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CONTINUED $\mathbf{A_2}$-FRACTIONS AND SINGULAR FUNCTIONS

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In the article we deepen the metric component of theory of infinite A_2-continued fractions $\dfrac{1}{a_1+\frac{1}{a_2+\frac{1}{\ddots}}} \equiv [0; a_1, a_2, ..., a_n, ...]$ with a two-element alphabet $A_2 = \{\frac{1}{2}, 1\}$, $a_n \in A_2$, and establish the normal property of numbers of the segment $I = [\frac{1}{2}; 1]$ in terms of their A_2-representations: $x = [0; a_1, a_2, ..., a_n, ...]$. It is proved that almost all (in the sense of the Lebesgue measure) numbers of segment I in their A_2-representations use each of the tuples of elements of the alphabet of arbitrary length as consecutive digits of the representation infinitely many times.

We consider a function f defined by equality $f(x = [0; a_1, a_2, ..., a_n, ...]) = e^{\sum\limits_{n=1}^{\infty}(2a_n-1)v_n}$, where $v_1 + v_2 + ... + v_n + ...$ is a given absolutely convergent series. For the function f, structural and functional relationships are indicated as well as necessary and sufficient conditions for continuity (which are: $v_n = \frac{v_1(-1)^{n-1}}{2^{n-1}}$, $v_1 \in \mathbb{R}$) and monotonicity are found. In the case of the continuity of the function f, we give the expression of its derivative and prove the singularity (the equality of the derivative to zero almost everywhere in the sense of the Lebesgue measure) using the above-mentioned normal property of numbers in terms of their A_2-representation. The relation between this new strictly monotonic singular function and the classical strictly increasing Minkowski question-mark function is indicated.

Introduction. A continuous function is called a *singular function* if it is not constant and has a derivative which is equal to zero almost everywhere (in the sense of the Lebesgue measure). More than a hundred years ago, the first examples of singular functions and singular probability distributions were published. They appeared against the background of different scientific interests, in various branches of mathematics (they are set theory and the Cantor function [9], number theory and the Minkowski function [6, 9], function theory and the Sierpiński function [16], the theory of infinite Bernoulli convolutions, and a cascade of different functions). The newly created theory of the Lebesgue measure (1902) became a theoretical basis and a powerful tool of developing the theory of such functions. At the same time, there is still no general theory of singular functions; individual theories have been developing fragmentary for a long time. The first significant outbreak of interest in singular probability distributions, their characteristic functions, the Fourier transform took place in the 30-40s of the previous century [5, 9]. The Jessen-Wintner theorem on the

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Lebesgue purity of the distribution of the sum with probability 1 of a convergent series whose elements are independent discretely distributed random variables [4] was fundamental in this regard. Many continuum classes of singular functions related to different encoding systems (representations) of real numbers have been studied today [9]. Monotonic [15], non-monotonic [13] and nowhere monotonic [11, 14] are among them.

In 1911, H. Minkowski [6] gave an original definition of continuous strictly increasing function in terms of Farey fractions and the operation "mediant sum of fractions". This function establishes a one-to-one correspondence between all quadratic irrationalities of the segment $[0;1]$ and rational numbers of the same segment (this is the so-called "question-mark function" $?(x)$). In 1938, Denjoy [2] was the first to prove the singularity of the Minkowski function. In 1943, Salem found an analytical representation of this function in terms of elementary continued fractions:

$$?(x=[0;_1, a_2, ..., a_n, ...]) = 2^{1-a_1} - 2^{1-a_1-a_2} + ... + (-1)^{n+1} 2^{1-a_1-a_2-...-.a_n} + ...,$$

$a_n \in \mathbb{N}$, and gave own proof of singularity.

Today it is known that the Minkowski function is the unique continuous solution of the system of two functional equations:

$$f\Big(\frac{x}{1+x}\Big) = \frac{1}{2} f(x), \quad f(1-x) = 1 - f(x).$$

The Minkowski function allows various generalizations. Since $?(x)$ is the distribution function of the random variable $\xi = [0; \xi_1, \xi_2, ..., \xi_k, ...]$, the elements of the continued representation of which are independent and identically distributed random variables taking the values $1, 2, ..., k, ...$ with probabilities $p_1 = 2^{-1}$, $p_2 = 2^{-2}$,...,$p_k = 2^{-k}$, ... respectively, we have one of the simplest generalizations of the Minkowski function by replacing probabilities with $p_k = (1-q)q^{k-1}$, where $q \in (0;1)$. In the papers [1, 10], a more general construction of the random variable ξ, when the digits ξ_k are independent and not identically distributed, was studied. As a result, another generalization of the Minkowski function was obtained.

In the paper [12], a one-parameter generalization φ_μ of the Minkowski function is constructed as a unique continuous solution of the system of functional equations

$$\begin{cases} \varphi_\mu(\frac{x}{1+x}) = 1 - \mu\varphi_\mu(x), \\ \varphi_\mu(1-x) = 1 - \varphi_{1-\mu}(x), \end{cases} \qquad \mu \in (0;1).$$

The function φ_μ is singular strictly increasing for each value μ, moreover $\varphi_{\frac{1}{2}}(x) = ?(x)$.

This article is devoted to a new singular function defined in terms of non-elementary continued fractions, the elements of which are the numbers $\frac{1}{2}$ and 1.

1. Basic concepts and facts of the theory of continued A_2-fractions. Let $A \equiv \{0; 1\}$ be an alphabet, $L \equiv A \times A \times \ldots$ be a space of sequences of elements of the alphabet (zeros and ones); $A_2 \equiv \{\frac{1}{2}; 1\}$, $L_2 \equiv A_2 \times A_2 \times \ldots$.

It is known [3] that, for any $x \in [\frac{1}{2}; 1]$, there exists a sequence $(a_k) \in L_2$ such that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \equiv [0; a_1, a_2, \ldots, a_k, \ldots]. \tag{1}$$

The infinite continued fraction (1) is called A_2-*continued fraction* (A_2-fraction), and its symbolic notation $[0; a_1, ..., a_k, ...]$ is called A_2-representation of number x. There are numbers that have two A_2-representation, because $[0; a_1, ..., a_m, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, ..., a_m, 1, (1, \frac{1}{2})]$ (here the parentheses mean the period). We say that these numbers are A_2-*binary*. The remaining numbers have a single A_2-representation. We say that these numbers are A_2-*unary*.

The theory of A_2-continued fractions started in the work [3]. We will recall the key concepts and facts of this theory, which we will use further.

A_2-representation is easily recoded by alphabet A, that is:

$$x = [0; a_1, a_2, \ldots, a_k, \ldots] \equiv \Delta^A_{\alpha_1\alpha_2\ldots\alpha_k\ldots},$$

where $\alpha_k = 2a_k - 1, k \in \mathbb{N}$. It is called A-*representation of number* x.

Definition 1. A-*cylinder of rank* m *with a base* $c_1...c_m$ is a set

$$\Delta^A_{c_1c_2\ldots c_m} = \{x : x = \Delta^A_{c_1c_2\ldots c_m\beta_1\ldots\beta_n\ldots}, (\beta_n) \in L\}$$

of all numbers $x \in [\frac{1}{2}; 1]$ such that the first m digits of their representation are equal to $c_1, c_2, ..., c_m$ respectively.

Clearly, $\Delta^A_{c_1\ldots c_m} = \Delta^A_{c_1\ldots c_m0} \cup \Delta^A_{c_1\ldots c_m1}$; $[\frac{1}{2}; 1] = \bigcup\limits_{c_1\in A} \ldots \bigcup\limits_{c_m\in A} \Delta^A_{c_1\ldots c_m}$;

1) A-cylinder $\Delta^A_{c_1\ldots c_m}$ is a segment with endpoints: $\Delta^A_{c_1\ldots c_m(01)}$ and $\Delta^A_{c_1\ldots c_m(10)}$, and endpoint is left or right if the number m is even or odd;
2) the length of the cylinder $\Delta^A_{c_1\ldots c_m}$ is calculated by the formula

$$|\Delta^A_{c_1\ldots c_m}| = \frac{1}{(q_{m-1} + q_m)(q_{m-1} + 2q_m)} \leq \frac{1}{q^2_{m-1}},$$

   where q_m is a denominator of rank m of convergent of a continued A_2-fraction, i.e., the denominator of a rational number, which is the value of the expression $[0; a_1, a_2, ..., a_m]$, that is calculated by formulas $q_0 = 1$, $q_1 = a_1$, $q_{n+1} = a_{n+1}q_n + q_{n-1}$, where $a_n = \frac{c_n+1}{2}$;
3) basic metric relation for A-representation of numbers is calculated by formulas:

$$\frac{|\Delta^A_{c_1\ldots c_mc}|}{|\Delta^A_{c_1\ldots c_m}|} = \frac{1 + a\frac{q_{m-1}}{q_m}}{2a^2 + 1 + 2a\frac{q_{m-1}}{q_m}}, \quad a = \frac{c+1}{2}, \tag{2}$$

   in particular,

$$\frac{|\Delta^A_{c_1\ldots c_m0}|}{|\Delta^A_{c_1\ldots c_m}|} = \frac{2 + \frac{q_{m-1}}{q_m}}{3 + 2\frac{q_{m-1}}{q_m}}, \quad \frac{|\Delta^A_{c_1\ldots c_m1}|}{|\Delta^A_{c_1\ldots c_m}|} = \frac{1 + \frac{q_{m-1}}{q_m}}{3 + 2\frac{q_{m-1}}{q_m}},$$

   where

$$\frac{|\Delta^A_{c_1\ldots c_m0}|}{|\Delta^A_{c_1\ldots c_m1}|} = \frac{2 + \frac{q_{m-1}}{q_m}}{1 + \frac{q_{m-1}}{q_m}} = 1 + \frac{1}{1 + \frac{q_{m-1}}{q_m}}.$$

2. Normal properties of numbers in terms of their A-representation.

Definition 2. The property B of the A-representation of the number $x \in [\frac{1}{2}; 1]$ is called *normal* if the set H_B of numbers possessing the property is a set of complete Lebesgue measure, i.e., $\lambda([\frac{1}{2}; 1] \setminus H_B) = 0$.

Theorem 1. *Let $(c_1, ..., c_p)$ be an ordered set of zeros and ones. The set*

$$D_0 = \{x : x = \Delta^A_{\alpha_1\alpha_2...\alpha_n...}, \overline{\alpha_{k+1}\alpha_{k+2}...\alpha_{k+p}} \neq \overline{c_1...c_p} \quad \forall k \in \mathbb{N}\}$$

of all numbers of the segment $[\frac{1}{2}; 1]$ whose A-representation does not contain the set $c_1...c_p$ as consecutive digits of the representation is of zero Lebesgue measure.

Proof. Remark that in cases where $p = 1$ or $p = 2$ and $c_1 \neq c_2$ the statement is obvious because the set D_0 is countable. We exclude these cases in the sequel.

Since

$$D_0 \subset F \equiv \{x : x = \Delta^A_{\alpha_1...\alpha_n...}, \text{ where } \overline{\alpha_{kp+1}\alpha_{kp+2}...\alpha_{(k+1)p}} \neq \overline{c_1...c_p} \; \forall k \in \mathbb{N}\},$$

to prove the statement it is enough to prove that $\lambda(F) = 0$.

Let $F_0 \equiv [\frac{1}{2}; 1]$, $F_m \equiv \{x : x = \Delta^A_{\alpha_1...\alpha_n...}, \overline{\alpha_{kp+1}\alpha_{kp+2}...\alpha_{(k+1)p}} \neq \overline{c_1...c_p}, k = \overline{1, m}\}$.

It is obvious that $F \subset F_{m+1} \subset F_m$ and $F = \bigcap\limits_{m=1}^{\infty} F_m = \lim\limits_{m\to\infty} F_m$. Then

$$\lambda(F) \leq \lambda(F_m) \;\; \forall m \in N \text{ and } \lambda(F) = \lim_{m\to\infty} \lambda(F_m).$$

Since $F_1 = [\frac{1}{2}; 1] \setminus \Delta^A_{c_1...c_p}$, we have $\lambda(F_1) = \frac{1}{2} - |\Delta^A_{c_1...c_p}| > 0$.

If we write the Lebesgue measure of the set F_m in the form

$$\lambda(F_m) = \frac{1}{2}\frac{\lambda(F_m)}{\lambda(F_{m-1})} \cdot \frac{\lambda(F_{m-1})}{\lambda(F_{m-2})} \cdot ... \cdot \frac{\lambda(F_2)}{\lambda(F_1)} \cdot \frac{\lambda(F_1)}{\lambda(F_0)},$$

we obtain

$$\lambda(F) = \lim_{m\to\infty} \lambda(F_m) = \frac{1}{2} \prod_{m=1}^{\infty} \frac{\lambda(F_m)}{\lambda(F_{m-1})}.$$

Taking into account the equalities $F_{m-1} \setminus F_m \equiv \overline{F}_m$, we obtain $F_m = F_{m-1} \setminus \overline{F}_m$. Then

$$\lambda(F) = \frac{1}{2} \prod_{m=1}^{\infty} \frac{\lambda(F_{m-1}) - \lambda(\overline{F}_m)}{\lambda(F_{m-1})} = \frac{1}{2} \prod_{m=1}^{\infty} [1 - \frac{\lambda(\overline{F}_m)}{\lambda(F_{m-1})}].$$

Since

$$\frac{|\Delta^A_{\alpha_1...\alpha_n c_1...c_p}|}{|\Delta^A_{\alpha_1...\alpha_n}|} = \frac{|\Delta^A_{\alpha_1...\alpha_n c_1...c_p}|}{|\Delta^A_{\alpha_1...\alpha_n c_1...c_{p-1}}|} \cdot \frac{|\Delta^A_{\alpha_1...\alpha_n c_1...c_{p-1}}|}{|\Delta^A_{\alpha_1...\alpha_n c_1...c_{p-2}}|} \cdot ... \cdot \frac{|\Delta^A_{\alpha_1...\alpha_n c_1}|}{|\Delta^A_{\alpha_1...\alpha_n}|},$$

from the basic metric relation for A_2-continued fractions and its corollaries it follows that there exist two positive constants d_1, d_2, $d_1 < d_2$, such that

$$0 < d_1 < \frac{|\Delta^A_{\alpha_1...\alpha_p...\alpha_{kp+1}...\alpha_{(k+1)p}c_1...c_p}|}{|\Delta^A_{\alpha_1...\alpha_p...\alpha_{kp+1}...\alpha_{(k+1)p}}|} < d_2 < 1.$$

Hence, $\frac{\lambda(\overline{F}_m)}{\lambda(F_{m-1})} \geq 1 - d_2 > 0$; and then $\sum_{m=1}^{\infty} \frac{\lambda(\overline{F}_m)}{\lambda(F_{m-1})} = \infty$. Taking into account the connection of divergence of infinite products and their corresponding series, we have $\lambda(F) = 0$. Therefore, $\lambda(D_0) = 0$ and $\lambda([\frac{1}{2}; 1] \setminus D_0) = \frac{1}{2}$. $\square$

Corollary 1. *The equality $\lambda(D_0 \cap \Delta^A_{c_1...c_m}) = 0$ is satisfied for any tuple $c_1...c_m$.*

Theorem 2. *Let $(a_1, ..., a_m)$ be an arbitrary ordered tuple of zeros and ones. Let H be a set of all numbers of the segment $[\frac{1}{2}; 1]$ such that their A-representation contains this tuple infinitely many times. Then H is a set of full measure, i.e.,*

$$\lambda\Big(\Big[\frac{1}{2}; 1\Big] \setminus H\Big) = 0. \tag{3}$$

Proof. By the previous theorem, it follows that the set $[\frac{1}{2}; 1] \setminus D_0$ is a set of full measure.

Let D_k be a set of numbers that use the tuple of digits $a_1...a_m$ exactly k times. Then it is obvious that there exists $n_0 \in \mathbb{N}$ such that numbers of all A-cylinders of rank $n > n_0$ never use this tuple of digits. Then, by the corollary of the previous theorem and the σ-additivity of the Lebesgue measure, it follows that $\lambda(D_k) = 0$. Then $\lambda(\bigcup_{k=1}^{\infty} D_k) = 0$ as a measure of the countable union of sets of zero measure. Thus, since

$$\Big[\frac{1}{2}; 1\Big] \setminus H = D_0 \cup D_1 \cup D_2 \cup ...,$$

we have the equality (3). This completes the proof. $\square$

Corollary 2. *The property of the number $x \in [\frac{1}{2}; 1]$ to use each of the possible tuples of digits as consecutive digits of the A-representation infinitely many times is normal.*

3. Quasi-exponential functions related to the A-representation of numbers. We consider the class of functions defined by equality

$$f(x = \Delta^A_{\alpha_1\alpha_2...\alpha_k...}) = \prod_{k=1}^{\infty} \lambda_k^{\alpha_k(x)}, \tag{4}$$

where (λ_k) is a given sequence of positive numbers such that the infinite product $P \equiv \prod_{k=1}^{\infty} \lambda_k$ is absolutely convergent. For the binary representation of the argument, functions of this type were considered by B. Sendov in his work [17], for Q_2-representation, they were considered in work [7], for Q_2^*-representation, they were considered in work [8].

Remark 1. For the function f to be well defined by equality (4) we use only one of two existing representation for A-binary numbers, which has a period (10).

The following equations are obvious:

1) $f(\Delta^A_{(0)}) = 1, f(\Delta^A_{(1)}) = P,$
 $f(x = \Delta^A_{c_1...c_m1(10)}) = \lambda_1^{c_1}\lambda_2^{c_2}...\lambda_m^{c_m}\lambda_{m+1}\lambda_{m+2}\ldots\lambda_{m+2k}..., k \in \mathbb{N};$

2) $\dfrac{f(\Delta^A_{a_1a_2...a_k...})}{f(\Delta^A_{b_1b_2...b_k...})} = \prod\limits_{k=1}^{\infty} \lambda_k^{a_k-b_k}$, particularly $f(\Delta^A_{\alpha_1...\alpha_m c\alpha_{m+2}...}) = \lambda_{m+1}^{2c-1} f(\Delta^A_{\alpha_1...\alpha_m[1-c]\alpha_{m+2}...})$.

Since $\lambda_k > 0$, we have $\lambda_k = e^{v_k}$ for some real number v_k. Therefore, we can write equality (4) in the form

$$f(x = \Delta^A_{\alpha_1\alpha_2...\alpha_n...}) = \prod_{k=1}^{\infty} e^{v_k\alpha_k} = e^{\sum\limits_{k=1}^{\infty} v_k\alpha_k}. \tag{5}$$

Moreover, from the absolute convergence of the infinite product $\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_k \cdot ...$ it follows that the series $v_1 + v_2 + ... + v_k + ...$ is absolutely convergent.

Since the definition of the function f is based on a chain of dependencies

$$\left[\frac{1}{2};1\right] \ni x \leftrightarrow (\alpha_n) \in L \to \varphi(x) = \sum_{i=1}^{\infty} v_i \alpha_i(x) \to f(x) = e^{\varphi}(x),$$

function f is composite: $f(x) = e^{\varphi(x)}$, where the inner function is

$$\varphi(x = \Delta^A_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = v_1\alpha_1(x) + v_2\alpha_2(x) + \ldots + v_n\alpha_n(x) + \ldots . \tag{6}$$

Lemma 1. *Let $r_0 = v_1 + v_2 + \cdots + v_n + \ldots$ be absolutely convergent series, where $v_n \in \mathbb{R}$. In order that equality*

$$v_n + \sum_{k=1}^{\infty} v_{n+2k-1} = \sum_{k=1}^{\infty} v_{n+2k} \tag{7}$$

holds for any $n \in \mathbb{N}$, it is necessary and sufficient that the equality

$$v_n = \frac{v_1(-1)^{n-1}}{2^{n-1}}, \quad v_1 \in \mathbb{R}, \tag{8}$$

holds for any $n \in \mathbb{N}$.

Proof. Necessity. Assume that equality (7) is satisfied. We prove that equality (8) is satisfied for all $n \in \mathbb{N}$.

If $n = 1$ then equality (7) is equivalent to the $v_1 + v_2 - v_3 + v_4 - v_5 + \cdots = 0$. Adding $2(v_3 + v_5 + \ldots)$ to both parts, we get $r_0 = 2(v_3 + v_5 + \ldots)$. Hence,

$$v_1 + v_2 + v_4 + \cdots = \frac{r_0}{2} = v_3 + v_5 + v_7 + \ldots .$$

If $n = 2$ then we have $v_2 + v_3 + v_5 + v_7 + \ldots = v_4 + v_6 + v_8 + \ldots$. Therefore,

$$v_2 + v_3 - v_4 + v_5 - v_6 + v_7 - v_8 + \ldots = 0.$$

Adding $2(v_4 + v_6 + v_8 + \ldots)$ to both previous parts, we get

$$v_2 + v_3 + v_5 + \cdots = \frac{r_1}{2} = v_4 + v_6 + v_8 + \ldots .$$

Hence,

$$\frac{r_0}{2} = v_1 + v_2 + \frac{r_1}{2}, \quad \frac{r_0}{2} = v_1 + v_2 + \frac{r_0 - v_1}{2},$$
$$v_1 + v_2 - \frac{v_1}{2} = 0, \quad v_2 = -\frac{v_1}{2}.$$

For a general case, we add the right-hand side part of equality (7) to both parts of this equality. Then

$$\sum_{k=1}^{\infty} v_{m+2k} = \frac{r_{m-1}}{2} = v_m + \sum_{k=1}^{\infty} v_{m+2k-1}.$$

Then, for $n = m + 1$, we obtain

$$\frac{r_m}{2} = v_{m+1} + \sum_{k=1}^{\infty} v_{m+2k},$$

$$\frac{r_{m-1}}{2} = v_m + v_{m+1} + \frac{r_m}{2},$$
$$\frac{r_{m-1}}{2} = v_m + v_{m+1} + \frac{r_{m-1} - v_m}{2},$$
$$v_{m+1} = -\frac{v_m}{2}.$$

Hence, we receive equality (8) in n steps.

Sufficiency. If we substitute the expression $v_n = \frac{v_1(-1)^{n-1}}{2^{n-1}}$ directly in (7), we receive

$$v_n + \sum_{k=1}^{\infty} v_{n+2k-1} = \frac{v_1(-1)^{n-1}}{2^{n-1}} + \sum_{k=1}^{\infty} v_{n+2k-1} = \sum_{k=1}^{\infty} v_{n+2k}.$$ □

Theorem 3. *The function $f(x)$ is continuous at each A-unary point, and it is continuous at A-binary point $x^* = \Delta^A_{c_1\ldots c_m 0(01)} = \Delta^A_{c_1\ldots c_m 1(10)}$ if and only if the following equality holds:*

$$v_m + \sum_{k=1}^{\infty} v_{m+2k-1} = \sum_{k=1}^{\infty} v_{m+2k}.$$

Proof. Let $x_0 = \Delta^A_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ be an arbitrary A-unary number. Consider $x = \Delta^A_{a_1\ldots a_n\ldots}$ such that $x \neq x_0$. Then there exists m such that $a_m \neq \alpha_m$ but $a_i = \alpha_i$ if $i < m$. Moreover, $x \to x_0$ is equivalent to $m \to \infty$. Hence, we have

$$\frac{f(x)}{f(x_0)} = \prod_{i=1}^{m-1} \lambda_i^{a_i - \alpha_i} \cdot \lambda_m^{a_m - \alpha_m} \cdot \prod_{i=m+1}^{\infty} \lambda_i^{a_i - \alpha_i},$$

but $\prod\limits_{i=1}^{m-1} \lambda_i^{a_i-\alpha_i} = 1$, $\lim\limits_{m\to\infty} \lambda_m = 1 = \lim\limits_{m\to\infty} \prod\limits_{i=1}^{m-1} \lambda_i^{a_i-\alpha_i}$.

Therefore, $\lim\limits_{x\to x_0} f(x) = f(x_0)$, i.e., function f is continuous at the point x_0.

It is easy to prove that function f is continuous at the point x^* if and only if images of two formally different representations calculated by the formula (5) are equal:

$$f(\Delta^A_{c_1\ldots c_{m-1}1(10)}) = e^{c_1v_1+\cdots+c_{m-1}v_{m-1}+v_m+v_{m+1}+v_{m+3}+\cdots+v_{m+2k+1}+\cdots},$$

$$f(\Delta^A_{c_1\ldots c_{m-1}0(01)}) = e^{c_1v_1+\cdots+c_{m-1}v_{m-1}+v_{m+2}+v_{m+4}+\cdots+v_{m+2k}+\cdots}.$$

Obviously, these values are equal when

$$v_m + v_{m+1} + v_{m+3} + \cdots + v_{m+2k+1} + \cdots = v_{m+2} + v_{m+4} + \cdots + v_{m+2k} + \ldots,$$

in other words, when equality (7) is true. □

Corollary 3. *The function f defined by equality (4) is continuous on the segment $[\frac{1}{2}; 1]$ if and only if $\lambda_k = e^{\frac{c(-1)^{k-1}}{2^{k-1}}}$ for some $c \in \mathbb{R}$ and for all $k \in \mathbb{N}$.*

Theorem 4. *Each continuous function φ and f respectively defined by equality (6) and (4) respectively is strictly decreasing for $v_1 > 0$, strictly increasing for $v_1 < 0$, and a constant for $v_1 = 0$.*

Proof. It is clear that φ and f have the same type of monotonicity i.e. they are simultaneously decreasing or increasing. We prove the theorem for $v_1 > 0$ only (the proof is similar for $v_1 < 0$).

Let f be a continuous function defined by (4), and let (v_n) be the corresponding series, namely $\lambda_n = e^{v_n}$. Taking into account Lemma 1, we get $v_n = \frac{v_1(-1)^{n-1}}{2^{n-1}}$. To prove the theorem, it is sufficient to show that the function $\varphi(x = \Delta^A_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) = \alpha_1 v_1 + \ldots + \alpha_n v_n + \ldots$ is strictly decreasing. We consider two arbitrary different points of segment $[\frac{1}{2}; 1]$:

$$x_1 = \Delta^A_{c_1\ldots c_m 1\alpha_1\alpha_2\ldots} \quad \text{and } x_2 = \Delta^A_{c_1\ldots c_m 0\beta_1\beta_2\ldots}.$$

For them we have

$$\varphi(x_1) - \varphi(x_2) = \frac{v_1(-1)^m}{2^m} + v_1 \sum_{n=1}^{\infty} \frac{(\alpha_n - \beta_n)(-1)^{m+n}}{2^{n+m}}.$$

If m is an even number, then $x_1 < x_2$ and

$$\varphi(x_1) - \varphi(x_2) > \frac{v_1}{2^m} - v_1\Big(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \ldots\Big) = 0.$$

Hence, $\varphi(x_1) > \varphi(x_2)$. If m is an odd number, then $x_1 > x_2$ and

$$\varphi(x_1) - \varphi(x_2) < -\frac{v_1}{2^m} + v_1\Big(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2} + \ldots}\Big) = 0.$$

Hence, $\varphi(x_1) < \varphi(x_2)$.

Strict inequalities are a consequence of the fact that $x_1 \neq x_2$. Indeed, equality is possible only if $\alpha_1 = 1$ and $(\alpha_{2k-1}, \alpha_{2k}) = (\beta_{2k}, \beta_{2k+1}) = (1, 0)$ for all $k \in N$ or equivalently numbers x_1 and x_2 are A-binary numbers. Thus, φ is a strictly decreasing function. □

4. Singularity of continuous quasi-exponential functions. In the sequel, we consider the continuous functions φ and f defined by equalities (6) and (4) that is under condition (8). At the same time, we take $v_1 = \frac{1}{2}$, i.e., $v_n = \frac{(-1)^{n-1}}{2^n}$.

First, we prove a general additional statement.

Lemma 2. *Let x_0 be an internal point of the domain D_g of the continuous function g. If there exists a derivative $g'(x_0)$ of the function g at the point x_0, then it can be calculated by the formula*

$$g'(x_0) = \lim_{n\to\infty} \frac{g(w_n) - g(u_n)}{w_n - u_n}, \tag{9}$$

where $u_n \in D_g$, $w_n \in D_g$, $u_n \le x_0 \le w_n$, $0 < w_n - u_n \to 0$ $(n \to \infty)$.

If limit (9) does not exist for some sequences (u_n) and (w_n), then the classical derivative $g'(x_0)$ does not exist either.

Proof. Let $w_n \equiv x_0 + \tau_n$, $u_n \equiv x_0 - \varepsilon_n$, $\tau_n > 0$, $\varepsilon_n > 0$. Then

$$\delta \equiv \frac{g(w_n) - g(u_n)}{w_n - u_n} = \frac{g(x_0 + \tau_n) - g(x_0 - \varepsilon_n)}{\varepsilon_n + \tau_n} = \frac{g(x_0 + \tau_n) - g(x_0)}{\tau_n + \varepsilon_n} + \frac{g(x_0) - g(x_0 - \varepsilon_n)}{\tau_n + \varepsilon_n} =$$
$$= \frac{g(x_0 + \tau_n) - g(x_0)}{\tau_n} \cdot \frac{\tau_n}{\tau_n + \varepsilon_n} + \frac{g(x_0) - g(x_0 - \varepsilon_n)}{\varepsilon_n} \cdot \frac{\varepsilon_n}{\tau_n + \varepsilon_n}.$$

Since the derivative $g'(x_0)$ exists, we have

$$\frac{g(x_0+\tau_n)-g(x_0)}{\tau_n} = g'(x_0)+\alpha(x_0,\tau_n), \text{ where } \lim_{n\to\infty}\alpha(x_0,\tau_n)=0,$$

$$\frac{g(x_0)-g(x_0-\varepsilon_n)}{\varepsilon_n} = g'(x_0)+\beta(x_0,\varepsilon_n), \text{ where } \lim_{n\to\infty}\beta(x_0,\varepsilon_n)=0.$$

Then $\delta = [g'(x_0)+\alpha(x_0,\tau_n)]\frac{\tau_n}{\tau_n+\varepsilon_n} + [g'(x_0)+\beta(x_0,\varepsilon_n)]\frac{\varepsilon_n}{\tau_n+\varepsilon_n} =$

$$= g'(x_0)+\alpha(x_0,\tau_n)\cdot\frac{\tau_n}{\tau_n+\varepsilon_n}+\beta(x_0,\varepsilon_n)\cdot\frac{\varepsilon_n}{\tau_n+\varepsilon_n}.$$

Taking into account the inequalities $0<\frac{\tau_n}{\tau_n+\varepsilon_n}=\frac{1}{1+\frac{\varepsilon_n}{\tau_n}}<1$, $0<\frac{\varepsilon_n}{\tau_n+\varepsilon_n}=\frac{1}{\frac{\tau_n}{\varepsilon_n}+1}<1$, we have

$$\lim_{n\to\infty}\frac{g(w_n)-g(u_n)}{w_n-u_n}=g'(x_0).$$

The second part of the lemma is a consequence of the first part. □

Corollary 4. *If there exists a finite derivative of the function* (6) *at the point* $x_0=\Delta^A_{\alpha_1\ldots\alpha_n\ldots}$, *then it can be calculated by each of the formulas*

$$\varphi'(x_0)=\lim_{n\to\infty}\frac{\varphi(\Delta^A_{\alpha_1\ldots\alpha_n(01)})-\varphi(\Delta^A_{\alpha_1\ldots\alpha_n(10)})}{\Delta^A_{\alpha_1\ldots\alpha_n(01)}-\Delta^A_{\alpha_1\ldots\alpha_n(10)}}=$$

$$=-\lim_{k\to\infty}\frac{1}{2^{2k}|\Delta^A_{\alpha_1\ldots\alpha_{2k}}|}=-2\lim_{k\to\infty}\prod_{n=1}^{2k}\frac{|\Delta^A_{\alpha_1\ldots\alpha_{n-1}}|}{2|\Delta^A_{\alpha_1\ldots\alpha_{n-1}\alpha_n}|}=-2\prod_{n=1}^{\infty}\frac{|\Delta^A_{\alpha_1\ldots\alpha_{n-1}}|}{2|\Delta^A_{\alpha_1\ldots\alpha_{n-1}\alpha_n}|}. \tag{10}$$

If limit (10) *does not exist, then the classical derivative* $\varphi'(x_0)$ *does not exist.*

The limit (10) is called *cylindrical derivative* of function φ at the point x_0.

Theorem 5. *If* φ *and* f *are continuous functions defined by equalities* (6) *and* (4), *then, for almost all numbers* $x\in[\frac{1}{2};1]$ *(in the sense of the Lebesgue measure), the equalities* $\varphi'(x)=0=f'(x)$ *are satisfied.*

Proof. Since the function f is continuous and monotonic, by the famous Lebesgue theorem, it follows that f has finite derivative at almost all internal points of the domain. We denote the set of all such points by V.

Let x_0 be a point, where there exists a finite derivative $\varphi'(x_0)$ (of $f'(x_0)$ respectively) and x_0 have the normal property of the A-representation, which is stated by Theorem 2. Then $x_0\in W=H\cap V$ and the set W is of full Lebesgue measure ($\lambda(W)=\frac{1}{2}$) as the intersection of two sets of full measure.

Taking into account the corollary of the previous lemma, we have

$$\varphi'(x_0)=-2\prod_{n=1}^{\infty}\frac{|\Delta^A_{\alpha_1\ldots\alpha_{n-1}}|}{2|\Delta^A_{\alpha_1\ldots\alpha_n}|}, \tag{11}$$

where $\alpha_n=\alpha_n(x_0)$ is an n -th digit of A-representation of a number x_0. Since, taking into account equalities (2)), the multiplier

$$\delta_n=\frac{|\Delta^A_{\alpha_1\ldots\alpha_{n-1}}|}{2|\Delta_{\alpha_1\ldots\alpha_n}|}=\frac{2a^2+1+2a\frac{q_{n-1}}{q_n}}{2(1+a\frac{q_{n-1}}{q_n})}, \text{ where } a\in\left\{\frac{1}{2},1\right\},$$

$$\delta_n = \frac{2 + \frac{q_{n-1}}{q_n}}{\frac{3}{2} + \frac{q_{n-1}}{q_n}},\ \text{if } a = \frac{1}{2},\ \text{and } \delta_n = \frac{2 + 2\frac{q_{n-1}}{q_n}}{3 + 2\frac{q_{n-1}}{q_n}},\ \text{if } a = 1,$$

does not tend to 1 for $n \to \infty$, that is, the necessary condition of convergence of the infinite product (11) is not satisfied, then $\varphi'(x_0) = 0$, and therefore $f'(x_0) = 0$. Hence, the function f is singular. $\square$

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