1 = 0. \tag{2}$$

These coefficients are significant for various estimates in the theory of univalent functions; in this regard see [11, Chapter 2]. The logarithmic coefficients γ_n of an arbitrary function $f \in \mathcal{S}$ satisfy the inequality $\sum\limits_{n=1}^{\infty} |\gamma_n|^2 \le \pi^2/6$, and the equality holds for the Koebe function. For $f \in \mathcal{S}^*$, the relation $|\gamma_n| \le 1/n$ holds but it does not hold for the whole family $\mathcal{S}$ (see [4, Theorem 8.4]). The best upper bounds for univalent functions' logarithmic coefficients for $n \ge 3$ remain a conundrum, though, apparently.

If f is given by (1) then, by matching the coefficients of z^n in (2) for $n = 1, 2, 3$ it concludes

$$2\gamma_1 = a_2, \quad 2\gamma_2 = a_3 - \frac{1}{2}a_2^2, \quad 2\gamma_3 = a_4 - a_2a_3 + \frac{1}{3}a_2^3. \tag{3}$$

Further, due to the significance of the recent studies about the logarithmic coefficients, the problem obtaining the sharp bounds for the second Hankel determinant of these coefficients, that is $H_{2,1}(F_f/2)$ was reported in the papers [1, 8, 9] for several subfamilies of analytic functions, where the second Hankel determinant for $F_f/2$, by utilizing the relations (3), will be

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{4}$$

Note that $H_{2,1}(F_f/2)$ is invariant under rotations (see [9]).

The main goal of this study is to get an upper bound of $H_{2,1}(F_f/2)$ for the families $\mathcal{S}_s^*(\psi)$ and $\mathcal{C}_s(\psi)$. Our results have some special corollaries with several applications for other well-known classes, some of which are extensions of those reported in earlier papers.

2. Main results. In order to get the upper bound of $H_{2,1}(F_f/2)$ for the mentioned categories, we need the following lemma. Let Ω denote the family of all analytic functions ϑ in $\mathbb{D}$ with $\vartheta(0) = 0$, and $|\vartheta(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 1 ([6], Lemma 2.1). *If $\vartheta(z) = \sum_{n=1}^{\infty} \vartheta_n z^n \in \Omega$, then for some p, q, with $|p| \le 1$ and $|q| \le 1$*

$$\vartheta_2 = p\left(1 - \vartheta_1^2\right), \quad \vartheta_3 = \left(1 - \vartheta_1^2\right)\left(1 - |p|^2\right)q - \vartheta_1\left(1 - \vartheta_1^2\right)p^2.$$

Theorem 1. *If $f \in \mathcal{S}_s^*(\psi)$, with $\psi(z) = 1 + \sum_{n=1}^{\infty} G_n z^n$, then*

$$|H_{2,1}(F_f/2)| \le \frac{|G_1|}{16} \cdot \begin{cases} \dfrac{4MO - N^2}{4M}, & \text{if} \quad M < 0 \text{ and } M \le -\dfrac{N}{2} \le 0; \\ \max\{O; M + N + O\}, & \text{otherwise.} \end{cases}$$

where

$$M := \left|\frac{G_1G_2}{4} + \frac{G_3}{2} + \frac{G_1^3}{48} - \frac{G_2^2}{G_1}\right| - |G_2| - \frac{|G_1|^2}{4} + \frac{|G_1|}{2},$$
$$N := |G_2| + \frac{|G_1|^2}{4} - \frac{3|G_1|}{2}, \; O := |G_1|. \tag{5}$$

Proof. If $f \in \mathcal{S}_s^*(\psi)$, then there is a function $\omega \in \Omega$, with $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$, $z \in \mathbb{D}$, such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \psi(\omega(z)) = \tag{6}$$
$$= 1 + G_1\omega_1 z + \left(G_1\omega_2 + G_2\omega_1^2\right)z^2 + \left(G_1\omega_3 + 2\omega_1\omega_2G_2 + G_3\omega_1^3\right)z^3 + \dots, \; z \in \mathbb{D},$$

where $\psi(z) = 1 + \sum_{n=1}^{\infty} G_n z^n$, $z \in \mathbb{D}$. Since ψ is univalent in $\mathbb{D}$, it follows that $G_1 = \psi'(0) \ne 0$.

The function $f \in \mathcal{S}_s^*(\psi)$ has the form (1), and equating the coefficients of z^n in (6) for $n = 1, 2, 3$ we have

$$\begin{cases} 2a_2 = G_1\omega_1, \quad 2a_3 = G_1\omega_2 + G_2\omega_1^2, \\ 4a_4 - 2a_2a_3 = G_1\omega_3 + 2\omega_1\omega_2G_2 + G_3\omega_1^3. \end{cases}$$

From these relations it follows that

$$\begin{cases} 2a_2 = G_1\omega_1, \quad 2a_3 = G_1\omega_2 + G_2\omega_1^2, \\ 4a_4 = G_1\left[\omega_3 + \left(\dfrac{G_1}{2} + \dfrac{2G_2}{G_1}\right)\omega_1\omega_2 + \left(\dfrac{G_2}{2} + \dfrac{G_3}{G_1}\right)\omega_1^3\right]. \end{cases}$$

Therefore, after considering the above values in (4) we obtain

$$|H_{2,1}(F_f/2)| = \frac{1}{16}\left|\left(\frac{G_1^2G_2}{4}+\frac{G_3G_1}{2}+\frac{G_1^4}{48}-G_2^2\right)\omega_1^4+\left(\frac{G_1^3}{4}-G_1G_2\right)\omega_1^2\omega_2+\frac{G_1^2}{2}\omega_1\omega_3-G_1^2\omega_2^2\right|.$$

Applying Lemma 1 for some x, s, with $|x|\leq 1$ and $|s|\leq 1$, from the above relation it follows that

$$\begin{aligned}|H_{2,1}(F_f/2)| = \frac{|G_1|}{16}\Bigg|&\left(\frac{G_1G_2}{4}+\frac{G_3}{2}+\frac{G_1^3}{48}-\frac{G_2^2}{G_1}\right)\omega_1^4+\left(\frac{G_1^2}{4}-G_2\right)x\omega_1^2\left(1-\omega_1^2\right)+\\ &+\left(-G_1(1-\omega_1^2)-\frac{G_1}{2}\omega_1^2\right)x^2\left(1-\omega_1^2\right)+\frac{G_1}{2}\omega_1\left(1-\omega_1^2\right)\left(1-|x|^2\right)s\Bigg|.\end{aligned}$$

Since it is well-known that $|\omega_1|\leq 1$ and as $H_{2,1}(F_f/2)$ and $\omega(z)$ are invariant under the rotations (see [9]). Thus, we can assume $\omega := \omega_1 \in [0,1]$ (see [5, Theorem 3. p. 80]) in order to simplify the calculation. Hence, we have

$$\begin{aligned}|H_{2,1}(F_f/2)| &= \frac{|G_1|}{16}\Bigg|\left(\frac{G_1G_2}{4}+\frac{G_3}{2}+\frac{G_1^3}{48}-\frac{G_2^2}{G_1}\right)\omega^4+\left(\frac{G_1^2}{4}-G_2\right)x\omega^2\left(1-\omega^2\right)+\\ &+\left(-G_1(1-\omega^2)-\frac{G_1}{2}\omega^2\right)x^2\left(1-\omega^2\right)+\frac{G_1}{2}\omega\left(1-\omega^2\right)\left(1-|x|^2\right)s\Bigg|\leq\\ &\leq\frac{|G_1|}{16}\Bigg[\left|\frac{G_1G_2}{4}+\frac{G_3}{2}+\frac{G_1^3}{48}-\frac{G_2^2}{G_1}\right|\omega^4+\left(\frac{|G_1|^2}{4}+|G_2|\right)|x|\omega^2\left(1-\omega^2\right)+\\ &+\left(|G_1|(1-\omega^2)+\frac{|G_1|}{2}\omega^2\right)|x|^2\left(1-\omega^2\right)+\frac{|G_1|}{2}\omega\left(1-\omega^2\right)\left(1-|x|^2\right)\Bigg]=\\ &=\frac{|G_1|}{16}\Bigg[\left|\frac{G_1G_2}{4}+\frac{G_3}{2}+\frac{G_1^3}{48}-\frac{G_2^2}{G_1}\right|\omega^4+\left(\frac{|G_1|^2}{4}+|G_2|\right)\omega^2\left(1-\omega^2\right)|x|+\\ &+\frac{|G_1|}{2}(1-\omega)(2+\omega)\left(1-\omega^2\right)|x|^2+\frac{|G_1|}{2}\omega\left(1-\omega^2\right)\Bigg]=:T_\omega(\nu),\end{aligned}$$

where $\nu := |x| \in [0,1]$. An easy analysis reveals that T_ω is an increasing function of ν and so it attains its maximum at $\nu = 1$, that is $\max\{T_\omega(\nu) : \nu\in[0,1]\} = T_\omega(1) =: H(\omega)$, where

$$\begin{aligned}H(\omega) = \frac{|G_1|}{16}\Bigg[&\left(\left|\frac{G_1G_2}{4}+\frac{G_3}{2}+\frac{G_1^3}{48}-\frac{G_2^2}{G_1}\right|-|G_2|-\frac{|G_1|^2}{4}+\frac{|G_1|}{2}\right)\omega^4+\\ &+\left(|G_2|+\frac{|G_1|^2}{4}-\frac{3|G_1|}{2}\right)\omega^2+|G_1|\Bigg].\end{aligned}$$

For the simplicity, if we denote $\kappa := \omega^2 \in [0,1]$ and set the values of M, N, and O like in (5), then $H(\kappa) = \frac{|G_1|}{16}\left(M\kappa^2+N\kappa+O\right),\ \kappa\in[0,1]$.

It is easy to show that

$$\max\left\{M\kappa^2+N\kappa+O : \kappa\in[0,1]\right\} = \begin{cases}\dfrac{4MO-N^2}{4M}, & \text{if } \quad M<0 \text{ and } M\leq -\dfrac{N}{2}\leq 0,\\ \max\{O; M+N+O\}, & \text{otherwise.}\end{cases}$$

Hence, we obtain

$$|H_{2,1}(F_f/2)| \le \frac{|G_1|}{16} \cdot \begin{cases} \dfrac{4MO - N^2}{4M}, & \text{if} \quad M < 0 \text{ and } M \le -\dfrac{N}{2} \le 0, \\ \max\{O; M+N+O\}, & \text{otherwise.} \end{cases}$$

where M, N, and O are given by (5). □

If we take in Theorem 1 the function $\psi(z) := (1 + (1-2\sigma)z)/(1-z)$ where $G_1 = G_2 = G_3 = 2(1-\sigma)$ with $\sigma \in [0,1)$, we get the next corollary for $f \in \mathcal{S}_s^*(\sigma)$ [2]:

Corollary 1. *If $f \in \mathcal{S}_s^*(\sigma)$, then*

$$|H_{2,1}(F_f/2)| \le \frac{(1-\sigma)^2}{4}.$$

This result is sharp for $f(z) = \frac{z}{(1-z^2)^{(1-\sigma)}} = z + (1-\sigma)z^3 + \ldots,\ z \in \mathbb{D}$.

Proof. For $\psi(z) := (1 + (1-2\sigma)z)/(1-z)$ where $\sigma \in [0,1)$, we obtain $-\frac{N}{2} = \frac{\sigma(1-\sigma)}{2} \not\le 0$ and with $M + N + O = \frac{(1-\sigma)}{6}\left|1 + \sigma^2 - 8\sigma\right| \le 2(1-\sigma) = O$ for $\sigma \in (0,1)$ and $M = -\frac{11}{6} \le -\frac{N}{2} = 0 \le 0$ for $\sigma = 1$. To prove the second part of the corollary, a computation shows that for $f(z) = \frac{z}{(1-z^2)^{(1-\sigma)}}$,

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) = \operatorname{Re}\left(\frac{1 + (1-2\sigma)z^2}{1-z^2}\right) > \sigma,\ z \in \mathbb{D},$$

that is $f \in \mathcal{S}_s^*(\sigma)$. It is easy to check with $a_2 = 0$ and $a_3 = (1-\sigma)$ we obtain

$$|H_{2,1}(F_f/2)| = \left|\frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right)\right| = \frac{1}{4}\left|a_3^2\right| = \frac{(1-\sigma)^2}{4},$$

which shows that our estimation is sharp. □

For $\sigma = 0$ the result of the Corollary 1 becomes Theorem 2.2 from [1].

Setting in Theorem 1 the function

$$\psi(z) = \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} + \ldots,\ z \in \mathbb{D},$$

because $-\frac{N}{2} = \frac{9}{32} \not\le 0$ with $M + N < 0$ we obtain the next outcome for $f \in \mathcal{S}_{s,L}^*$ [7].

Corollary 2. *If $f \in \mathcal{S}_{s,L}^*$, then*

$$|H_{2,1}(F_f/2)| \le \frac{1}{64}.$$

This bound is sharp for the function $f_2 \in \mathcal{A}$ given by

$$\frac{2zf_2'(z)}{f_2(z) - f_2(-z)} = \sqrt{1+z^2},$$

that is for $f_2(z) = z + z^3/4 + \ldots$.

Taking in Theorem 1 the function

$$\psi(z) := \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \left(\frac{4}{3}\alpha^3 + \frac{2}{3}\alpha\right) z^3 + \ldots,\ z \in \mathbb{D},\ 0 < \alpha \leq 1,$$

because $-\frac{N}{2} = \frac{3\alpha(1-\alpha)}{2} \nleq 0$ with $M + N < 0$ for $\sigma \in (0, 1)$ and $M = -\frac{11}{6} \leq -\frac{N}{2} = 0 \leq 0$ for $\alpha = 1$, we obtain the next outcome:

Corollary 3. *If* $\mathcal{S}_s^*\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$, *then*

$$|H_{2,1}(F_f/2)| \leq \frac{\alpha^2}{4}.$$

This bound is sharp for the function $f_\alpha \in \mathcal{A}$ *given by*

$$\frac{2zf_\alpha'(z)}{f_\alpha(z) - f_\alpha(-z)} = \left(\frac{1+z^2}{1-z^2}\right)^{\alpha}, \tag{7}$$

that is for $f_\alpha(z) = z + \alpha z^3 + \ldots$.

Theorem 2. *If* $f \in \mathcal{C}_s(\psi)$ *with* $\psi(z) = 1 + \sum\limits_{n=1}^{\infty} G_n z^n$, *then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{4} \cdot \begin{cases} \frac{4WY - X^2}{4W}, & \text{if } W < 0 \text{ and } W \leq -\frac{X}{2} \leq 0; \\ \max\{Y; W + X + Y\}, & \text{otherwise,} \end{cases}$$

where

$$W := \left|\frac{G_1^2 G_2}{128} + \frac{G_3 G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right| - \frac{|G_1|^3}{128} - \frac{7|G_2 G_1|}{288} + \frac{28|G_1|^2}{2304},$$
$$X := \frac{|G_1|^3}{128} + \frac{7|G_2 G_1|}{288} - \frac{92|G_1|^2}{2304},\ Y := \frac{|G_1|^2}{36}. \tag{8}$$

Proof. If $f \in \mathcal{C}_s(\psi)$, then there exists a function $\varpi \in \Omega$, with $\varpi(z) = \sum_{n=1}^{\infty} \omega_n z^n$, $z \in \mathbb{D}$, such that

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} = \psi(\varpi(z))$$
$$= 1 + G_1\omega_1 z + \left(G_1\omega_2 + G_2\omega_1^2\right) z^2 + \left(G_1\omega_3 + 2\omega_1\omega_2 G_2 + G_3\omega_1^3\right) z^3 + \ldots,\ z \in \mathbb{D}.$$

After some calculations it results

$$\begin{cases} 4a_2 = G_1\omega_1, \quad 6a_3 = G_1\omega_2 + G_2\omega_1^2, \\ 16a_4 = G_1\left[\omega_3 + \left(\frac{G_1}{2} + \frac{2G_2}{G_1}\right)\omega_1\omega_2 + \left(\frac{G_2}{2} + \frac{G_3}{G_1}\right)\omega_1^3\right]. \end{cases}$$

Therefore, after replacing in (4) we have

$$|H_{2,1}(F_f/2)| = \frac{1}{4}\left|\left(\frac{G_1^2 G_2}{128} + \frac{G_3 G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right)\omega_1^4 + \left(\frac{G_1^3}{128} - \frac{7G_1 G_2}{288}\right)\omega_1^2\omega_2 + \frac{G_1^2}{64}\omega_1\omega_3 - \frac{G_1^2}{36}\omega_2^2\right|.$$

Using Lemma 1 for some x, s, with $|x| \le 1$ and $|s| \le 1$, from the above relation it follows that

$$|H_{2,1}(F_f/2)| = \frac{1}{4}\Bigg| \left(\frac{G_1^2G_2}{128} + \frac{G_3G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right)\omega_1^4 + \left(\frac{G_1^3}{128} - \frac{7G_1G_2}{288}\right)x\omega_1^2\left(1-\omega_1^2\right) + \\ + \left(-\frac{G_1^2}{36}(1-\omega_1^2) - \frac{G_1^2}{64}\omega_1^2\right)x^2\left(1-\omega_1^2\right) + \frac{G_1^2}{64}\omega_1\left(1-\omega_1^2\right)\left(1-|x|^2\right)s\Bigg|.$$

We may assume that $\omega := \omega_1 \in [0,1]$ so we have

$$|H_{2,1}(F_f/2)| \le \frac{1}{4}\Bigg[\left|\frac{G_1^2G_2}{128} + \frac{G_3G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right|\omega^4 + \left(\frac{|G_1|^3}{128} + \frac{7|G_2G_1|}{288}\right)|x|\omega^2\left(1-\omega^2\right) + \\ + \left(\frac{|G_1|^2}{36}(1-\omega^2) + \frac{|G_1|^2}{64}\omega^2\right)|x|^2\left(1-\omega^2\right) + \frac{|G_1|^2}{64}\omega\left(1-\omega^2\right)\left(1-|x|^2\right)\Bigg] = \\ = \frac{1}{4}\Bigg[\left|\frac{G_1^2G_2}{128} + \frac{G_3G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right|\omega^4 + \left(\frac{|G_1|^3}{128} + \frac{7|G_2G_1|}{288}\right)\omega^2\left(1-\omega^2\right)|x| + \\ + \frac{|G_1|^2}{2304}(1-\omega)(64+28\omega)\left(1-\omega^2\right)|x|^2 + \frac{|G_1|^2}{64}\omega\left(1-\omega^2\right)\Bigg] =: J_\omega(\lambda),$$

where $\lambda := |x| \in [0,1]$. The function J_ω is an increasing function of λ and so it attains its maximum at $\lambda = 1$, that is $\max\{J_\omega(\lambda) : \lambda \in [0,1]\} = J_\omega(1) =: L(\omega)$, where

$$L(\omega) = \frac{1}{4}\Bigg[\left(\left|\frac{G_1^2G_2}{128} + \frac{G_3G_1}{64} + \frac{G_1^4}{3072} - \frac{G_2^2}{36}\right| - \frac{|G_1|^3}{128} - \frac{7|G_2G_1|}{288} + \frac{28|G_1|^2}{2304}\right)\omega^4 + \\ + \left(\frac{|G_1|^3}{128} + \frac{7|G_2G_1|}{288} - \frac{92|G_1|^2}{2304}\right)\omega^2 + \frac{|G_1|^2}{36}\Bigg].$$

If we set $u := \omega^2 \in [0,1]$ and set the values of W, X, and Y in (8), then

$$L(u) = \frac{1}{4}\left(Wu^2 + Xu + Y\right),\ u \in [0,1].$$

□

Corollary 4 ([1], Theorem 2.4)**.** *If $f \in \mathcal{K}_s$, then $|H_{2,1}(F_f/2)| \le \frac{1}{36}$. This result is sharp for $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$.*

REFERENCES

1. V. Allu, V. Arora, A. Shaji, *On the second Hankel determinant of logarithmic coefficients for certain univalent functions*, Mediterr. J. Math., **20** (2023), 81. https://doi.org/10.1007/s00009-023-02272-x
2. S.Z.H. Bukhari, T. Bulboacă, M.S. Shabbir, *Subordination and superordination results for analytic functions with respect to symmetrical points*, Quaest. Math., **41** (2018), №1, 65–79. https://doi.org/10.2989/16073606.2017.1372528

3. R.N. Das, P. Singh, *On subclasses of schlicht mapping*, Indian J. Pure Appl. Math., **8** (1977), №8, 864–872.
4. P.L. Duren, Univalent functions, Springer, Amsterdam, 1983.
5. A.W. Goodman, Univalent functions, Mariner, Tampa, 1983.
6. S. Kanas, E.A. Adegani, A. Zireh, *An unified approach to second Hankel determinant of bi-subordinate functions*, Mediterr. J. Math., **14** (2017), 233. https://doi.org/10.1007/s00009-017-1031-6
7. K. Khatter, V. Ravichandran, S. Sivaprasad Kumar, *Estimates for initial coefficients of certain starlike functions with respect to symmetric points*, in: Cushing, J., Saleem, M., Srivastava, H., Khan, M., Merajuddin, M., (eds) Applied Analysis in Biological and Physical Sciences, Springer Proceedings in Mathematics & Statistics, V.186, Springer, New Delhi, 2016.
8. B. Kowalczyk, A. Lecko, *Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha*, Bull. Malays. Math. Sci. Soc., **45** (2022), 727–740. https://doi.org/10.1007/s40840-021-01217-5
9. B. Kowalczyk, A. Lecko, *Second Hankel determinant of logarithmic coefficients of convex and starlike functions*, Bull. Aust. Math. Soc., **105** (2022), 458–467. https://doi.org/10.1017/S0004972721000836
10. W.C. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Internat. Press, Cambridge, MA, USA (1992), 157–169.
11. I.M. Milin, Univalent functions and orthonormal systems, Amer. Math. Soc. Transl. Math. Monogr, 49, Proovidence, RI, 1977.
12. C. Pommerenke, *On the Hankel determinants of Univalent functions*, Mathematika, **14** (1967), 108–112.
13. V. Ravichandran, *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), **20** (2004), №1, 31–37.
14. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, **11** (1959), 72–75.

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