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GENERALIZED DERIVATIONS OF ORDER 2 ON MULTILINEAR POLYNOMIALS IN PRIME RINGS


Let $R$ be a prime ring of characteristic different from 2 with a right Martindale quotient ring $Q_r$ and an extended centroid $C$. Let $F$ be a non zero generalized derivation of $R$ and $S$ be the set of evaluations of a non central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$. Let $p, q \in R$ be such that
\[
pF^2(u) + F^2(u)q = 0 \text{ for all } u \in S.
\]
Then for all $x \in R$ one of the followings holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,
2. $p = -q \in C$,
3. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 + a^2q = 0$.

1. Introduction. Throughout the article we suppose that $R$ is an associative ring. A ring $R$ is said to be prime if for $x, y \in R$ and $xRy = 0$ implies either $x = 0$ or $y = 0$. Let $Q_r$ denotes the right Martindale’s ring of quotient of a prime ring $R$ and $C$ denotes the center of $Q_r$. The ring $Q_r$ is prime with unity containing $R$ and $C$ is a field which is known as the extended centroid of $R$ (for more details see [2]). An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is said to generalized derivation associative with a derivation $d$ if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. For $a \in R$, the mapping $d(x) = [a, x]$ for all $x \in R$, is a derivation, which is said to be an inner derivation induced by $a$. If a derivation is not an inner derivation it is called an outer derivation. A generalized derivation $F$ is said to be generalized inner derivation if its associated derivation $d$ is inner, otherwise $F$ is called an outer derivation.

By [20] we know that every generalized derivation $F$ on a dense right ideal of $R$ can be uniquely extended to $Q_r$ and hence any generalized derivation $F$ of $R$ can be implicitly assumed to be defined on the whole $Q_r$ and has the form $F(x) = qx + d(x)$ for all $x \in R$, where $q$ is a fixed element of $Q_r$ and $d$ is an associated derivation of $F$. For $b \in Q_r$ if $d(x) = [b, x]$ for all $x \in R$ then $d$ is said to be $X$-inner derivation otherwise $X$-outer derivation. The notion of generalized derivation has been introduced by Brešar in [3]. Such maps are extensively studied in ring theory and theory of operator algebras. Further the study of generalized derivations are extended to Banach algebras (see [1, 25, 26]).

In 1957, Posner [24] proved that if $d$ is a non zero derivation on a prime ring $R$ such that $[d(x), x] \in Z(R)$ for all $x \in R$ then $R$ is a commutative ring. Further more generalization

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of Posner’s result can be found in several ways (see [5, 6, 23] where further references can be found). In fact, it was the starting point of much research article concerning the study of various kind of additive mappings satisfying appropriate algebraic conditions on some subset of a prime or semiprime ring \( R \). In [16], Hvala extended the Posner’s result by replacing derivations with generalized derivations. Recently more results on composition of generalized derivations on prime rings have been found in [10, 11, 12].

In [27], Tiwari et al. state as “Let \( R \) be a prime ring of characteristic different from 2 with the Utumi quotient ring \( U \) and the extended centroid \( C \), \( f(x_1, \ldots, x_n) \) be a multilinear polynomial over \( C \), which is not central valued on \( R \). Suppose that \( d \) is a non-zero derivation of \( R \), \( F \) and \( G \) are two generalized derivations of \( R \) such that \( d\{F(u)u - uG^2(u)\} = 0 \) for all \( u \in f(R) \) and find all possible forms of \( d, F \) and \( G \).” In the same time Eroglu et al. [14] proved the following result:

**Theorem 1.** Let \( R \) be a prime ring with the extended centroid \( C \), \( Q \) maximal right ring of quotients of \( R \), \( RC \) central closure of \( R \) such that \( \dim_{C}(RC) > 4 \), \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( C \) which is not central valued on \( R \) and \( f(R) \) the set of all evaluations of the multilinear polynomial \( f(x_1, \ldots, x_n) \) in \( R \). Suppose that \( G \) is a non-zero generalized derivation of \( R \) such that \( G^2(u)u \in C \) for all \( u \in f(R) \) then one of the followings conditions holds:

(i) there exists \( a \in Q \) such that \( a^2 = 0 \) and either \( G(x) = ax \) for all \( x \in R \) or \( G(x) = xa \) for all \( x \in R \),

(ii) there exists \( a \in Q \) such that \( 0 \neq a^2 \in C \) and either \( G(x) = ax \) for all \( x \in R \) or \( G(x) = xa \) for all \( x \in R \),

(iii) \( \text{char}(R) = 2 \) and one of the followings holds:

i) there exist \( a, b \in Q \) such that \( G(x) = ax + xb \) for all \( x \in R \) and \( a^2 = b^2 \in C \),

ii) there exist \( a, b \in Q \) such that \( G(x) = ax + xb \) for all \( x \in R \) and \( a^2, b^2 \in C \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \),

iii) there exists \( a \in Q \) and \( X \)-outer derivation \( d \) of \( R \) such that \( G(x) = ax + d(x) \) for all \( x \in R \), \( d^2 = 0 \) and \( a^2 + d(a) = 0 \),

iv) there exists \( a \in Q \) and \( X \)-outer derivation \( d \) of \( R \) such that \( G(x) = ax + d(x) \) for all \( x \in R \), \( d^2 = 0 \) and \( a^2 + d(a) \in C \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \).

More recently, in [8] Filippis et al. proved the following result.

**Theorem 2.** Let \( R \) be a prime ring of characteristic different from 2, \( Q_r \) be its right Martindale quotient ring and \( C \) be its extended centroid, \( G \) be a non-zero \( X \)-generalized skew derivation of \( R \), and \( S \) be the set of the evaluations of a multilinear polynomial \( f(x_1, \ldots, x_n) \) over \( C \) with \( n \) non-commuting variables. Let \( u, v \in R \) be such that \( uG(x)x + G(x)xy = 0 \) for all \( x \in S \). Then one of the following statements holds:

(a) \( v \in C \) and there exist \( a, b, c \in Q_r \) such that \( G(x) = ax + bxc \) for any \( x \in R \) with \( (u + v)a = (u + v)b = 0 \),

(b) \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \) and there exists \( a \in Q_r \) such that \( G(x) = ax \) for all \( x \in R \) with \( ua + av = 0 \).

Motivated by Theorem 1 and Theorem 2 we prove the following main theorem.
Theorem (Main Theorem). Let $R$ be a prime ring of characteristic different from 2 with the right Martindale quotient ring $Q_r$ and the extended centroid $C$. Let $F$ be a non-zero generalized derivation of $R$ and $S$ be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$. Let $p, q \in R$ be such that $pF^2(u)u + F^2(u)uq = 0$ for all $u \in S$. Then for all $x \in R$ one of the followings holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,
2. $p = -q \in C$,
3. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 + a^2q = 0$.

2. Preliminaries. In all that follows, $R$ always denotes a prime ring and $Q_r$ its right Martindale’s ring of quotients, $C$ is the center of $Q_r$, $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $C$. The definition and axiomatic formulation of right Martindale ring $Q_r$ can be found in [2] and [4].

We have the following properties which we need:

1. $R \subseteq Q_r$;
2. $Q_r$ is a prime ring with identity;
3. The center of $Q_r$ is denoted by $C$ and is called the extended centroid of $R$. $C$ is a field.

Moreover, we will use frequently some important theory of generalized polynomial identities and differential identities. We recall some facts.

Fact-1: Let $X = \{x_1, x_2, \ldots\}$, the countable set consisting of the non-commuting indeterminates $x_1, x_2, \ldots$ Let $C\{X\}$ be the free algebra over $C$ in the set $X$. We denote $T = Q_r \ast_C C\{X\}$, the free product of the $C$-algebra $Q_r$ and $C\{X\}$. The elements of $T$ are called the generalized polynomials with coefficients in $Q_r$. Let $B$ be a set of $C$-independent vectors of $Q_r$. Then any element $f \in T$ can be represented in this form $f = \sum \alpha_i m_i$, where $\alpha_i \in C$ and $m_i$ are $B$-monomials of the form $q_0y_1y_2\cdots y_{n}q_n$, with $q_0, q_1, \ldots, q_n \in B$ and $y_1, y_2, \ldots, y_n \in X$. Any generalized polynomial $f = \sum \alpha_i m_i$ is trivial i.e., zero element in $T$ if and only if $\alpha_i = 0$ for each $i$. For detail study we refer to [4].

Fact-2: Every derivation $d$ of $R$ can be uniquely extended to a derivation of $Q_r$ ([2]).

Fact-3: If $I$ is a two-sided ideal of $R$, then $R$, $I$ and $Q_r$ satisfy the same differential identities ([19]).

Fact-4: If $I$ is a two-sided ideal of $R$, then $R$, $I$ and $Q_r$ satisfies the same generalized polynomial identities with coefficients in $Q_r$ ([2]).

Fact-5: (Kharchenko [18, Theorem 2]) Let $R$ be a prime ring, $d$ a non-zero derivation on $R$ and $I$ a non-zero ideal of $R$. If $I$ satisfies the differential identity

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \quad \text{for any } r_1, \ldots, r_n \in I,$$

then either

(i) $I$ satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, x_1, \ldots, x_n) = 0$

or

(ii) $d$ is $Q_r$-inner i.e., for some $q \in Q_r$, $d(x) = [q, x]$ and $I$ satisfies the generalized polynomial identity $f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0$.

Fact-6: We shall use the following notation:

$$f(x_1, \ldots, x_n) = x_1x_2\cdots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_{\sigma} x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$$
for some $\alpha_\sigma \in C$ and the symmetric group $S_n$ of $n$ symbols.

**Fact-7:** Every generalized derivation $F$ on ring $R$ can be extended to $Q_r$ and assumes the form $F(x) = ax + db$ for some $a \in Q_r$ and $d$ is a derivation on $Q_r$.

Let $d$ be a derivation. We denote by $f^d(x_1, \ldots, x_n)$ and $f^{d^2}(x_1, \ldots, x_n)$ the polynomials obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $\alpha_\sigma$ with $d(\alpha_\sigma)$ and $d^2(\alpha_\sigma)$, respectively. Then we have

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n),$$

$$d^2(f(x_1, \ldots, x_n)) = f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) +$$

$$+ \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n).$$

3. **The case when $F$ is an inner generalized derivation.** First, we study the situation when $F$ is an inner generalized derivation, that is, $F(x) = ax + xb$ for all $x \in R$, for some $a, b \in Q_r$. Then $pF^2(f(r))f(r) + F^2(f(r))f(r)q = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$ implies

$$pa^2f(r)^2 + 2pa\bar{f}(r)bf(r) + pf(r)b^2f(r) + a^2f(r)^2q + 2af(r)bf(r)q + f(r)b^2f(r)q = 0.$$ 

This gives

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, where $a' = pa^2, b' = 2pa, c = a^2, c' = b^2$.

In this section we prove the following proposition.

**Proposition 1.** Let $R$ be a prime ring of characteristic different from 2 with the right Martindale quotient ring $Q_r$ and the extended centroid $C$. Let $S$ be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$. Suppose that $F$ is a generalized inner derivation of $R$ defined as $F(x) = ax + xb$ for all $x \in R$ such that $pF^2(u)u + F^2(u)uq = 0$ for all $u \in f(R)$ and for some $p, q \in Q_r$. Then either $F(x) = (a+b)x$ or $F(x) = x(a+b)$, $a+b \in Q_r$ and one of the followings holds:

1. $(a+b)^2 = 0$,
2. $p = -q \in C$,
3. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $p(a+b)^2 + (a+b)^2q = 0$.

To prove Proposition 1 we need the following.

**Lemma 1** ([7], Lemma 1). Let $C$ be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that the matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ have all non-zero entries.

**Proposition 2.** Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field $C$, $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over $C$ and $a, b, p, q, a', b', c' \in R$. If

$$a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$ then either $a$ or $b$ or $q$ is central.
Proposition 3. By our assumption, $R$ satisfies the generalized identity
\begin{align*}
a'f(x_1, \ldots, x_n)^2 + b'f(x_1, \ldots, x_n)bf(x_1, \ldots, x_n) + \\
+ pf(x_1, \ldots, x_n)c'f(x_1, \ldots, x_n) + cf(x_1, \ldots, x_n)^2q + \\
+ 2af(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)q + f(x_1, \ldots, x_n)c'f(x_1, \ldots, x_n)q = 0. \tag{3}
\end{align*}
We assume that $a \notin Z(R)$, $b \notin Z(R)$ and $q \notin Z(R)$. Then we shall prove that this case leads to a contradiction.

Since $a \notin Z(R)$, $b \notin Z(R)$ and $q \notin Z(R)$, by Lemma 1 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ and $q_1 = \phi(q)$ have all non-zero entries. Clearly $a_1$, $b_1$, $q_1$, $p_1 = \phi(p)$, $a'_1 = \phi(a')$, $b'_1 = \phi(b')$ and $c'_1 = \phi(c')$ must satisfy the condition (3). Without loss of generality we may replace $a, b, p, q, a', b', c'$ with $a_1, b_1, p_1, q_1, a'_1, b'_1, c'_1$ respectively.

Here $e_{ij}$ denotes the matrix whose $(i, j)$-entry is 1 and rest entries are zero. Since $f(x_1, \ldots, x_n)$ is not central, by [19] (see also [21]), there exist $u_1, \ldots, u_n \in M_m(C)$ and $\gamma \in C \setminus \{0\}$ such that $f(u_1, \ldots, u_n) = \gamma e_{st}$, with $s \neq t$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all $C$-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence, by (3) we have
\begin{align*}
a'\gamma e_{ij}^2 + b'\gamma e_{ij}be_{ij} + p\gamma e_{ij}c'\gamma e_{ij} + ce_{ij}^2q + 2a\gamma e_{ij}be_{ij}q + e_{ij}c'\gamma e_{ij}q = 0
\end{align*}
Right and left multiplying by $e_{ij}$, we obtain $2a_{ji}b_{ji}q_{ji}e_{ij} = 0$. Since char($R$) $\neq 2$, thus we have $a_{ji}b_{ji}q_{ji}e_{ij} = 0$. This gives a contradiction, since $a$, $b$ and $q$ have all non-zero entries. Thus, we conclude that either $a$ or $b$ or $q$ is central. \hfill $\square$

Proposition 3. Let $R = M_n(C)$ be the ring of all matrices over the field $C$ with char($R$) $\neq 2$ and $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over $C$ and $a, b, p, q, a', b', c' \in R$. If
\begin{align*}
a'f(r)^2 + b'f(r)bf(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)bf(r)q + f(r)c'f(r)q = 0
\end{align*}
for all $r = (r_1, \ldots, r_n) \in R^n$ then either $a$ or $b$ or $q$ is central.

Proof. First, if we assume that the case $C$ is infinite, then the conclusions hold by Proposition 2.

Now let $C$ be finite and $K$ be an infinite field which is an extension of the field $C$. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$ if and only if it is central-valued on $\overline{R}$. Suppose that the generalized polynomial $Q(r_1, \ldots, r_n)$ such that
\begin{align*}
Q(r_1, \ldots, r_n) = a'f(r_1, \ldots, r_n)^2 + b'f(r_1, \ldots, r_n)bf(r_1, \ldots, r_n) + \\
+ pf(r_1, \ldots, r_n)c'f(r_1, \ldots, r_n) + cf(r_1, \ldots, r_n)^2q + \\
+ 2af(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)q + f(r_1, \ldots, r_n)c'f(r_1, \ldots, r_n)q
\end{align*}
is a generalized polynomial identity for $R$.

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Hence the complete linearization of $Q(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ in $2n$ indeterminates, moreover
\begin{align*}
\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n) = 2^nQ(r_1, \ldots, r_n).
\end{align*}
It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for both $R$ and $\overline{R}$. By assumption char($R$) $\neq 2$ we obtain $Q(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 2. \hfill $\square$
Lemma 2. Let $R$ be a prime ring of characteristic different from 2 with right Martindale quotient ring $Q_r$ and extended centroid $C$, $f(x_1, \ldots, x_n)$ a non central multilinear polynomial over $C$. Suppose that $a, b, p, q, a', b', c' \in R$. If

$$a'f(r)^2 + b'f(r)b(f(r) + pf(r)c'f(r) + cf(r)^2q + 2af(r)b(f(r)q + f(r)c'f(r)q = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$ then either $a$ or $b$ or $q$ is central.

Proof. Let $a \notin C$, $b \notin C$ and $q \notin C$. By hypothesis, we have

$$h(x_1, \ldots, x_n) = a'f(x_1, \ldots, x_n)^2 + b'f(x_1, \ldots, x_n)b(f(x_1, \ldots, x_n) +$$

$$+ pf(x_1, \ldots, x_n)c'f(x_1, \ldots, x_n) + cf(x_1, \ldots, x_n)^2q +$$

$$+ 2af(x_1, \ldots, x_n)b(f(x_1, \ldots, x_n)q + f(x_1, \ldots, x_n)c'f(x_1, \ldots, x_n)q = 0 \quad (5)$$

for all $x_1, \ldots, x_n \in R$. Since $R$ and $Q_r$ satisfy same generalized polynomial identity (GPI) (see [4]), $Q_r$ satisfies $h(x_1, \ldots, x_n) = 0$. Suppose that $h(x_1, \ldots, x_n)$ is a trivial GPI for $Q_r$. Let $T = Q_r \ast_C C\{x_1, \ldots, x_n\}$, the free product of $Q_r$ and $C\{x_1, \ldots, x_n\}$, the free $C$-algebra in noncommuting indeterminates $x_1, \ldots, x_n$. Then, $h(x_1, \ldots, x_n)$ is the zero element in $T = Q_r \ast_C C\{x_1, \ldots, x_n\}$. Since $a \notin C$, $b \notin C$ and $q \notin C$, the term $2af(x_1, \ldots, x_n)b(f(x_1, \ldots, x_n)q$ appears nontrivially in $h(x_1, \ldots, x_n)$. This gives a contradiction.

Next, suppose that $h(x_1, \ldots, x_n)$ is a non-trivial GPI for $Q_r$. In the case $C$ is infinite, we have $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $Q_r$ and $Q_r \otimes_C \overline{C}$ are prime and centrally closed [13, Theorems 2.5 and 3.5], we may replace $R$ by $Q_r$ or $Q_r \otimes_C \overline{C}$ according to $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $h(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$. By Martindale’s theorem [22], $R$ is then a primitive ring with nonzero socle $soc(R)$ and with $C$ as its associated division ring. Then, by Jacobson’s theorem [17, p.75], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Assume first that $V$ is finite dimensional over $C$, that is, $\dim_C V = m$. By density of $R$, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on $R$, $R$ must be noncommutative and so $m \geq 2$. In this case, by Proposition 3, we get that either $a$ or $b$ or $q$ is in $C$, a contradiction. If $V$ is infinite dimensional over $C$, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since $a, b$ and $q$ are not in $C$, there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$ and $[q, h_3] \neq 0$. By Litoff’s Theorem [15], there exists idempotent $e \in soc(R)$ such that $ah_1, h_1a, bh_2, h_2b, qh_3, h_3q, h_1, h_2, h_3 \in eRe$. Since $R$ satisfies generalized identity

$$e\{a'f(x_1 \ldots, x_n)e^2 + b'f(x_1 \ldots, x_n)e)bf(x_1 \ldots, x_n)c'f(x_1 \ldots, x_n)e +$$

$$+ pf(x_1 \ldots, x_n)ce(x_1 \ldots, x_n) + cf(x_1 \ldots, x_n)^2q +$$

$$+ 2af(x_1 \ldots, x_n)b(x_1 \ldots, x_n)q + f(x_1 \ldots, x_n)c'f(x_1 \ldots, x_n)q\}e,$$

the subring $eRe$ satisfies

$$ea'f(x_1 \ldots, x_n)^2 + eb'f(x_1 \ldots, x_n)e)bf(x_1 \ldots, x_n) +$$

$$+ epef(x_1 \ldots, x_n)e)ce(x_1 \ldots, x_n) + eef(x_1 \ldots, x_n)^2e +$$

$$+ 2aeaf(x_1 \ldots, x_n)ebf(x_1 \ldots, x_n)e + f(x_1 \ldots, x_n)e)ce(x_1 \ldots, x_n)e.$$

Then by the above finite dimensional case, either $eae$ or $ebe$ or $ege$ is central element of $eRe$. This leads a contradiction, since

$$ah_1 = (eae)h_1 = h_1a = h_1a, bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$$

and $gh_3 = (ege)h_3 = h_3(ege) = h_3g$. Thus, we have proved that either $a$ or $b$ or $q$ is in $C$. \qed
The following Lemma 3 is consequence of Proposition 5.1 from [8].

**Lemma 3.** Let $R$ be a noncommutative prime ring of characteristics different from 2, $Q_r$ be its right Martindale quotient ring and $C$ be its extended centroid. Suppose that $F$ is a generalized derivation defined as $F(x) = ax + xb$ for all $x \in R$ and some $a, b \in Q_r$. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$. If $pF(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) + F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n)q = 0$ for all $(r_1, \ldots, r_n) \in R^n$ then one of the following statements holds: 1. $b \in C$ and $a + b = 0$, 2. $q, b \in C$ and $(p + q)(a + b) = 0$, 3. $q \in C$ and $(p + q)a = (p + q) = 0$, 4. $b \in C$, $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $p(a + b) + (a + b)q = 0$.

**Proof of Proposition 1.** From the given hypothesis we have
\[
pa^2f(r)^2 + 2paf(r)bf(r) + pf(r)b^2f(r) + a^2f(r)^2q + 2af(r)bf(r)q + f(r)b^2f(r)q = 0.
\]
By Lemma 2, we have either $a \in C$ or $b \in C$ or $q \in C$.

**Case I.** Suppose $a \in C$ then $F(x) = x(a + b)$ and so $F^2(x) = x(a + b)^2$ is a generalized inner derivation. From our hypothesis we get $p(x(a + b)^2)x + (x(a + b)^2)xq = 0$. Application of Lemma 3 implies one of the followings statements: 1. $(a + b)^2 = 0$, 2. $p = −q \in C$.

**Case II.** Suppose $b \in C$ then $F(x) = (a + b)x$ and so $F^2(x) = (a + b)^2x$. From our hypothesis we get $p((a + b)^2)x + ((a + b)^2)xq = 0$. Application of Lemma 3 implies one of the followings statements: 1. $(a + b)^2 = 0$, 2. $p = −q \in C$.

**Case III.** Suppose $q \in C$ then from our hypothesis $Q_r$ satisfies
\[
2(p + q)(a^2x^2 + 2axbx + xb^2x) = 0.
\]
Since $\text{char}R \neq 2$ we get
\[
(p + q)(a^2x^2 + 2axbx + xb^2x) = 0. \tag{6}
\]
For $x = 1$ one has
\[
(p + q)(a^2 + 2ab + b^2) = 0. \tag{7}
\]
Now in (6) replace $x$ by $x + 1$ we obtain
\[
(p + q)((a^2x^2 + 2axbx + xb^2x) + (a^2 + 2ab + b^2) + 2(a^2 + 2ab + b^2)x - b^2x + 2axb + xb^2 = 0. \tag{8}
\]
Using (6) and (7) in (8) we have $(p + q)((−b^2x + 2axb + xb^2) = 0$. Hence, if $b^2 \in C$ then $(p + q)axb = 0$. Therefore by primeness of $R$ we get either $b = 0$ or $(p + q)a = 0$. If $b = 0$ then result follows from case II. If $(p + q)a = 0$ then from (6) we obtain $(p + q)(xb^2)x = 0$ and result follows as of case I.

If $b^2 \notin C$ then $\{1, b, b^2\}$ is linearly independent over $C$ and from
\[
\{(p + q)(−b^2x)\}.1 + \{(p + q)2ax\}.b + \{(p + q)x\}.b^2 = 0
\]
we get $(p + q)2axb = 0$ and again result follows.

4. **Proof of the Main Theorem.**

**Proof.** By Fact 7 every generalized derivation $F$ on $R$ can be uniquely extended to a generalized derivation of $Q_r$ and thus can be defined on the whole $Q_r$ of the form $F(x) = ax + d(x)$ for all $x \in R$ and for some $a \in Q_r$ and $d$ is a derivation of $Q_r$. If $d$ is zero or inner derivation
then we are done by Proposition 1. So assume \( d \neq 0 \) is an outer derivation. Then from our hypothesis we get

\[
p(a^2 + d(a))f(x_1, \ldots, x_n)^2 + 2pd(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) + \\
+ pd^2(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) + (a^2 + d(a))f(x_1, \ldots, x_n)^2q + \\
+ 2ad(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)q + d^2(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)q = 0.
\]

Replacing the value of \( d(f(x_1, \ldots, x_n)) \) and \( d^2(f(x_1, \ldots, x_n)) \) from (1) and (2) in above expression we obtain

\[
p(a^2 + d(a))f(x_1, \ldots, x_n)^2 + 2pa\left[f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)\right]f(x_1, \ldots, x_n) + \\
+ p\left[f^d(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) + \\
+ \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n)\right]f(x_1, \ldots, x_n) + (a^2 + d(a))f(x_1, \ldots, x_n)^2q + \\
+ 2a\left[f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)\right]f(x_1, \ldots, x_n)q + \\
+ \left[f^{dd}(x_1, \ldots, x_n) + 2\sum_i f^{dd}(x_1, \ldots, d(x_i), \ldots, x_n) + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) + \\
+ \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n)\right]f(x_1, \ldots, x_n)q = 0. \quad (9)
\]

By Kharchenko’s theorem we replace \( d(x_i) \) by \( y_i \) and \( d^2(x_i) \) by \( z_i \) in equation (9) we get

\[
p(a^2 + d(a))f(x_1, \ldots, x_n)^2 + 2pa\left[f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)\right]f(x_1, \ldots, x_n) + \\
+ p\left[f^d(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, z_i, \ldots, x_n) + \\
+ \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n)\right]f(x_1, \ldots, x_n) + (a^2 + d(a))f(x_1, \ldots, x_n)^2q + \\
+ 2a\left[f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)\right]f(x_1, \ldots, x_n)q + \\
+ \left[f^{dd}(x_1, \ldots, x_n) + 2\sum_i f^{dd}(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, z_i, \ldots, x_n) + \\
+ \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n)\right]f(x_1, \ldots, x_n)q = 0. \quad (10)
\]

for all \( x_i, y_i, z_i \in R \) and these are independent variables. In particular, \( Q_r \) satisfies the blended component

\[
p \sum f(x_1, \ldots, z_i, \ldots, x_n)f(x_1, \ldots, x_n) + \sum f(x_1, \ldots, z_i, \ldots, x_n)f(x_1, \ldots, x_n)q = 0.
\]

Replacing in above expression \( z_i \) with \( [c, x_i] \), for a fixed element \( c \in Q_r \setminus C \) we see that

\[
p[c, f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n) + [c, f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n)q
\]

is a generalized identity for \( Q_r \). Since \( c \notin C \), application of Lemma 3 gives \( p = -q \in C \). □
Corollary 1. Let $R$ be a prime ring of characteristic different from 2 with right Martindale quotient ring $Q_r$ and extended centroid $C$. Let $F$ be a non zero generalized derivation of $R$ and $S$ be the set of evaluations of a non central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$. Let $p \in R$ be such that $pF^2(u)u = 0$ for all $u \in S$. Then for all $x \in R$ one of the following holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,
2. $p = 0$,
3. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $a \in Q_r$ such that $F(x) = ax$ with $pa^2 = 0$.

Corollary 2. Let $R$ be a prime ring of characteristic different from 2 with the right Martindale quotient ring $Q_r$ and the extended centroid $C$. Let $F$ be a non zero generalized derivation of $R$ and $S$ be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$ such that $F^2(u)u \in C$ for all $u \in S$. Then for all $x \in R$ one of the following holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ and $a^2 = 0$,
2. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $a \in Q_r$ such that $F(x) = ax$ with $a^2 \in C$.

Corollary 3. Let $R$ be a prime ring of characteristic different from 2 with the right Martindale quotient ring $Q_r$ and the extended centroid $C$. Let $F$ and $G$ be two non-zero generalized derivations of $R$ and $S$ be the set of evaluations of a non-central valued multilinear polynomial $f(x_1, \ldots, x_n)$ over $C$ such that $[F^2(u), G(v)v] = 0$ for all $u, v \in S$. Then one of the following holds:

1. there exists $a \in Q_r$ such that $F(x) = ax$ or $F(x) = xa$ for all $x \in R$ and $a^2 = 0$,
2. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $a \in Q_r$ such that $F(x) = ax$ for all $x \in R$ with $a^2 \in C$,
3. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $c \in C$ such that $G(x) = cx$ for all $x \in R$,
4. $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $a \in Q_r, p \in Q_r \setminus C$ such that $F(x) = ax$ for all $x \in R$ with $[p, a^2] = 0$.

Proof. If $F^2(u)u \in C$ then from Corollary 2 we get our conclusions (1) or (2). If $G(u)u \in C$ then from [9] we get our conclusion (3). If $G(u)u \notin C$ then there exists $p \notin C$ such that $G(v)v = p$ for all $v \in S$ and $[F^2(u), p] = 0$ then from Main Theorem we get our conclusions (1) or (4).

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