УДК 517.53

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ON REGULAR VARIATION OF ENTIRE DIRICHLET SERIES

P. V. Filevych, O. B. Hrybel. *On regular variation of entire Dirichlet series*, Mat. Stud. **58** (2022), 174–181.

Let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a nonnegative sequence increasing to $+\infty$ such that $\lambda_{n+1} \sim \lambda_n$ as $n \to \infty$, and let $\rho \geq 1$ be a constant. Put

$$\omega(\lambda) = \varlimsup_{n\to\infty} \frac{\ln\ln n}{\ln \lambda_n}, \quad C(\rho) = \begin{cases} \rho/(\rho-1), & \text{when } \rho > 1, \\ +\infty, & \text{when } \rho = 1. \end{cases}$$

Consider an entire (absolutely convergent in $\mathbb{C}$) Dirichlet series F with the exponents λ_n, i.e., of the form $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, and, for all $\sigma \in \mathbb{R}$, put $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ and $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$. Previously, the first of the authors and M.M. Sheremeta proved that if $\omega(\lambda) < C(\rho)$, then the regular variation of the function $\ln \mu(\sigma, F)$ with index ρ implies the regular variation of the function $\ln M(\sigma, F)$ with index ρ, and constructed examples of entire Dirichlet series F, for which $\ln \mu(\sigma, F)$ is a regularly varying function with index ρ, and $\ln M(\sigma, F)$ is not a regularly varying function with index ρ. For the exponents of the constructed series we have $\lambda_n = \ln\ln n$ for all $n \geq n_0$ in the case $\rho = 1$, and $\lambda_n \sim (\ln n)^{(\rho-1)/\rho}$ as $n \to \infty$ in the case $\rho > 1$. In the present article we prove that the exponents of entire Dirichlet series with the same property can form an arbitrary sequence $\lambda = (\lambda_n)_{n=0}^{\infty}$ not satisfying $\omega(\lambda) < C(\rho)$. More precisely, if $\omega(\lambda) \geq C(\rho)$, then there exists a regularly varying function $\Phi(\sigma)$ with index ρ such that, for an arbitrary positive function $l(\sigma)$ on $[a, +\infty)$, there exists an entire Dirichlet series F with the exponents λ_n, for which $\ln \mu(\sigma, F) \sim \Phi(\sigma)$ as $\sigma \to +\infty$ and $M(\sigma, F) \geq l(\sigma)$ for all $\sigma \geq \sigma_0$.

1. Introduction. Let $l(\sigma)$ be a positive and measurable function on $[a, +\infty)$. The function $l(\sigma)$ is said to be *slowly varying* ([1]) if $l(c\sigma) \sim l(\sigma)$ as $\sigma \to +\infty$ for any $c > 0$, and is said to be *regularly varying* ([1]) if, for some real ρ, we have $l(\sigma) = \sigma^\rho \alpha(\sigma)$, where $\alpha(\sigma)$ is a slowly varying function, and ρ is called the *index of regular variation* ([1]).

Problems of finding conditions for regular variation of the main characteristics of entire functions, presented by power series, Dirichlet series or Taylor-Dirichlet series, were considered, in particular, in the articles [2–6]. In this article, we give some addentum to the results from [4].

We denote by Λ the class of all nonnegative sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ increasing to $+\infty$, i.e. $0 \leq \lambda_0 < \lambda_n < \lambda_{n+1}$ $(n \geq 1)$ and $\lim\limits_{n\to+\infty} \lambda_n = +\infty$.

Let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ. Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

2010 *Mathematics Subject Classification:* 30B50, 30D15, 30D20.
Keywords: slowly varying function; regularly varying function; Dirichlet series; supremum modulus; maximal term; central index; Young-conjugate function.
doi:10.30970/ms.58.2.174-181

and by $\sigma_a(F)$ denote the abscissa of absolute convergence of this series. Put

$$\beta(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

For $\sigma < \sigma_a(F)$ we denote

$$M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}.$$

If $\beta(F) > -\infty$, then for all $\sigma < \beta(F)$ we put

$$\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \ge 0\}, \quad \nu(\sigma, F) = \max\{n \ge 0 : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}.$$

By $\mathcal{D}(\lambda)$ we denote the class of all Dirichlet series of the form (1) that do not reduce to exponential polynomials and satisfy $\sigma_a(F) = +\infty$, and let $\mathcal{D}^*(\lambda)$ be the class of all Dirichlet series of the form (1) that do not reduce to exponential polynomials and satisfy $\beta(F) = +\infty$. It is well known that $\mathcal{D}(\lambda) \subset \mathcal{D}^*(\lambda)$, and $\mathcal{D}(\lambda) = \mathcal{D}^*(\lambda)$ if and only if $\ln n = O(\lambda_n)$ as $n \to \infty$.

We put $\mathcal{D} = \cup_{\lambda\in\Lambda}\mathcal{D}(\lambda)$ and $\mathcal{D}^* = \cup_{\lambda\in\Lambda}\mathcal{D}^*(\lambda)$. Then $\mathcal{D}$ is a proper subset of $\mathcal{D}^*$.

If $F \in \mathcal{D}^*$, then $\sigma = o(\ln \mu(\sigma, F))$ as $\sigma \to +\infty$. Therefore, the function $\ln \mu(\sigma, F)$ cannot be regularly varying with index $\rho < 1$, because for each slowly varying function $\alpha(\sigma)$ we have $\ln \alpha(\sigma) = o(\ln \sigma)$ as $\sigma \to +\infty$ (see, for example, [1]). Necessary and sufficient conditions for the function $\ln \mu(\sigma, F)$ to be regularly varying with index $\rho \ge 1$ was given in [4] (for series of the Taylor-Dirichlet type, see a similar statement in [5]).

Theorem A ([4])**.** *Let $\rho \ge 1$ and let $F \in \mathcal{D}^*$ be a Dirichlet series of the form* (1)*. Then the following statements are equivalent:*

(i) $\ln \mu(\sigma, F)$ is a regularly varying function with index ρ;

(ii) $\lambda_{\nu(\sigma,F)}$ is a regularly varying function with index $\rho - 1$;

(iii) $\sigma\lambda_{\nu(\sigma,F)}/\ln \mu(\sigma, F) \to \rho$ as $\sigma \to +\infty$;

(iv) there exists an increasing sequence $(n_k)_{k=0}^{\infty}$ of nonnegative integers such that

$$\varkappa_k := \frac{\ln |a_{n_k}| - \ln |a_{n_{k+1}}|}{\lambda_{n_{k+1}} - \lambda_{n_k}} \uparrow +\infty, \quad k \to \infty; \tag{2}$$

$$|a_n|e^{\varkappa_k\lambda_n} \le |a_{n_k}|e^{\varkappa_k\lambda_{n_k}}, \quad n_k < n < n_{k+1}, \quad k \ge 0; \tag{3}$$

$$c_k := \frac{\varkappa_k\lambda_{n_{k+1}}}{\varkappa_k\lambda_{n_{k+1}} + \ln |a_{n_{k+1}}|} \to \rho, \quad k \to \infty; \tag{4}$$

$$d_k := \frac{\varkappa_k\lambda_{n_k}}{\varkappa_k\lambda_{n_k} + \ln |a_{n_k}|} \to \rho, \quad k \to \infty. \tag{5}$$

Note that conditions (2), (4) and (5) imply the relation

$$\lambda_{n+1} \sim \lambda_n, \quad n \to +\infty. \tag{6}$$

Therefore, if (6) does not hold and $\rho \ge 1$, then there does not exist a Dirichlet series $F \in \mathcal{D}^*$ of the form (1) such that the function $\ln \mu(\sigma, F)$ is regularly varying function with index ρ.

The following theorem together with Theorem A gives conditions for the regular variation of the function $\ln M(\sigma, F)$.

Theorem B ([4])**.** *Let $\rho \ge 1$ and let $F \in \mathcal{D}$ be a Dirichlet series of the form* (1)*. If*

$$\varlimsup_{n\to\infty} \frac{\ln \ln n}{\ln \lambda_n} < C(\rho) := \begin{cases} \rho/(\rho - 1), & \text{when } \rho > 1, \\ +\infty, & \text{when } \rho = 1. \end{cases}, \tag{7}$$

then $\ln M(\sigma, F)$ is a regularly varying function with index ρ if and only if $\ln \mu(\sigma, F)$ is a regularly varying function with the same index.

Examples of Dirichlet series $F \in \mathcal{D}$ of the form (1), for which $\ln\mu(\sigma, F)$ is a regularly varying function with given index $\rho \geq 1$, and $\ln M(\sigma, F)$ is not a regularly varying function with the same index, were also constructed in [4]. For the exponents of the constructed series we have $\lambda_n = \ln\ln n$ for all $n \geq n_0$ in the case $\rho = 1$, and $\lambda_n \sim (\ln n)^{(\rho-1)/\rho}$ as $n \to \infty$ in the case $\rho > 1$. The following theorem shows that the exponents of Dirichlet series with the same property can form an arbitrary sequence $\lambda = (\lambda_n)_{n=0}^{\infty}$ from the class Λ satisfying (6) and not satisfying (7).

Theorem 1. *Let $\rho \geq 1$ and let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ, for which (6) holds, and (7) is false. Then there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$ such that $\ln\mu(\sigma, F)$ is a regularly varying function with index ρ, and $\ln M(\sigma, F)$ is not a regularly varying function.*

Theorem 1 follows from the following stronger theorem.

Theorem 2. *Let $\rho \geq 1$ and let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ, for which (6) holds, and (7) is false. Then there exists a regularly varying function $\Phi(\sigma)$ with index ρ such that, for an arbitrary positive function $l(\sigma)$ on $[a, +\infty)$, there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$, for which $\ln\mu(\sigma, F) \sim \Phi(\sigma)$ as $\sigma \to +\infty$ and $M(\sigma, F) \geq l(\sigma)$ for all $\sigma \geq \sigma_0$.*

2. Auxiliary results. We write $\Phi \in \Omega$ if $\Phi(\sigma)$ is a continuous function on $[a, +\infty)$ satysfying $\Phi(\sigma)/\sigma \to +\infty$ as $\sigma \to +\infty$. If $\Phi \in \Omega$, then let $\widetilde{\Phi}(x)$ be the Young-conjugate function of $\Phi(\sigma)$, i.e.

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) : \sigma \in [a, +\infty)\}, \quad x \in \mathbb{R}.$$

The following lemma is well known (see, for example, [7]).

Lemma 1. *Let $\Phi \in \Omega$ and let $F \in \mathcal{D}^*$ be a Dirichlet series of the form (1). Then we have $\ln\mu(\sigma, F) \leq \Phi(\sigma)$ for all $\sigma \geq \sigma_0$ if and only if $\ln|a_n| \leq -\widetilde{\Phi}(\lambda_n)$ for all $n \geq n_0$.*

Let $q > 1$ and let $\Phi(\sigma) = \sigma^q$ for all $\sigma \geq 0$. Then $\Phi \in \Omega$ and, as it is easy to show,

$$\widetilde{\Phi}(x) = (q-1)(x/q)^{\rho/(\rho-1)}, \quad x \geq 0.$$

Using this fact and Lemma 1, we obtain the following statement.

Lemma 2. *Let $\rho \geq 1$ and $F \in \mathcal{D}^*$ be a Dirichlet series of the form (1). Then*

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\mu(\sigma, F)}{\ln\sigma} = \rho$$

if and only if

$$\varliminf_{n\to\infty} \frac{\ln\ln(1/|a_n|)}{\ln\lambda_n} = C(\rho).$$

Theorem C ([7,8])**.** *Let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ and let $G \in \mathcal{D}^*(\lambda)\backslash\mathcal{D}(\lambda)$ be a Dirichlet series of the form*

$$G(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n} \tag{8}$$

such that $b_n \geq 0$ for all integers $n \geq 0$. Then, for an arbitrary positive function $l(\sigma)$ on $[a, +\infty)$, there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$ of the form (1) such that either $a_n = b_n$ or $a_n = 0$ for each integer $n \geq 0$ and $M(\sigma, F) \geq l(\sigma)$ for all $\sigma \geq \sigma_0$.

Lemma 3. *Let $a > 1$ and let $\beta(\sigma)$ be an arbitrary positive, bounded function on $[a, +\infty)$ such that*

$$\ln \beta(\sigma) = o(\ln \sigma), \quad \sigma \to +\infty. \tag{9}$$

Then there exists a slowly increasing, continuous function $\alpha(\sigma)$ on $[a, +\infty)$ such that we have $\alpha(\sigma) \geq \beta(\sigma)$ for all $\sigma \geq a$.

Proof. We may assume without loss of generality that $\beta(\sigma) > 1$ on $[a, +\infty)$. Then the function $\eta(\sigma) = \ln \beta(\sigma)/\ln \sigma$ will be positive and bounded from above by some constant M on $[a, +\infty)$. We choose a sequence $(\varepsilon_k)_{k=0}^{\infty}$ decreasing to 0 such that $\varepsilon_0 = M$. It is clear that there exists a sequence $(\sigma_k)_{k=0}^{\infty}$ increasing to $+\infty$ such that $\sigma_0 = a$, $\eta(\sigma) \leq \varepsilon_k$ for all $\sigma \geq \sigma_k$ and every integer $k \geq 1$, and in addition $\ln \sigma_{k-1} = o(\ln \sigma_k)$ as $k \to +\infty$.

We put $\alpha(\sigma) = \sigma^{\varepsilon_0}$ for all $\sigma \in [\sigma_0, \sigma_1)$. Let $k \geq 1$ be an integer, and

$$\alpha(\sigma) = \exp\left(\varepsilon_k \ln \sigma + \sum_{j=1}^{k} (\varepsilon_{j-1} - \varepsilon_j) \ln \sigma_j\right)$$

for all $\sigma \in [\sigma_k, \sigma_{k+1})$. Then, as it is easy to see, $\alpha(\sigma_k - 0) = \alpha(\sigma_k)$ and the function $\alpha(\sigma)$ is continuous and increasing on $[\sigma_k, \sigma_{k+1})$. This implies that the function $\alpha(\sigma)$ is continuous and increasing on $[a, +\infty)$.

Next, if $\sigma \in [\sigma_k, \sigma_{k+1})$ for some integer $k \geq 0$, then

$$\ln \alpha(\sigma) \geq \varepsilon_k \ln \sigma \geq \eta(\sigma) \ln \sigma = \ln \beta(\sigma).$$

Therefore, $\alpha(\sigma) \geq \beta(\sigma)$ for all $\sigma \in [a, +\infty)$.

It remains to prove that $\alpha(\sigma)$ is a slowly varying function. We put $\delta(\sigma) = \ln \alpha(\sigma)/\ln \sigma$ for all $\sigma \in [a, +\infty)$ and prove, first of all, that the function $\delta(\sigma)$ is nonincreasing on $[a, +\infty)$ and $\delta(\sigma) \to 0$ as $\sigma \to +\infty$. Since $\delta(\sigma) = \varepsilon_0$ for all $\sigma \in [\sigma_0, \sigma_1)$ and

$$\delta(\sigma) = \varepsilon_k + \frac{1}{\ln \sigma} \sum_{j=1}^{k} (\varepsilon_{j-1} - \varepsilon_j) \ln \sigma_j$$

for all $\sigma \in [\sigma_k, \sigma_{k+1})$ and every integer $k \geq 1$, the function $\delta(\sigma)$ is nonincreasing on each of the intervals $[\sigma_k, \sigma_{k+1})$. Then the continuity of the function $\delta(\sigma)$ on $[a, +\infty)$ implies that this function is nonincreasing on $[a, +\infty)$. In addition, for every integer $k \geq 2$ we have

$$\delta(\sigma_k) = \varepsilon_k + \frac{1}{\ln \sigma_k} \sum_{j=1}^{k-1} (\varepsilon_{j-1} - \varepsilon_j) \ln \sigma_j + \varepsilon_{k-1} - \varepsilon_k \leq$$

$$\leq \frac{\ln \sigma_{k-1}}{\ln \sigma_k} \sum_{j=1}^{k-1} (\varepsilon_{j-1} - \varepsilon_j) + \varepsilon_{k-1} = \frac{(\varepsilon_0 - \varepsilon_{k-1}) \ln \sigma_{k-1}}{\ln \sigma_k} + \varepsilon_{k-1}.$$

Recalling that $\ln \sigma_{k-1} = o(\ln \sigma_k)$ as $k \to +\infty$, we see that $\delta(\sigma_k) \to 0$ as $k \to +\infty$, and therefore $\delta(\sigma) \to 0$ as $\sigma \to +\infty$.

Since the function $\alpha(\sigma)$ is increasing and the function $\delta(\sigma)$ is nonincreasing on $[a, +\infty)$, for each $\sigma \geq a$ we obtain

$$0 \leq \ln \alpha(2\sigma) - \ln \alpha(\sigma) = \delta(2\sigma) \ln 2\sigma - \delta(\sigma) \ln \sigma \leq \delta(\sigma) \ln 2\sigma - \delta(\sigma) \ln \sigma = \delta(\sigma) \ln 2.$$

Hence $\alpha(2\sigma) \sim \alpha(\sigma)$ as $\sigma \to +\infty$, because $\delta(\sigma) \to 0$ as $\sigma \to +\infty$. Therefore, since the function $\alpha(\sigma)$ is monotonic on $[a, +\infty)$, this function is slowly varying. □

Lemma 4 ([10]). *Let $\lambda=(\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ and let F be a Dirichlet series of the form* (1)*. If there exists an increasing sequence $(n_k)_{k=0}^{\infty}$ of nonnegative integers such that $a_n=0$ for all $n<n_0$, $a_{n_k}\ne 0$ for every integer $k\ge 0$, and the conditions* (2) *and* (3) *hold, then $F\in\mathcal{D}^*$ and, moreover, $\nu(\sigma,F)=n_0$ for all $\sigma<\varkappa_0$ and $\nu(\sigma,F)=n_{k+1}$ for all $\sigma\in[\varkappa_k,\varkappa_{k+1})$ and every integer $k\ge 0$.*

Lemma 5. *Let $\gamma(\sigma)$ be a continuous, increasing to $+\infty$ function on $[a,+\infty)$ and let $\lambda=(\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ satisfying* (6)*. Then there exists a Dirichlet series $F\in\mathcal{D}(\lambda)$ such that $\lambda_{\nu(\sigma,F)}\sim\gamma(\sigma)$ as $\sigma\to+\infty$.*

Proof. Suppose without loss of generality that $\gamma(a)<0$. Let $n_0=0$ and, for every integer $k\ge 0$, set $n_{k+1}=\min\{n>n_k:\lambda_n\ge\lambda_{n_k}+1\}$. It is clear that the sequence $\eta=(\lambda_{n_k})_{k=0}^{\infty}$ belongs to the class Λ and for it we have

$$\lambda_{n_{k+1}}\sim\lambda_{n_k},\quad \ln k=O(\lambda_{n_k}) \tag{10}$$

as $k\to\infty$. Put $a_0=a_{n_0}=1$ and let

$$\varkappa_k=\gamma^{-1}(\lambda_{n_k}),\quad a_{n_{k+1}}=\prod_{j=0}^{k}e^{-\varkappa_j(\lambda_{n_{j+1}}-\lambda_{n_j})}.$$

for every integer $k\ge 0$. Let also $a_n=0$ for all integers $n\ge 0$ such that λ_n is not a member of the sequence η. Consider Dirichlet series (1) with the coefficients a_n defined in this way. Since this series can also be written in the form

$$F(s)=\sum_{k=0}^{\infty}a_{n_k}e^{s\lambda_{n_k}},$$

by Lemma 4 we obtain $F\in\mathcal{D}^*(\eta)$. But, by the second of relations (10), $D^*(\eta)=D(\eta)$. Therefore, $F\in\mathcal{D}(\lambda)$. In addition, by Lemma 4, for all $\sigma\in[\varkappa_k,\varkappa_{k+1})$ and every integer $k\ge 0$, we obtain

$$\gamma(\sigma)<\gamma(\varkappa_{k+1})=\lambda_{n_{k+1}}=\lambda_{\nu(\sigma,F)}=\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}}\gamma(\varkappa_k)\le\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}}\gamma(\sigma).$$

Therefore, according to the first of relations (10), we have $\lambda_{\nu(\sigma,F)}\sim\gamma(\sigma)$ as $\sigma\to+\infty$. □

3. Proof of Theorem 2. Let $\rho\ge 1$ and let $\lambda=(\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ, for which (6) holds, and (7) is false. First of all, we show that there exists a Dirichlet series $G\in\mathcal{D}^*(\lambda)\backslash\mathcal{D}(\lambda)$ of the form (8) such that $b_n\ge 0$ for all integers $n\ge 0$, and for this series we have

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\ln\mu(\sigma,G)}{\ln\sigma}=\rho. \tag{11}$$

Since (7) is false, for an arbitrary constant $\gamma\in[1,C(\rho))$ the set of those integers $n\ge 0$, for which $\ln n\ge\lambda_n^{\gamma}$, is infinite. Therefore, if we fix some sequence $(\gamma_k)_{k=0}^{\infty}$ of points in $[1,C(\rho))$ increasing to $C(\rho)$, we can find an increasing sequence $(n_k)_{k=0}^{\infty}$ of integers such that $n_0=0$ and, for each integer $k\ge 0$, the inequalities $n_{k+1}\ge 2n_k$ and $\ln n_{k+1}\ge\lambda_{n_{k+1}}^{\gamma_k}$ hold. For all integers $k\ge 0$ and $n\in[n_k,n_{k+1})$, we put $b_n=\exp(-\lambda_n^{\gamma_k})$ and consider Dirichlet series (8) with such coefficients $b_n \geq 0$. Then, as it is easy to see, $\beta(G) = +\infty$, that is, $G \in \mathcal{D}^*(\lambda)$. If $n \in [n_k, n_{k+1})$, then

$$\frac{\ln\ln(1/|b_n|)}{\ln\lambda_n} = \gamma_k \uparrow C(\rho), \quad k \to +\infty.$$

Using Lemma 2, we obtain (11). In addition, for each integer $k \geq 0$ we have

$$\sum_{n=n_k}^{n_{k+1}-1} b_n \geq (n_{k+1} - n_k) b_{n_{k+1}} \geq \frac{n_{k+1)}}{2} b_{n_{k+1}} \geq \frac{1}{2}.$$

This implies that series (8) diverges at the point $s = 0$. Therefore, $G \notin \mathcal{D}(\lambda)$.

Next, let $a > 0$ be a fixed number such that $\mu(a, G) > e$. Put $\beta(\sigma) = 2\ln\mu(\sigma, G)/\sigma^\rho$ for all $\sigma \geq a$. Then (11) implies (9), and therefore, according to Lemma 3, there exists a slowly increasing to $+\infty$, continuous function $\alpha(\sigma)$ on $[a, +\infty)$ such that $\alpha(\sigma) \geq \beta(\sigma)$ for all $\sigma \geq a$. Put $\gamma(\sigma) = \rho\sigma^{\rho-1}\alpha(\sigma)$ for each $\sigma \geq a$ and let

$$\Phi(\sigma) = \int_a^\sigma \gamma(t)dt, \quad \sigma \geq a.$$

By Lemma 5, there exists a Dirichlet series $F_1 \in \mathcal{D}(\lambda)$ of the form, say,

$$F_1(s) = \sum_{n=0}^{\infty} a_{1,n} e^{s\lambda_n}$$

such that $\lambda_{\nu(\sigma,F_1)} \sim \gamma(\sigma)$ as $\sigma \to +\infty$, and $a_{1,n} \geq 0$ for every integer $n \geq 0$. It is clear that $\lambda_{\nu(\sigma,F_1)}$ is a regularly varying function with index $\rho - 1$. By L'Hôpital's rule and Theorem A, we have

$$\Phi(\sigma) \sim \ln\mu(\sigma, F_1) \sim \sigma\lambda_{\nu(\sigma,F_1)}/\rho \sim \sigma^\rho\alpha(\sigma), \quad \sigma \to +\infty, \tag{12}$$

that is, $\Phi(\sigma)$ is a regularly varying function with index ρ.

Now let $l(\sigma)$ be an arbitrary positive function on $[a, +\infty)$. According to Theorem C, there exists a Dirichlet series $F_2 \in \mathcal{D}(\lambda)$ of the form

$$F_2(s) = \sum_{n=0}^{\infty} a_{2,n} e^{s\lambda_n}$$

such that $a_{2,n} = b_n$ or $a_{2,n} = 0$ for every integer $n \geq 0$, and $M(\sigma, F_2) = F_2(\sigma) \geq l(\sigma)$ for all $\sigma \geq \sigma_0$.

For each integer $n \geq 0$ we put $a_n = a_{1,n} + a_{2,n}$ and consider Dirichlet series (1) with such coefficients a_n. It is clear that $F \in \mathcal{D}(\lambda)$ and

$$M(\sigma, F) = F_1(\sigma) + F_2(\sigma) \geq F_2(\sigma) \geq l(\sigma), \quad \sigma \geq \sigma_0.$$

In addition, using (12), for all sufficiently large σ we have

$$\mu(\sigma, F_2) \leq \mu(\sigma, G) = \exp(\sigma^\rho\beta(\sigma)/2) \leq \exp(\sigma^\rho\alpha(\sigma)/2) \leq \mu(\sigma, F_1),$$

and hence $\mu(\sigma, F_1) < \mu(\sigma, F) \leq 2\mu(\sigma, F_1)$. Therefore, $\ln\mu(\sigma, F) \sim \ln\mu(\sigma, F_1) \sim \Phi(\sigma)$ as $\sigma \to +\infty$. Theorem 2 is proved.

4. Some open problems. In this section, we formulate some unsolved problems that naturally arise in connection with the above results.

Problem 1. *Let $\rho \geq 1$. Describe the class Λ_ρ of all sequences $\lambda \in \Lambda$ for which there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$ such that $\ln M(\sigma, F)$ is a regularly varying function with index ρ.*

The following statement gives the solution of a similar problem for $\ln \mu(\sigma, F)$.

Proposition 1. *Let $\rho \geq 1$ and let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ. Then there exists a Dirichlet series $F \in \mathcal{D}(\lambda)$ ($F \in \mathcal{D}^*(\lambda)$) such that $\ln \mu(\sigma, F)$ is a regularly varying function with index ρ if and only if* (6) *holds.*

As we noted above, the necessity of condition (6) in Proposition 1 directly follows from Theorem A as a consequence of conditions (2), (4) and (5). The sufficiency of condition (6) is a direct consequence of Lemma 5, although theorem A can also be used for its justification.

Note that in case of a positive answer to the question of the following problem, the class Λ_ρ will coincide with the class of all sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ from the class Λ satisfying (6).

Problem 2. *Let $\rho \geq 1$ and let $F \in \mathcal{D}$ be a Dirichlet series such that $\ln M(\sigma, F)$ is a regularly varying function with index ρ. Does* (6) *necessarily hold then?*

Let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence from the class Λ. Consider an arbitrary Dirichlet series F of the form (1). If $\sigma_a(F) > -\infty$, then for all $\sigma < \sigma_a(F)$ we put

$$\mathfrak{M}(\sigma, F) = \sum_{n=0}^{\infty} |a_n| \, e^{\sigma\lambda_n}.$$

Note that if $a_n \geq 0$ for every integer $n \geq 0$, then $M(\sigma, F) = \mathfrak{M}(\sigma, F)$ for all $\sigma < \sigma_a(F)$. For an arbitrary integer $n \geq 0$ we put $T_n = \sum_{k=n}^{\infty} |a_k|$ and consider the Dirichlet series

$$F^*(s) = \sum_{n=0}^{\infty} T_n e^{s\lambda_n}.$$

It is well known that $F \in \mathcal{D}$ if and only if $F^* \in \mathcal{D}^*$ (see, for example, [9]).

If $F \in \mathcal{D}$ is a Dirichlet series of the form (1) with nonnegative coefficients a_n, then in this case the answer to the question of Problem 2 is positive. This follows from the following statement.

Proposition 2. *Let $\rho \geq 1$ and let $F \in \mathcal{D}$ be a Dirichlet series of the form* (1). *Then the following statements are equivalent:*

(i) $\ln \mathfrak{M}(\sigma, F)$ is a regularly varying function with index ρ;

(ii) $\ln \mu(\sigma, F^*)$ is a regularly varying function with index ρ;

(iii) there exists an increasing sequence $(n_k)_{k=0}^{\infty}$ of nonnegative integers such that

$$\varkappa_k := \frac{\ln |T_{n_k}| - \ln |T_{n_{k+1}}|}{\lambda_{n_{k+1}} - \lambda_{n_k}} \uparrow +\infty, \quad k \to \infty;$$

$$|T_n| e^{\varkappa_k \lambda_n} \leq |T_{n_k}| e^{\varkappa_k \lambda_{n_k}}, \quad n_k < n < n_{k+1}, \quad k \geq 0;$$

$$c_k := \frac{\varkappa_k \lambda_{n_{k+1}}}{\varkappa_k \lambda_{n_{k+1}} + \ln |T_{n_{k+1}}|} \to \rho, \quad k \to \infty;$$

$$d_k := \frac{\varkappa_k \lambda_{n_k}}{\varkappa_k \lambda_{n_k} + \ln |T_{n_k}|} \to \rho, \quad k \to \infty.$$

The equivalence of statements (i) and (ii) in Proposition 2 is easy to prove by using the following inequalities

$$\mu(\sigma, F^*) \leq \mathfrak{M}(\sigma, F) \leq \frac{\sigma + \varepsilon}{\varepsilon} \mu(\sigma + \varepsilon, F^*),$$

which hold for arbitrary $\sigma \geq 0$ and $\varepsilon > 0$ (see [9]). The equivalence of statements (ii) and (iii) follows from Theorem A applied to the series F^*.

In connection with Theorem 1, the following problem arises.

Problem 3. *Let $\rho \geq 1$ and let $\lambda = (\lambda_n)_{n=0}^{\infty}$ be an arbitrary sequence from the class Λ, for which (6) holds, and (7) is false. Does there exist a Dirichlet series $F \in \mathcal{D}(\lambda)$ such that $\ln M(\sigma, F)$ is a regularly varying function with index ρ, and $\ln \mu(\sigma, F)$ is not a regularly varying function?*

Note that examples of Dirichlet series $F \in \mathcal{D}$ of the form (1), for which $\ln M(\sigma, F)$ is a regularly varying function with given index $\rho > 1$, and $\ln \mu(\sigma, F)$ is not a regularly varying function with the same index, were constructed in [4]. For the exponents of the constructed series we have $\lambda_n \sim (\ln n)^{(q-1)/q}$ as $n \to \infty$, where $1 < q \leq \rho$.

REFERENCES

1. E. Seneta, Regularly Varying Functions, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
2. M.V. Zabolotskyi, M.M. Sheremeta, *On the slow growth of the main characteristics of entire functions,* Math. Notes, **65** (1999), №2, 168–174. https://doi.org/10.1007/BF02679813
3. P.V. Filevych, M.M. Sheremeta, *On the regular variation of main characteristics of an entire function,* Ukr. Math. J., **55** (2003), №6, 1012–1024. https://doi.org/10.1023/B:UKMA.0000010600.46493.2c
4. P.V. Filevych, M.M. Sheremeta, *Regularly increasing entire Dirichlet series,* Math. Notes, **74** (2003), №1, 110–122. https://doi.org/10.1023/A:1025027418525
5. M.M. Dolynyuk, O.B. Skaskiv, *About the regular growth of some positive functional series*, Nauk. Visn. Cherniv. National Univer. Mat., (2006), Iss. 314–315, 50–58. https://bmj.fmi.org.ua/index.php/adm/article/view/519
6. T.Ya. Hlova, P.V. Filevych, *Paley effect for entire Dirichlet series,* Ukr. Math. J., **67** (2015), №6, 838–852. https://doi.org/10.1007/s11253-015-1117-x
7. T.Ya. Hlova, P.V. Filevych, *Generalized types of the growth of Dirichlet series,* Carpathian Math. Publ., **7** (2015), №2, 172–187. https://doi.org/10.15330/cmp.7.2.172-187
8. T.Ya. Hlova, P.V. Filevych, *The growth of entire Dirichlet series in terms of generalized orders,* Sb. Math., **209** (2018), №2, 241–257. https://doi.org/10.1070/SM8644
9. M.M. Sheremeta, *On the growth of an entire Dirichlet series,* Ukr. Math. J., **51** (1999), №8, 1296–1302. https://doi.org/10.1007/BF02592520
10. P.V. Filevych, *On Valiron's theorem on the relations between the maximum modulus and the maximal term of an entire Dirichlet series,* Russ. Math., **48** (2004), №4, 63–69.

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Received 28.08.2022

Revised 15.12.2022