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A NEW MODEL OF THE FREE MONOGENIC DIGROUP

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It is well-known that one of open problems in the theory of Leibniz algebras is to find a suitable generalization of Lie's third theorem which associates a (local) Lie group to any Lie algebra, real or complex. It turns out, this is related to finding an appropriate analogue of a Lie group for Leibniz algebras. Using the notion of a digroup, Kinyon obtained a partial solution of this problem, namely, an analogue of Lie's third theorem for the class of so-called split Leibniz algebras. A digroup is a nonempty set equipped with two binary associative operations, a unary operation and a nullary operation satisfying additional axioms relating these operations. Digroups generalize groups and have close relationships with the dimonoids and dialgebras, the trioids and trialgebras, and other structures. Recently, G. Zhang and Y. Chen applied the method of Gröbner–Shirshov bases for dialgebras to construct the free digroup of an arbitrary rank, in particular, they considered a monogenic case separately. In this paper, we give a simpler and more convenient digroup model of the free monogenic digroup. We construct a new class of digroups which are based on commutative groups and show how the free monogenic group can be obtained from the free monogenic digroup by a suitable factorization.

1. Introduction. The notion of a digroup first implicitly appeared in Loday's work [5]. Later, Phillips gave a simple basis of independent axioms for the variety of digroups. Recall that a nonempty set G equipped with two binary operations $\vdash$ and $\dashv$, a unary operation $^{-1}$, and a nullary operation 1, is called a *digroup* [8, Theorem 2] if the following conditions hold:

(G_1) $(G,\vdash)$ and $(G,\dashv)$ are semigroups,

(G_2) $x\vdash(x\dashv z)=(x\vdash x)\dashv z$,

(G_3) $1\vdash x=x=x\dashv 1$,

(G_4) $x\vdash x^{-1}=1=x^{-1}\dashv x$.

An element 1 is called a *bar-unit* of the digroup and x^{-1} is said to be *inverse* to x with respect to 1. Other than in groups, a digroup can have many bar-units (see Example 1 (b) bellow). If binary operations of a digroup coincide, the digroup becomes a group. Thus, digroups are generelizations of groups.

Example 1. (a) Let V be a finite dimensional vector space and φ be an idempotent (i.e., $\varphi^2=\varphi$) linear operator of V. Define operations $\vdash$ and $\dashv$ on V by $x\vdash y:=x\varphi+y$, $x\dashv y:=x+y\varphi$ for all $x,y\in V$. Take the null-vector 0 in V as a bar-unit and put $g^{-1}=-g$ for all $g\in V$. Then $(V,\vdash,\dashv,^{-1},0)$ is a digroup [2, Example 3.2].

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(b) Let $(G,+)$ be a semigroup. Define $x \vdash y := y$ and $x \dashv y := x$. Select any element of G as 1 and define all x^{-1} as 1. Then $(G, \vdash, \dashv, ^{-1}, 1)$ becomes a digroup [13, Example 3.1] in which every element is a bar-unit, and 1 is the inverse of every element. Observe that $(G, \dashv, \vdash, ^{-1}, 1)$ does not yield a digroup; hence the axioms for a digroup are not "self-dual".

Digroups are closely related to dimonoids [5] which also play an important role in problems from the theory of Leibniz algebras and they have been studied by many authors (see, e.g., [1, 6, 14, 17]). One of the first results about digroups is the proof of the fact that Cayley's theorem for groups has an analogue in the class of all digroups [4]. Kinyon modified Loday's terminology and showed that every digroup is a product of a group and a trivial digroup [3]. Examples of different digroups can be found in [13]. Digroup analogues of some structure results of group theory were obtained in [7]. Some properties of generalized digroups and also generalized dimonoids were investigated in [9, 15]. The free digroup of an arbitrary rank was constructed in [11], where in particular the free monogenic digroup was separately described in a slightly cumbersome way. For other recent works on digroups see, for instance, [10, 16]. The main purpose of this paper is to obtain a clearer description of the free monogenic digroup.

The paper is organised as follows. In Section 2, we give a known construction of the free digroup of rank 1 and a description of its halo and group parts. In Section 3, we present a new model of the free monogenic digroup which is a simpler and more convenient digroup construction. Besides that, we find a new class of non-commutative digroups which are defined by commutative groups and describe the least group congruence on the free monogenic digroup.

2. The free monogenic digroup. Let k be an arbitrary field. Recall that a dialgebra is a k-module equipped with two binary associative operations $\vdash$ and $\dashv$, satisfying axioms of a dimonoid (see, e.g., [5]). For every dialgebra (dimonoid) D, any parenthesizing $x_1 \vdash ... \vdash x_i \dashv ... \dashv x_n$ of elements $x_1, x_2, ..., x_n \in D$ gives the same element in D which is denoted by $[x_1x_2 \ ... \ x_n]_i$. For $n = 1$, the notation $[x_1]_1$ means simply x_1.

Observe that by the definition of digroups and dialgebras, the classes of all digroups and the ones of all dialgebras form varieties in the sense of universal algebra. So free objects exist in both varieties, and they are uniquely determined (up to isomorphism) by the cardinality of their free generating sets.

Let $Di\langle X\rangle$ be the free dialgebra over k generated by a set X, and X^+ (X^*) be the free semigroup (the free monoid) on X, and ε is the empty word of X^*. As usual, the length of $u \in X^+$ is denoted by $|u|$. It is well-known [12] that

$$[X^+]_w = \{[u]_n \mid u \in X^+, 1 \leq n \leq |u|\}$$

is a free dimonoid on X and a k-basis of $Di\langle X\rangle$, where for all $[u]_i, [v]_j \in [X^+]_w$, operations $\vdash, \dashv$ are defined as follows

$$[u]_i \vdash [v]_j = [uv]_{|u|+j}, \quad [u]_i \dashv [v]_j = [uv]_i.$$

Here for elements $[u]_i$ from the free dimonoid $[X^+]_w$, a number i corresponds to i-th letter of the word u represented in the canonical form. For example, $[x^3xx^2]_2 = x \vdash x \dashv x \dashv x \dashv x \dashv x = [x^6]_2$, where $x \in X$.

Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{Z}$ be the set of all integers. For any set X, let $X^{-1} = \{x^{-1} | x \in X\}$. For the sake of convenience, we will write x^{-n} $(x^{-1} \notin X)$ instead of $(x^{-1})^n$ for all $n \in \mathbb{N}$. For every $i \in \mathbb{Z}$, under the length of x^i we mean the absolute value $|i|$ of i. Let $x^0 = \varepsilon$ for all $x \in X$.

Definition 1. Let $\mathcal{D} = (D, \vdash, \dashv, \perp, 1)$ be an arbitrary digroup. The set of all bar-units of $\mathcal{D}$ is called the *halo part* and it is denoted by $E(\mathcal{D})$ or simply by E; the set of all inverse elements of $\mathcal{D}$ is called the *group part* and denoted by $J(\mathcal{D})$ or simply by J.

It should be noted that the group part of any digroup $(D, \vdash, \dashv, \perp, 1)$ is a group in which operations $\vdash$ and $\dashv$ coincide [3, Lemma 4.5 (3)].

The following statement describes the structure of a free monogenic digroup generated by a single element, and its group and halo parts.

Lemma 1 ([11], Corollary 3.7)**.** *Let $X = \{x\}$ and $e \notin X \cup X^{-1}$ be an arbitrary symbol. The free monogenic digroup on X is the set $F(X) = \Omega_e \cup \Omega_x \cup \Omega_{x^{-1}}$, where $\Omega_e = \{[ex^n]_1 \mid n \geq 0\}$, $\Omega_x = \{[x^i x x^j]_{|i|+1} \mid i, j \in \mathbb{Z}\}$, and $\Omega_{x^{-1}} = \{[x^{-m}]_1 \mid m \geq 1\}$, with e as a bar-unit and operations $\succ, \prec$, and $\perp$ defined by the rule:*

$\succ$	$[ex^{n'}]_1$	$[x^{i'}xx^{j'}]_{\|i'\|+1}$	$[x^{-m'}]_1$
$[ex^n]_1$	$[ex^{n+n'}]_1$	$[x^{n+i'}xx^{j'}]_{\|n+i'\|+1}$	$[ex^p]_1, p \geq 0$ $[x^p]_1, p < 0$
$[x^ixx^j]_{\|i\|+1}$	$[ex^s]_1, s \geq 0$ $[x^s]_1, s < 0$	$[x^{t+i'}xx^{j'}]_{\|t+i'\|+1}$	$[ex^q]_1, q \geq 0$ $[x^q]_1, q < 0$
$[x^{-m}]_1$	$[ex^{p'}]_1, p' \geq 0$ $[x^{p'}]_1, p' < 0$	$[x^{-m+i'}xx^{j'}]_{\|-m+i'\|+1}$	$[x^{-m-m'}]_1$

$\prec$	$[ex^{n'}]_1$	$[x^{i'}xx^{j'}]_{\|i'\|+1}$	$[x^{-m'}]_1$
$[ex^n]_1$	$[ex^{n+n'}]_1$	$[ex^{s'}]_1, s' \geq 0$ $[x^{s'}]_1, s' < 0$	$[ex^p]_1, p \geq 0$ $[x^p]_1, p < 0$
$[x^ixx^j]_{\|i\|+1}$	$[x^ixx^{j+n'}]_{\|i\|+1}$	$[x^ixx^{j+t'}]_{\|i\|+1}$	$[x^ixx^{j-m'}]_{\|i\|+1}$
$[x^{-m}]_1$	$[ex^{p'}]_1, p' \geq 0$ $[x^{p'}]_1, p' < 0$	$[ex^{q'}]_1, q' \geq 0$ $[x^{q'}]_1, q' < 0$	$[x^{-m-m'}]_1$

$$([ex^n]_1)^{\perp} = [x^{-n}]_1, \quad ([x^{-m}]_1)^{\perp} = [ex^m]_1,$$

$$([x^ixx^j]_{|i|+1})^{\perp} = \begin{cases} [ex^{-t}]_1, & t \leq 0; \\ [x^{-t}]_1, & t > 0. \end{cases}$$

Here $[ex^n]_1, [ex^{n'}]_1 \in \Omega_e$, and $[x^ixx^j]_{|i|+1}, [x^{i'}xx^{j'}]_{|i'|+1} \in \Omega_x$, and $[x^{-m}]_1, [x^{-m'}]_1 \in \Omega_{x^{-1}}$, and $t = i + j + 1$, $t' = i' + j' + 1$, $p = n - m'$, $p' = -m + n'$, $q = t - m'$, $q' = -m + t'$, $s = t + n'$, $s' = n + t'$.

In addition, the sets $J = \{[ex^n]_1 \mid n \geq 0\} \cup \{[x^{-m}]_1 \mid m \geq 1\}$ and

$$E = \{e\} \cup \{[x^{-n}x^n]_{n+1} \mid n \geq 1\} \cup \{[x^nx^{-n}]_n \mid n \geq 1\}$$

are the group part and, respectively, the halo part of the digroup $\mathcal{F}(X) = (F(X), \succ, \prec, \perp, e)$.

3. A new model of the free digroup of rank 1. Let $(G,+)$ be an arbitrary commutative group with the unit 0. Denote by $D(G)$ the union of G and of the direct product $G\times G$, that is, $D(G)=G\cup(G\times G)$, and let $\eta\in G$. Further we extend the operation of an addition $+$ on G to two binary operations $\vdash,\dashv$ on $D(G)$ and define a unary operation $\dagger$ on $D(G)$ as follows:

$$a\vdash(b,c)=(a+b,c),\quad (b,c)\vdash a=b+c+a+\eta,\quad (a,b)\vdash(c,d)=(a+b+c+\eta,d),$$
$$a\dashv(b,c)=a+b+c+\eta,\quad (b,c)\dashv a=(b,c+a),\quad (a,b)\dashv(c,d)=(a,b+c+d+\eta),$$

and

$$x^{\dagger}=\begin{cases}-x, & x\in G;\\ -x_1-x_2-\eta, & x=(x_1,x_2)\in G\times G\end{cases}$$

for all $a,b,c,d\in G$ and $x\in D(G)$.

The obtained algebra $(D(G),\vdash,\dashv,\dagger,0)$ is denoted by $\mathcal{D}_\eta(\mathcal{G})$.

Proposition 1. *For any commutative group G and every $\eta\in G$, the algebra $\mathcal{D}_\eta(\mathcal{G})$ is a digroup.*

Proof. Firstly, we show that $(D(G),\vdash)$ is a semigroup. Let $a,\ b,\ c\in D(G)$. The case $a,b,c\in G$ is trivial.

For $a=(a_1,a_2)\in G\times G$ and $b,c\in G$, we have

$$a\vdash(b\vdash c)=a_1+a_2+b+c+\eta=(a_1+a_2+b+\eta)\vdash c=(a\vdash b)\vdash c.$$

Let $a,c\in G$ and $b=(b_1,b_2)\in G\times G$. Then

$$a\vdash(b\vdash c)=a\vdash(b_1+b_2+c+\eta)=a+b_1+b_2+c+\eta=(a+b_1,b_2)\vdash c=(a\vdash b)\vdash c.$$

If $a,b\in G$ and $c=(c_1,c_2)\in G\times G$, we have

$$a\vdash(b\vdash c)=a\vdash(b+c_1,c_2)=(a+b+c_1,c_2)=(a\vdash b)\vdash c.$$

Now let $a=(a_1,a_2),\ b=(b_1,b_2)\in G\times G$ and $c\in G$. Then

$$\begin{aligned}a\vdash(b\vdash c)&=(a_1,a_2)\vdash(b_1+b_2+c+\eta)=a_1+a_2+b_1+b_2+c+2\eta=\\&=(a_1+a_2+b_1+\eta,b_2)\vdash c=(a\vdash b)\vdash c.\end{aligned}$$

If $a=(a_1,a_2),c=(c_1,c_2)\in G\times G$ and $b\in G$, then

$$\begin{aligned}a\vdash(b\vdash c)&=(a_1,a_2)\vdash(b+c_1,c_2)=(a_1+a_2+b+c_1+\eta,c_2)=\\&=(a_1+a_2+b+\eta)\vdash(c_1,c_2)=(a\vdash b)\vdash c.\end{aligned}$$

For $a\in G$ and $b=(b_1,b_2)\,,c=(c_1,c_2)\in G\times G$, we have

$$\begin{aligned}a\vdash(b\vdash c)&=a\vdash(b_1+b_2+c_1+\eta,c_2)=(a+b_1+b_2+c_1+\eta,c_2)=\\&=(a+b_1,b_2)\vdash(c_1,c_2)=(a\vdash b)\vdash c.\end{aligned}$$

Finally, for $a=(a_1,a_2),b=(b_1,b_2),c=(c_1,c_2)\in G\times G$,

$$\begin{aligned}a\vdash(b\vdash c)&=(a_1,a_2)\vdash(b_1+b_2+c_1+\eta,c_2)=(a_1+a_2+b_1+b_2+c_1+2\eta,c_2)=\\&=(a_1+a_2+b_1+\eta,b_2)\vdash(c_1,c_2)=(a\vdash b)\vdash c.\end{aligned}$$

Thus, $(D(G), \vdash)$ is a semigroup. Analogously, one can prove that $(D(G), \dashv)$ is a semigroup, too. So, the axiom (G_1) holds.

Now let $a, c \in D(G)$. It is obvious that (G_2) holds if $a, c \in G$. For $a = (a_1, a_2) \in G \times G$ and $c \in G$, we have

$$a \vdash (a \dashv c) = (a_1, a_2) \vdash (a_1, a_2 + c) = (2a_1 + a_2 + \eta, a_2 + c) =$$
$$= (2a_1 + a_2 + \eta, a_2) \dashv c = (a \vdash a) \dashv c.$$

If $a \in G$ and $c = (c_1, c_2) \in G \times G$, then

$$a \vdash (a \dashv c) = a \vdash (a + c_1 + c_2 + \eta) = 2a + c_1 + c_2 + \eta = 2a \dashv (c_1, c_2) = (a \vdash a) \dashv c.$$

For $a = (a_1, a_2), c = (c_1, c_2) \in G \times G$ we obtain that

$$a \vdash (a \dashv c) = (a_1, a_2) \vdash (a_1, a_2 + c_1 + c_2 + \eta) =$$
$$= (2a_1 + a_2 + \eta,\ a_2 + c_1 + c_2 + \eta) = (2a_1 + a_2 + \eta, a_2) \dashv (c_1, c_2) = (a \vdash a) \dashv c.$$

Thus, the axiom (G_2) holds.

Further we show that for the algebra $(D(G), \vdash, \dashv)$ there exists at least one bar-unit. Indeed, a bar-unit of $(D(G), \vdash, \dashv)$ is, for example, $0 \in G$ since for all $a \in G$ and $b = (b_1, b_2) \in G \times G$ we have

$$0 \vdash a = 0 + a = a = a + 0 = a \dashv 0, \quad 0 \vdash b = (0 + b_1, b_2) = b = (b_1, b_2 + 0) = b \dashv 0.$$

It means that (G_3) holds, too. So we take 0 as the element determined by the nullary operation.

At the end, we check the last axiom (G_4). For every $x \in G$ there exists an inverse element $x^\dagger = -x \in G$ such that

$$x + x^\dagger = 0 = x^\dagger + x.$$

In addition, for every pair $(x, y) \in G \times G$ there exists an inverse element $(x, y)^\dagger = -x - y - \eta \in G$ such that

$$(x, y) \vdash (x, y)^\dagger = (x, y) \vdash (-x - y - \eta) =$$
$$= x + y + (-x - y - \eta) + \eta = 0 = (-x - y - \eta) \dashv (x, y) = (x, y)^\dagger \dashv (x, y).$$

By the definition, $\mathcal{D}_\eta(\mathcal{G})$ is a digroup. $\square$

So, Proposition 1 gives a new class of non-commutative digroups (both digroup operations are not commutative) which are defined by commutative groups.

The following remark can be checked directly.

Remark 1. For the digroup $\mathcal{D}_\eta(\mathcal{G})$ we obtain that the group part $J = G$ and the halo part $E = \{0, (x, -x - \eta) \mid x \in G\}$.

Definition 2. Let $\mathcal{G} = (G, \vdash, \dashv, ^{-1}, 1_G)$ and $\mathcal{G}' = (G', \vdash', \dashv', ^{-1'}, 1_{G'})$ be arbitrary digroups with bar-units 1_G and $1_{G'}$, respectively. A mapping ϕ from $\mathcal{G}$ to $\mathcal{G}'$ is called a *digroup homomorphism* if for all $x, y \in G$ and any $* \in \{\vdash, \dashv\}$, the following conditions hold: $(x * y)\phi = x\phi *' y\phi$, $x^{-1}\phi = (x\phi)^{-1'}$, and $1_G\phi = 1_{G'}$.

Let $\mathcal{Z} = (\mathbb{Z}, +)$ be the additive group of all integers and let $\eta = 1 \in \mathbb{Z}$. The main result of this paper is the following statement.

Theorem 1. *The free monogenic digroup $\mathcal{F}(X) = (F(X), \succ, \prec, \perp, e)$ is isomorphic to the algebra $\mathcal{D}_1(\mathcal{Z}) = (D(\mathbb{Z}), \vdash, \dashv, \dagger, 0)$.*

Proof. Define a mapping ξ from the free monogenic digroup $\mathcal{F}(X)$ into the algebra $\mathcal{D}_1(\mathcal{Z})$ in the following way

$$a\xi = \begin{cases} n, & \text{if } a = [ex^n]_1, n \geq 0, \\ -m, & \text{if } a = [x^{-m}]_1, m \geq 1, \\ (i,j), & \text{if } a = [x^i x x^j]_{|i|+1}. \end{cases}$$

It is not hard to see that ξ is a bijection. We show that ξ is a semigroup homomorphism from $(F(X), \succ)$ to $(D(\mathbb{Z}), \vdash)$.

Let $[ex^n]_1, [ex^{n'}]_1 \in \Omega_e$, and $[x^i x x^j]_{|i|+1}, [x^{i'} x x^{j'}]_{|i'|+1} \in \Omega_x$, and $[x^{-m}]_1, [x^{-m'}]_1 \in \Omega_{x^{-1}}$. Using the definition of operations $\succ, \prec$, and auxiliary denotations from Lemma 1, we consider the following 9 cases.

1) $a = [ex^n]_1, b = [ex^{n'}]_1$, then

$$(a \succ b)\xi = [ex^{n+n'}]_1\xi = n + n' = a\xi \vdash b\xi.$$

2) $a = [ex^n]_1, b = [x^{i'} x x^{j'}]_{|i'|+1}$. Then

$$(a \succ b)\xi = [x^{n+i'} x x^{j'}]_{|n+i'|+1}\xi = (n + i', j') = n \vdash (i', j') = a\xi \vdash b\xi.$$

3) $a = [ex^n]_1, b = [x^{-m'}]_1$. In this case,

$$(a \succ b)\xi = p = n - m' = a\xi \vdash b\xi.$$

4) $a = [x^i x x^j]_{|i|+1}, b = [ex^{n'}]_1$, then we have

$$(a \succ b)\xi = s = t + n' = (i,j) \vdash n' = a\xi \vdash b\xi.$$

5) $a = [x^i x x^j]_{|i|+1}, b = [x^{i'} x x^{j'}]_{|i'|+1}$, then

$$(a \succ b)\xi = [x^{t+i'} x x^{j'}]_{|t+i'|+1}\xi = (t + i', j') = (i,j) \vdash (i', j') = a\xi \vdash b\xi.$$

6) $a = [x^i x x^j]_{|i|+1}, b = [x^{-m'}]_1$. Then

$$(a \succ b)\xi = q = t - m' = (i,j) \vdash (-m') = a\xi \vdash b\xi.$$

7) $a = [x^{-m}]_1, b = [ex^{n'}]_1$. For this case,

$$(a \succ b)\xi = p' = -m + n' = a\xi \vdash b\xi.$$

8) $a = [x^{-m}]_1, b = [x^{i'} x x^{j'}]_{|i'|+1}$, then we obtain

$$(a \succ b)\xi = [x^{-m+i'} x x^{j'}]_{|-m+i'|+1}\xi = (-m + i', j') = (-m) \vdash (i', j') = a\xi \vdash b\xi.$$

9) $a = [x^{-m}]_1, b = [x^{-m'}]_1$, then

$$(a \succ b)\xi = [x^{-m-m'}]_1\xi = -m - m' = a\xi \vdash b\xi.$$

So, ξ is a semigroup homomorphism. The fact that ξ is a homomorphism from $(F(X), \prec)$ to $(D(\mathbb{Z}), \dashv)$ can be proved analogously.

Finally, we note that $e\xi = [e]_1\xi = [ex^0]_1\xi = 0$ and in addition,

$$([ex^n]_1\xi)^\dagger = n^\dagger = -n = [x^{-n}]_1\xi = ([ex^n]_1)^\perp\xi,$$
$$([x^{-m}]_1\xi)^\dagger = (-m)^\dagger = m = [ex^m]_1\xi = ([x^{-m}]_1)^\perp\xi,$$
$$([x^ixx^j]_{|i|+1}\xi)^\dagger = (i,j)^\dagger = -i-j-1 = -t = \begin{cases} [ex^{-t}]_1\xi = ([x^ixx^j]_{|i|+1})^\perp\xi, & t \leq 0, \\ [x^{-t}]_1\xi = ([x^ixx^j]_{|i|+1})^\perp\xi, & t > 0. \end{cases}$$

□

From the last theorem it follows that the digroup $\mathcal{D}_1(\mathcal{Z})$ is the free monogenic digroup up to an isomorphism and it is generated by $(0,0)$. Observe that the obtained digroup model $\mathcal{D}_1(\mathcal{Z})$ is a simpler and more convenient construction.

Remark 2. Theorem 1 is similar to the statement about that the free group of rank 1 is isomorphic to the additive group $\mathcal{Z}$ of all integers. Note that the group $\mathcal{Z}$ is commutative while the free monogenic digroup $\mathcal{D}_1(\mathcal{Z})$ is not commutative.

Finally, we show how the free monogenic group can be obtained from the free monogenic digroup by a suitable factorization. It is obvious that the free group of rank 1 one can consider as a digroup in which the binary operations coincide and 0 is the fixed bar-unit.

A congruence ρ on a digroup $\mathcal{D} = (D, \vdash, \dashv, \perp, 1)$ is called a *group congruence* if the binary operations of $\mathcal{D}/\rho$ coincide and $\mathcal{D}/\rho$ is a group.

Proposition 2. *A mapping ψ of the free monogenic digroup $\mathcal{D}_1(\mathcal{Z})$ into the free monogenic group $\mathcal{Z}$ defined as follows:*

$$x\psi = \begin{cases} x_1 + x_2 + 1, & x = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z}; \\ x, & x \in \mathbb{Z}, \end{cases}$$

is an epimorphism inducing the least group congruence on $\mathcal{D}_1(\mathcal{Z})$.

Proof. Let $a, b \in D(\mathcal{Z})$ be arbitrary elements. The case $a, b \in \mathbb{Z}$ is trivial. For $a = (a_1, a_2)$ and $b \in \mathbb{Z}$, we have

$$(a \vdash b)\psi = (a_1 + a_2 + b + 1)\psi = a_1 + a_2 + b + 1 = a\psi + b\psi,$$
$$(a \dashv b)\psi = (a_1, a_2 + b)\psi = a_1 + a_2 + b + 1 = a\psi + b\psi.$$

If $a \in \mathbb{Z}$ and $b = (b_1, b_2)$, then

$$(a \vdash b)\psi = (a + b_1, b_2)\psi = a + b_1 + b_2 + 1 = a\psi + b\psi,$$
$$(a \dashv b)\psi = (a + b_1 + b_2 + 1)\psi = a + b_1 + b_2 + 1 = a\psi + b\psi.$$

For the case $a = (a_1, a_2), b = (b_1, b_2)$, we obtain

$$(a \vdash b)\psi = (a_1 + a_2 + b_1 + 1, b_2)\psi = a_1 + a_2 + b_1 + 1 + b_2 + 1 = a\psi + b\psi,$$
$$(a \dashv b)\psi = (a_1, a_2 + b_1 + b_2 + 1)\psi = a_1 + a_2 + b_1 + b_2 + 2 = a\psi + b\psi.$$

Thus, ψ is a homomorphism which obviously is surjective. In addition, $0\psi = 0$, and $a^{-1}\psi = -a = (a\psi)^{-1}$ if $a \in \mathbb{Z}$, and

$$b^{-1}\psi = (-b_1 - b_2 - 1)\psi = -b_1 - b_2 - 1 = (b_1 + b_2 + 1)^{-1} = (b\psi)^{-1}$$

for all $b = (b_1, b_2) \in \mathbb{Z} \times \mathbb{Z}$.

Since $\mathcal{D}_1(\mathcal{Z})$ and $\mathcal{Z}$ are free algebras, and $\mathcal{Z}$ is a group, then the kernel of the epimorphism ψ is the least group congruence on $\mathcal{D}_1(\mathcal{Z})$. $\square$

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