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GENERALIZED DERIVATIONS OF ORDER 2 ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Let R be a prime ring of characteristic different from 2 with a right Martindale quotient ring Q_r and an extended centroid C . Let F be a non zero generalized derivation of R and S be the set of evaluations of a non central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C . Let $p, q \in R$ be such that

$$pF^2(u)u + F^2(u)uq = 0 \text{ for all } u \in S.$$

Then for all $x \in R$ one of the followings holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,
2. $p = -q \in C$,
3. $f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 + a^2q = 0$.

1. Introduction. Throughout the article we suppose that R is an associative ring. A ring R is said to be prime if for $x, y \in R$ and $xRy = 0$ implies either $x = 0$ or $y = 0$. Let Q_r denotes the right Martindale's ring of quotient of a prime ring R and C denotes the center of Q_r . The ring Q_r is prime with unity containing R and C is a field which is known as the extended centroid of R (for more details see [2]). An additive mapping $d : R \rightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is said to generalized derivation associative with a derivation d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. For $a \in R$, the mapping $d(x) = [a, x]$ for all $x \in R$, is a derivation, which is said to be an inner derivation induced by a . If a derivation is not an inner derivation it is called an outer derivation. A generalized derivation F is said to be generalized inner derivation if its associated derivation d is inner, otherwise F is called an outer derivation.

By [20] we know that every generalized derivation F on a dense right ideal of R can be uniquely extended to Q_r and hence any generalized derivation F of R can be implicitly assumed to be defined on the whole Q_r and has the form $F(x) = qx + d(x)$ for all $x \in R$, where q is a fixed element of Q_r and d is an associated derivation of F . For $b \in Q_r$ if $d(x) = [b, x]$ for all $x \in R$ then d is said to be X -inner derivation otherwise X -outer derivation. The notion of generalized derivation has been introduced by Brešar in [3]. Such maps are extensively studied in ring theory and theory of operator algebras. Further the study of generalized derivations are extended to Banach algebras (see [1, 25, 26]).

In 1957, Posner [24] proved that if d is a non zero derivation on a prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$ then R is a commutative ring. Further more generalization

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of Posner's result can be found in several ways (see [5, 6, 23] where further references can be found). In fact, it was the starting point of much research article concerning the study of various kind of additive mappings satisfying appropriate algebraic conditions on some subset of a prime or semiprime ring R . In [16], Hvala extended the Posner's result by replacing derivations with generalized derivations. Recently more results on composition of generalized derivations on prime rings have been found in [10, 11, 12].

In [27], Tiwari et al. state as "Let R be a prime ring of characteristic different from 2 with the Utumi quotient ring U and the extended centroid C , $f(x_1, \dots, x_n)$ be a multilinear polynomial over C , which is not central valued on R . Suppose that d is a non-zero derivation of R , F and G are two generalized derivations of R such that $d\{F(u)u - uG^2(u)\} = 0$ for all $u \in f(R)$ and find all possible forms of d , F and G ". In the same time Eroglu et al. [14] proved the following result:

Theorem 1. *Let R be a prime ring with the extended centroid C , Q maximal right ring of quotients of R , RC central closure of R such that $\dim_C(RC) > 4$, $f(x_1, \dots, x_n)$ a multilinear polynomial over C which is not central valued on R and $f(R)$ the set of all evaluations of the multilinear polynomial $f(x_1, \dots, x_n)$ in R . Suppose that G is a non-zero generalized derivation of R such that $G^2(u)u \in C$ for all $u \in f(R)$ then one of the followings conditions holds:*

- (I) *there exists $a \in Q$ such that $a^2 = 0$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$,*
- (II) *there exists $a \in Q$ such that $0 \neq a^2 \in C$ and either $G(x) = ax$ for all $x \in R$ or $G(x) = xa$ for all $x \in R$,*
- (III) *$\text{char}(R) = 2$ and one of the followings holds:*
 - i) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2 = b^2 \in C$,*
 - ii) *there exist $a, b \in Q$ such that $G(x) = ax + xb$ for all $x \in R$ and $a^2, b^2 \in C$ and $f(x_1, \dots, x_n)^2$ is central valued on R ,*
 - iii) *there exists $a \in Q$ and X -outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$ and $a^2 + d(a) = 0$,*
 - iv) *there exists $a \in Q$ and X -outer derivation d of R such that $G(x) = ax + d(x)$ for all $x \in R$, $d^2 = 0$ and $a^2 + d(a) \in C$ and $f(x_1, \dots, x_n)^2$ is central valued on R .*

More recently, in [8] Filippis et al. proved the following result.

Theorem 2. *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid, G be a non-zero X -generalized skew derivation of R , and S be the set of the evaluations of a multilinear polynomial $f(x_1, \dots, x_n)$ over C with n non-commuting variables. Let $u, v \in R$ be such that $uG(x)x + G(x)xv = 0$ for all $x \in S$. Then one of the following statements holds:*

- (a) *$v \in C$ and there exist $a, b, c \in Q_r$ such that $G(x) = ax + bxc$ for any $x \in R$ with $(u + v)a = (u + v)b = 0$,*
- (b) *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $G(x) = ax$ for all $x \in R$ with $ua + av = 0$.*

Motivated by Theorem 1 and Theorem 2 we prove the following main theorem.

Theorem (Main Theorem). *Let R be a prime ring of characteristic different from 2 with the right Martindale quotient ring Q_r and the extended centroid C . Let F be a non-zero generalized derivation of R and S be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C . Let $p, q \in R$ be such that $pF^2(u)u + F^2(u)uq = 0$ for all $u \in S$. Then for all $x \in R$ one of the followings holds:*

1. *there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,*
2. *$p = -q \in C$,*
3. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 + a^2q = 0$.*

2. Preliminaries. In all that follows, R always denotes a prime ring and Q_r its right Martindale's ring of quotients, C is the center of Q_r , $f(x_1, \dots, x_n)$ is a multilinear polynomial over C . The definition and axiomatic formulation of right Martindale ring Q_r can be found in [2] and [4].

We have the following properties which we need:

1. $R \subseteq Q_r$;
2. Q_r is a prime ring with identity;
3. The center of Q_r is denoted by C and is called the extended centroid of R . C is a field.

Moreover, we will use frequently some important theory of generalized polynomial identities and differential identities. We recall some facts.

Fact-1: Let $X = \{x_1, x_2, \dots\}$, the countable set consisting of the non-commuting indeterminates $x_1, x_2, \dots$. Let $C\{X\}$ be the free algebra over C in the set X . We denote $T = Q_r *_C C\{X\}$, the free product of the C -algebra Q_r and $C\{X\}$. The elements of T are called the generalized polynomials with coefficients in Q_r . Let B be a set of C -independent vectors of Q_r . Then any element $f \in T$ can be represented in this form $f = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B -monomials of the form $q_0 y_1 q_1 y_2 q_2 \cdots y_n q_n$, with $q_0, q_1, \dots, q_n \in B$ and $y_1, y_2, \dots, y_n \in X$. Any generalized polynomial $f = \sum_i \alpha_i m_i$ is trivial i.e., zero element in T if and only if $\alpha_i = 0$ for each i . For detail study we refer to [4].

Fact-2: Every derivation d of R can be uniquely extended to a derivation of Q_r ([2]).

Fact-3: If I is a two-sided ideal of R , then R , I and Q_r satisfy the same differential identities ([19]).

Fact-4: If I is a two-sided ideal of R , then R , I and Q_r satisfies the same generalized polynomial identities with coefficients in Q_r ([2]).

Fact-5: (Kharchenko [18, Theorem 2]) Let R be a prime ring, d a non-zero derivation on R and I a non-zero ideal of R . If I satisfies the differential identity

$$f(r_1, \dots, r_n, d(r_1), \dots, d(r_n)) = 0 \text{ for any } r_1, \dots, r_n \in I,$$

then either

- (i) I satisfies the generalized polynomial identity $f(r_1, \dots, r_n, x_1, \dots, x_n) = 0$

or

- (ii) d is Q_r -inner i.e., for some $q \in Q_r$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity $f(r_1, \dots, r_n, [q, r_1], \dots, [q, r_n]) = 0$.

Fact-6: We shall use the following notation:

$$f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and the symmetric group S_n of n symbols.

Fact-7: Every generalized derivation F on ring R can be extended to Q_r and assumes the form $F(x) = ax + d(x)$, for some $a \in Q_r$ and d is a derivation on Q_r .

Let d be a derivation. We denote by $f^d(x_1, \dots, x_n)$ and $f^{d^2}(x_1, \dots, x_n)$ the polynomials obtained from $f(x_1, \dots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$, respectively. Then we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n), \quad (1)$$

$$\begin{aligned} d^2(f(x_1, \dots, x_n)) &= f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) + \\ &+ \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n). \end{aligned} \quad (2)$$

3. The case when F is an inner generalized derivation. First, we study the situation when F is an inner generalized derivation, that is, $F(x) = ax + xb$ for all $x \in R$, for some $a, b \in Q_r$. Then $pF^2(f(r))f(r) + F^2(f(r))f(r)q = 0$ for all $r = (r_1, \dots, r_n) \in R^n$ implies

$$pa^2f(r)^2 + 2paf(r)bf(r) + pf(r)b^2f(r) + a^2f(r)^2q + 2af(r)bf(r)q + f(r)b^2f(r)q = 0.$$

This gives

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, where $a' = pa^2, b' = 2pa, c = a^2, c' = b^2$.

In this section we prove the following proposition.

Proposition 1. *Let R be a prime ring of characteristic different from 2 with the right Martindale quotient ring Q_r and the extended centroid C . Let S be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C . Suppose that F is a generalized inner derivation of R defined as $F(x) = ax + xb$ for all $x \in R$ such that $pF^2(u)u + F^2(u)uq = 0$ for all $u \in f(R)$ and for some $p, q \in Q_r$. Then either $F(x) = (a+b)x$ or $F(x) = x(a+b)$, $a+b \in Q_r$ and one of the followings holds:*

1. $(a+b)^2 = 0$,
2. $p = -q \in C$,
3. $f(x_1, \dots, x_n)^2$ is central valued on R and $p(a+b)^2 + (a+b)^2q = 0$.

To prove Proposition 1 we need the following.

Lemma 1 ([7], Lemma 1). *Let C be an infinite field and $m \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that the matrices $PA_1P^{-1}, \dots, PA_kP^{-1}$ have all non-zero entries.*

Proposition 2. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field C , $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C and $a, b, p, q, a', b', c' \in R$. If*

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$ then either a or b or q is central.

Proof. By our assumption, R satisfies the generalized identity

$$\begin{aligned} & a'f(x_1, \dots, x_n)^2 + b'f(x_1, \dots, x_n)bf(x_1, \dots, x_n) + \\ & + pf(x_1, \dots, x_n)c'f(x_1, \dots, x_n) + cf(x_1, \dots, x_n)^2q + \\ & + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n)q + f(x_1, \dots, x_n)c'f(x_1, \dots, x_n)q = 0. \end{aligned} \quad (3)$$

We assume that $a \notin Z(R)$, $b \notin Z(R)$ and $q \notin Z(R)$. Then we shall prove that this case leads to a contradiction.

Since $a \notin Z(R)$, $b \notin Z(R)$ and $q \notin Z(R)$, by Lemma 1 there exists a C -automorphism ϕ of $M_m(C)$ such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ and $q_1 = \phi(q)$ have all non-zero entries. Clearly $a_1, b_1, q_1, p_1 = \phi(p)$, $a'_1 = \phi(a')$, $b'_1 = \phi(b')$ and $c'_1 = \phi(c')$ must satisfy the condition (3). Without loss of generality we may replace a, b, p, q, a', b', c' with $a_1, b_1, p_1, q_1, a'_1, b'_1, c'_1$ respectively.

Here e_{ij} denotes the matrix whose (i, j) -entry is 1 and rest entries are zero. Since $f(x_1, \dots, x_n)$ is not central, by [19] (see also [21]), there exist $u_1, \dots, u_n \in M_m(C)$ and $\gamma \in C \setminus \{0\}$ such that $f(u_1, \dots, u_n) = \gamma e_{st}$, with $s \neq t$. Moreover, since the set $\{f(r_1, \dots, r_n) : r_1, \dots, r_n \in M_m(C)\}$ is invariant under the action of all C -automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \dots, r_n \in M_m(C)$ such that $f(r_1, \dots, r_n) = e_{ij}$. Hence, by (3) we have

$$a'e_{ij}^2 + b'e_{ij}be_{ij} + pe_{ij}c'e_{ij} + ce_{ij}^2q + 2ae_{ij}be_{ij}q + e_{ij}c'e_{ij}q = 0$$

Right and left multiplying by e_{ij} , we obtain $2a_{ji}b_{ji}q_{ji}e_{ij} = 0$. Since $\text{char}(R) \neq 2$, thus we have $a_{ji}b_{ji}q_{ji}e_{ij} = 0$. This gives a contradiction, since a, b and q have all non-zero entries. Thus, we conclude that either a or b or q is central. $\square$

Proposition 3. *Let $R = M_m(C)$ be the ring of all matrices over the field C with $\text{char}(R) \neq 2$ and $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C and $a, b, p, q, a', b', c' \in R$. If*

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$ then either a or b or q is central.

Proof. First, if we assume that the case C is infinite, then the conclusions hold by Proposition 2.

Now let C be finite and K be an infinite field which is an extension of the field C . Let $\bar{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \dots, x_n)$ is central-valued on R if and only if it is central-valued on $\bar{R}$. Suppose that the generalized polynomial $Q(r_1, \dots, r_n)$ such that

$$\begin{aligned} Q(r_1, \dots, r_n) &= a'f(r_1, \dots, r_n)^2 + b'f(r_1, \dots, r_n)bf(r_1, \dots, r_n) + \\ &+ pf(r_1, \dots, r_n)c'f(r_1, \dots, r_n) + cf(r_1, \dots, r_n)^2q + \\ &+ 2af(r_1, \dots, r_n)bf(r_1, \dots, r_n)q + f(r_1, \dots, r_n)c'f(r_1, \dots, r_n)q \end{aligned} \quad (4)$$

is a generalized polynomial identity for R .

Moreover, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $r_1, \dots, r_n$. Hence the complete linearization of $Q(r_1, \dots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \dots, r_n, x_1, \dots, x_n)$ in $2n$ indeterminates, moreover

$$\Theta(r_1, \dots, r_n, r_1, \dots, r_n) = 2^n Q(r_1, \dots, r_n).$$

It is clear that the multilinear polynomial $\Theta(r_1, \dots, r_n, x_1, \dots, x_n)$ is a generalized polynomial identity for both R and $\bar{R}$. By assumption $\text{char}(R) \neq 2$ we obtain $Q(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \bar{R}$ and then conclusion follows from Proposition 2. $\square$

Lemma 2. Let R be a prime ring of characteristic different from 2 with right Martindale quotient ring Q_r and extended centroid C , $f(x_1, \dots, x_n)$ a non central multilinear polynomial over C . Suppose that $a, b, p, q, a', b', c' \in R$. If

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$ then either a or b or q is central.

Proof. Let $a \notin C$, $b \notin C$ and $q \notin C$. By hypothesis, we have

$$\begin{aligned} h(x_1, \dots, x_n) &= a'f(x_1, \dots, x_n)^2 + b'f(x_1, \dots, x_n)bf(x_1, \dots, x_n) + \\ &\quad + pf(x_1, \dots, x_n)c'f(x_1, \dots, x_n) + cf(x_1, \dots, x_n)^2q + \\ &\quad + 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n)q + f(x_1, \dots, x_n)c'f(x_1, \dots, x_n)q = 0 \end{aligned} \quad (5)$$

for all $x_1, \dots, x_n \in R$. Since R and Q_r satisfy same generalized polynomial identity (GPI) (see [4]), Q_r satisfies $h(x_1, \dots, x_n) = 0$. Suppose that $h(x_1, \dots, x_n)$ is a trivial GPI for Q_r . Let $T = Q_r *_C C\{x_1, \dots, x_n\}$, the free product of Q_r and $C\{x_1, \dots, x_n\}$, the free C -algebra in noncommuting indeterminates $x_1, \dots, x_n$. Then, $h(x_1, \dots, x_n)$ is the zero element in $T = Q_r *_C C\{x_1, \dots, x_n\}$. Since $a \notin C$, $b \notin C$ and $q \notin C$, the term $2af(x_1, \dots, x_n)bf(x_1, \dots, x_n)q$ appears nontrivially in $h(x_1, \dots, x_n)$. This gives a contradiction.

Next, suppose that $h(x_1, \dots, x_n)$ is a non-trivial GPI for Q_r . In the case C is infinite, we have $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C . Since both Q_r and $Q_r \otimes_C \overline{C}$ are prime and centrally closed [13, Theorems 2.5 and 3.5], we may replace R by Q_r or $Q_r \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$. By Martindale's theorem [22], R is then a primitive ring with nonzero socle $\text{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [17, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C . Assume first that V is finite dimensional over C , that is, $\dim_C V = m$. By density of R , we have $R \cong M_m(C)$. Since $f(r_1, \dots, r_n)$ is not central valued on R , R must be noncommutative and so $m \geq 2$. In this case, by Proposition 3, we get that either a or b or q is in C , a contradiction. If V is infinite dimensional over C , then for any $e^2 = e \in \text{soc}(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since a, b and q are not in C , there exist $h_1, h_2, h_3 \in \text{soc}(R)$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$ and $[q, h_3] \neq 0$. By Litoff's Theorem [15], there exists idempotent $e \in \text{soc}(R)$ such that $ah_1, h_1a, bh_2, h_2b, qh_3, h_3q, h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned} &e\{a'f(ex_1e, \dots, ex_ne)^2 + b'f(ex_1e, \dots, ex_ne)bf(ex_1e, \dots, ex_ne) + \\ &\quad + pf(ex_1e, \dots, ex_ne)c'f(ex_1e, \dots, ex_ne) + cf(ex_1e, \dots, ex_ne)^2q + \\ &\quad + 2af(ex_1e, \dots, ex_ne)bf(ex_1e, \dots, ex_ne)q + f(ex_1e, \dots, ex_ne)c'f(ex_1e, \dots, ex_ne)q\}e, \end{aligned}$$

the subring eRe satisfies

$$\begin{aligned} &ea'ef(x_1, \dots, x_n)^2 + eb'ef(x_1, \dots, x_n)ebef(x_1, \dots, x_n) + \\ &\quad + epef(x_1, \dots, x_n)ec'ef(x_1, \dots, x_n) + ecef(x_1, \dots, x_n)^2eqe + \\ &\quad + 2eaeef(x_1, \dots, x_n)ebef(x_1, \dots, x_n)eqe + f(x_1, \dots, x_n)ec'ef(x_1, \dots, x_n)eqe. \end{aligned}$$

Then by the above finite dimensional case, either eae or ebe or eqe is central element of eRe . This leads a contradiction, since

$$ah_1 = (eae)h_1 = h_1eae = h_1a, \quad bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$$

and $qh_3 = (eqe)h_3 = h_3(eqe) = h_3q$. Thus, we have proved that either a or b or q is in C . $\square$

The following Lemma 3 is consequence of Proposition 5.1 from [8].

Lemma 3. *Let R be a noncommutative prime ring of characteristics different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that F is a generalized derivation defined as $F(x) = ax + xb$ for all $x \in R$ and some $a, b \in Q_r$. Let $f(x_1, \dots, x_n)$ be a multilinear polynomial over C . If $pF(f(r_1, \dots, r_n))f(r_1, \dots, r_n) + F(f(r_1, \dots, r_n))f(r_1, \dots, r_n)q = 0$ for all $(r_1, \dots, r_n) \in R^n$ then one of the following statements holds: 1. $b \in C$ and $a + b = 0$, 2. $q, b \in C$ and $(p + q)(a + b) = 0$, 3. $q \in C$ and $(p + q)a = (p + q)b = 0$, 4. $b \in C$, $f(x_1, \dots, x_n)^2$ is central valued on R and $p(a + b) + (a + b)q = 0$.*

Proof of Proposition 1. From the given hypothesis we have

$$pa^2f(r)^2 + 2paf(r)bf(r) + pf(r)b^2f(r) + a^2f(r)^2q + 2af(r)bf(r)q + f(r)b^2f(r)q = 0.$$

By Lemma 2, we have either $a \in C$ or $b \in C$ or $q \in C$.

Case I. Suppose $a \in C$ then $F(x) = x(a + b)$ and so $F^2(x) = x(a + b)^2$ is a generalized inner derivation. From our hypothesis we get $p(x(a + b)^2)x + (x(a + b)^2)xq = 0$. Application of Lemma 3 implies one of the followings statements: 1. $(a + b)^2 = 0$, 2. $p = -q \in C$.

Case II. Suppose $b \in C$ then $F(x) = (a + b)x$ and so $F^2(x) = (a + b)^2x$. From our hypothesis we get $p((a + b)^2x)x + ((a + b)^2x)xq = 0$. Application of Lemma 3 implies one of the followings statements: 1. $(a + b)^2 = 0$, 2. $p = -q \in C$,

3. $f(x_1, \dots, x_n)^2$ is central valued on R and $p(a + b)^2 + (a + b)^2q = 0$.

Case III. Suppose $q \in C$ then from our hypothesis Q_r satisfies

$$2(p + q)(a^2x^2 + 2axbx + xb^2x) = 0.$$

Since $\text{char} R \neq 2$ we get

$$(p + q)(a^2x^2 + 2axbx + xb^2x) = 0. \quad (6)$$

For $x = 1$ one has

$$(p + q)(a^2 + 2ab + b^2) = 0. \quad (7)$$

Now in (6) replace x by $x + 1$ we obtain

$$(p + q)\{(a^2x^2 + 2axbx + xb^2x) + (a^2 + 2ab + b^2) + 2(a^2 + 2ab + b^2)x - b^2x + 2axb + xb^2\} = 0. \quad (8)$$

Using (6) and (7) in (8) we have $(p + q)\{-b^2x + 2axb + xb^2\} = 0$. Hence, if $b^2 \in C$ then $(p + q)axb = 0$. Therefore by primeness of R we get either $b = 0$ or $(p + q)a = 0$. If $b = 0$ then result follows from case II. If $(p + q)a = 0$ then from (6) we obtain $(p + q)(xb^2)x = 0$ and result follows as of case I.

If $b^2 \notin C$ then $\{1, b, b^2\}$ is linearly independent over C and from

$$\{(p + q)(-b^2x)\}.1 + \{(p + q)2ax\}.b + \{(p + q)x\}.b^2 = 0$$

we get $(p + q)2axb = 0$ and again result follows. $\square$

4. Proof of the Main Theorem.

Proof. By Fact 7 every generalized derivation F on R can be uniquely extended to a generalized derivation of Q_r and thus can be defined on the whole Q_r of the form $F(x) = ax + d(x)$ for all $x \in R$ and for some $a \in Q_r$ and d is a derivation of Q_r . If d is zero or inner derivation

then we are done by Proposition 1. So assume $d \neq 0$ is an outer derivation. Then from our hypothesis we get

$$\begin{aligned} & p(a^2 + d(a))f(x_1, \dots, x_n)^2 + 2pad(f(x_1, \dots, x_n))f(x_1, \dots, x_n) + \\ & + pd^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n) + (a^2 + d(a))f(x_1, \dots, x_n)^2q + \\ & + 2ad(f(x_1, \dots, x_n))f(x_1, \dots, x_n)q + d^2(f(x_1, \dots, x_n))f(x_1, \dots, x_n)q = 0. \end{aligned}$$

Replacing the value of $d(f(x_1, \dots, x_n))$ and $d^2(f(x_1, \dots, x_n))$ from (1) and (2) in above expression we obtain

$$\begin{aligned} & p(a^2 + d(a))f(x_1, \dots, x_n)^2 + 2pa \left[f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n) \right] f(x_1, \dots, x_n) \\ & + p \left[f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) + \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \right. \\ & + \left. \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n) \right] f(x_1, \dots, x_n) + (a^2 + d(a))f(x_1, \dots, x_n)^2q + \\ & + 2a \left[f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n) \right] f(x_1, \dots, x_n)q + \\ & + \left[f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) + \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) + \right. \\ & + \left. \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n) \right] f(x_1, \dots, x_n)q = 0. \end{aligned} \quad (9)$$

By Kharchenko's theorem we replace $d(x_i)$ by y_i and $d^2(x_i)$ by z_i in equation (9) we get

$$\begin{aligned} & p(a^2 + d(a))f(x_1, \dots, x_n)^2 + 2pa \left[f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n) \right] f(x_1, \dots, x_n) + \\ & + p \left[f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, y_i, \dots, x_n) + \sum_i f(x_1, \dots, z_i, \dots, x_n) + \right. \\ & + \left. \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n) \right] f(x_1, \dots, x_n) + (a^2 + d(a))f(x_1, \dots, x_n)^2q + \\ & + 2a \left[f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, y_i, \dots, x_n) \right] f(x_1, \dots, x_n)q + \\ & + \left[f^{d^2}(x_1, \dots, x_n) + 2 \sum_i f^d(x_1, \dots, y_i, \dots, x_n) + \sum_i f(x_1, \dots, z_i, \dots, x_n) + \right. \\ & + \left. \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n) \right] f(x_1, \dots, x_n)q = 0 \end{aligned} \quad (10)$$

for all $x_i, y_i, z_i \in R$ and these are independent variables. In particular, Q_r satisfies the blended component

$$p \sum_i f(x_1, \dots, z_i, \dots, x_n) f(x_1, \dots, x_n) + \sum_i f(x_1, \dots, z_i, \dots, x_n) f(x_1, \dots, x_n)q = 0.$$

Replacing in above expression z_i with $[c, x_i]$, for a fixed element $c \in Q_r \setminus C$ we see that

$$p[c, f(x_1, \dots, x_n)]f(x_1, \dots, x_n) + [c, f(x_1, \dots, x_n)]f(x_1, \dots, x_n)q$$

is a generalized identity for Q_r . Since $c \notin C$, application of Lemma 3 gives $p = -q \in C$. $\square$

Corollary 1. *Let R be a prime ring of characteristic different from 2 with right Martindale quotient ring Q_r and extended centroid C . Let F be a non zero generalized derivation of R and S be the set of evaluations of a non central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C . Let $p \in R$ be such that $pF^2(u)u = 0$ for all $u \in S$. Then for all $x \in R$ one of the following holds:*

1. *there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,*
2. *$p = 0$,*
3. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 = 0$.*

Corollary 2. *Let R be a prime ring of characteristic different from 2 with the right Martindale quotient ring Q_r and the extended centroid C . Let F be a non zero generalized derivation of R and S be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C such that $F^2(u)u \in C$ for all $u \in S$. Then for all $x \in R$ one of the following holds:*

1. *there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,*
2. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $F(x) = ax$ with $a^2 \in C$.*

Corollary 3. *Let R be a prime ring of characteristic different from 2 with the right Martindale quotient ring Q_r and the extended centroid C . Let F and G be two non-zero generalized derivations of R and S be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \dots, x_n)$ over C such that $[F^2(u)u, G(v)v] = 0$ for all $u, v \in S$. Then one of the following holds:*

1. *there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ for all $x \in R$ and $a^2 = 0$,*
2. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $a \in Q_r$ such that $F(x) = ax$ for all $x \in R$ with $a^2 \in C$,*
3. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exists $c \in C$ such that $G(x) = cx$ for all $x \in R$,*
4. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exist $a \in Q_r, p \in Q_r \setminus C$ such that $F(x) = ax$ for all $x \in R$ with $[p, a^2] = 0$.*

Proof. If $F^2(u)u \in C$ then from Corollary 2 we get our conclusions (1) or (2). If $G(u)u \in C$ then from [9] we get our conclusion (3). If $G(u)u \notin C$ then there exists $p \notin C$ such that $G(v)v = p$ for all $v \in S$ and $[F^2(u)u, p] = 0$ then from Main Theorem we get our conclusions (1) or (4). $\square$

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