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# ON MODULUS INEQUALITY OF THE ORDER P FOR THE INNER DILATATION

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The article is devoted to mappings with bounded and finite distortion of planar domains. Our investigations are devoted to the connection between mappings of the Sobolev class and upper bounds for the distortion of the modulus of families of paths. For this class, we have proved the Poletsky-type inequality with respect to the so-called inner dilatation of the order $p$. We separately considered the situations of homeomorphisms and mappings with branch points. In particular, we have established that homeomorphisms of the Sobolev class satisfy the upper estimate of the distortion of the modulus at the inner and boundary points of the domain. In addition, we have proved that similar estimates of capacity distortion occur at the inner points of the domain for open discrete mappings. Also, we have shown that open discrete and closed mappings satisfy some estimates of the distortion of the modulus of families of paths at the boundary points. The results of the manuscript are obtained mainly under the condition that the so-called inner dilatation of mappings is locally integrable. The main approach used in the proofs is the choice of admissible functions, using the relations between the modulus and capacity, and connections between different modulus of families of paths (similar to Hesse, Ziemer and Shlyk equalities). In this context, we have obtained some lower estimate of the modulus of families of paths in Sobolev classes. The manuscript contains some examples related to applications of obtained results to specific mappings.

**1. Introduction.** This article is devoted to establishing estimates of the distortion of the modulus of families of paths under mappings. The study of modulus and capacity inequalities is fundamentally important, since they are the main tools for the study of various mappings. In this sense, we could point to a fairly wide range of applications in which the modulus and capacity estimates are used. For example, such estimates participate under establishing the equicontinuity of families of maps, the removability of an isolated singularity, the possibility of a continuous extension to the boundary, see, e.g. [2]–[4], [7], [11], [12], [16]–[19], [21], [22] and [36]. In addition, modulus (capacity) inequalities may be used in proofs of the existence theorems of Beltrami equations and the investigation of the Dirichlet problem for it (see, e.g., [5], [8], [19], [24] and [32]).

The main object of the study is the Sobolev classes on the plane. Our manuscript refers to the case when the inequality under study involves an inner dilatation of an arbitrary order. In addition, the mapping can admit branch points. Similar and close results of the authors may be found in [15], [27], [29] and [30].

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Let us turn to the definitions and the formulation of the main result. In what follows, $D$ is a domain in $\mathbb{C}$ and $dm(z)$ denotes the element of the Lebesgue measure on $\mathbb{C}$. As a rule, a mapping $f\colon D \to \mathbb{C}$ is assumed to be *sense-preserving,* moreover, we assume that $f$ has partial derivatives in real and imaginary parts almost everywhere. Put $z = x + iy$, $i^2 = -1$, $f_{\overline{z}} = (f_x + if_y)/2$ and $f_z = (f_x - if_y)/2$. Note that the Jacobian of $f$ at $z \in D$ is calculated by the formula

$$J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2\,.$$

Given $p \geqslant 1$, the *inner dilatation of the order* $p$ is defined as

$$K_{I,p}(z, f) = \frac{|f_z|^2 - |f_{\overline{z}}|^2}{(|f_z| - |f_{\overline{z}}|)^p} \tag{1}$$

for $J(z, f) \neq 0$; an addition, we set $K_{I,p}(z, f) = 1$ for $f'(z) = 0$ and $K_{I,p}(z, f) = \infty$ otherwise.

Recall the definition of Sobolev classes, which is the key to this manuscript. In what follows, $C_0^k(U)$ denotes the space of functions $u\colon U \to \mathbb{R}$ with a compact support in $U$, having $k$ partial derivatives with respect to any variable that are continuous in $U$. We also recall the concept of a generalized Sobolev derivative (see, for example, [23, Section 2, Ch. I]). Let $U$ be an open set, $U \subset \mathbb{C}$, $u\colon U \to \mathbb{R}$ some function, $u \in L^1_{\rm loc}(U)$. Suppose there is a function $v \in L^1_{\rm loc}(U)$ such that

$$\int\limits_U \frac{\partial \varphi}{\partial x_j}(z) u(z)\, dm(z) = -\int\limits_U \varphi(z) v(z)\, dm(z)$$

for any function $\varphi \in C_0^1(U)$, $j \in \{1, 2\}$. Then we say that the function $v$ is a generalized derivative of the first order of the function $u$ with respect to $x_j$ and denoted by $\frac{\partial u}{\partial x_j}(z) := v$. Here $z = x_1 + ix_2$, $i^2 = -1$.

A function $u \in W^{1,1}_{\rm loc}(U)$ if $u$ has generalized derivatives of the first order with respect to each of the variables in $U$, which are locally integrable in $U$.

A mapping $f\colon D \to \mathbb{C}$, $f(z) = u(z) + iv(z)$, belongs to the Sobolev class $W^{1,1}_{\rm loc}$, write $f \in W^{1,1}_{\rm loc}(D)$, if $u$ and $v$ have generalized partial derivatives of the first order, which are locally integrable in $D$ in the first degree. We write $f \in W^{1,k}_{\rm loc}(D)$, $k \in \mathbb{N}$, if all coordinate functions $f = (f_1, \ldots, f_n)$ have generalized partial derivatives of the first order, which are locally integrable in $D$ to the degree $k$.

Recall that a mapping $f$ between domains $D$ and $D'$ in $\mathbb{C}$ is of *finite distortion* if $f \in W^{1,1}_{\rm loc}$ and, besides that, there is a function $K(z) < \infty$ a.e. such that

$$\|f'(z)\|^2 \leqslant K(z) \cdot J(z, f)$$

for a.e. $z \in D$, where $\|f'(z)\| = |f_z| + |f_{\overline{z}}|$. For mappings of finite distortion, we refer to [11] and to the reference therein.

Let $Q\colon \mathbb{C} \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(z) \equiv 0$ for $z \in \mathbb{C} \setminus D$. Let $z_0 \in \overline{D}$, $z_0 \neq \infty$,

$$B(z_0, r) = \{z \in \mathbb{C}\colon |z - z_0| < r\}\,, \quad S(z_0, r) = \{z \in \mathbb{C}\colon |z - z_0| = r\}\,, \tag{2}$$

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C}\colon r_1 < |z - z_0| < r_2\}\,. \tag{3}$$

Recall that a *path* will be called a continuous mapping $\gamma\colon I\to\mathbb{R}^n$ of a segment, interval or half-interval $I\subset\mathbb{R}$ into $n$-dimensional Euclidean space $\mathbb{R}^n$. By a *family of paths* $\Gamma$ we mean some fixed set of paths $\gamma$. A Borel function $\rho\colon\mathbb{R}^n\to[0,\infty]$ is called *admissible* for $\Gamma$, abbr. $\rho\in\operatorname{adm}\Gamma$, if $\int_\gamma\rho(x)|dx|\geqslant 1$ for each (locally rectifiable) $\gamma\in\Gamma$. For $\alpha\geqslant 1$, we define the quantity

$$M_\alpha(\Gamma)\;=\;\inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^{\,\alpha}(x)\,dm(x) \tag{4}$$

and call $M_\alpha(\Gamma)$ *$\alpha$-modulus* of $\Gamma$ [35, 6.1]; here $m$ stands for the $n$-dimensional Lebesgue measure. If $\alpha=n$, we write $M(\Gamma):=M_n(\Gamma)$. Given $\alpha\geqslant 1$, a mapping $f\colon D\to\overline{\mathbb{C}}$ is called a *ring $Q$-mapping at a point $z_0\in\overline{D}\setminus\{\infty\}$ with respect to $\alpha$-modulus*, if the condition

$$M_\alpha(f(\Gamma(C_1,C_2,D)))\leqslant\int\limits_{A\cap D}Q(z)\cdot\eta^{\,\alpha}(|z-z_0|)\,dm(z) \tag{5}$$

holds for all $0<r_1<r_2<d_0:=\operatorname{dist}(z_0,\partial D)$, for any continua $C_1\subset\overline{B(z_0,r_1)}$, $C_2\subset D\setminus B(z_0,r_2)$ and all Lebesgue measurable functions $\eta\colon(r_1,r_2)\to[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{6}$$

A mapping $f$ is called a *ring $Q$-mapping in $D$ with respect to $\alpha$-modulus,* if condition (5) is satisfied at every point $z_0\in D$, and a *ring $Q$-mapping in $\overline{D}$ with respect to $\alpha$-modulus,* if the condition (5) holds at every point $z_0\in\overline{D}$. For the properties of such mappings see [19] and [24].

Recall that a pair $E=(A,\,C)$, where $A$ is an open set in $\mathbb{R}^n$, and $C$ is a compact subset of $A$, is called *condenser* in $\mathbb{R}^n$.

Let $G$ be an open subset of $\mathbb{R}^n$ and $I=\{x\in\mathbb{R}^n\colon a_j<x_j<b_j,\ j=1,\ldots,n\}$ an open $n$-dimensional interval. A mapping $f\colon I\to\mathbb{R}^n$ is said to *belong to the class $ACL$ (absolutely continuous on lines)* if it is absolutely continuous on almost all intervals of straight lines in $I$ which are parallel to coordinate axes. We say that a mapping $f\colon G\to\mathbb{R}^n$ *belongs to the class $ACL$* in $G$ if the restriction $f\big|_I$ belongs to the class $ACL$ for any interval $I$, $\overline{I}\subset G$.

A quantity

$$\operatorname{cap}_p E=\operatorname{cap}_p(A,\,C)=\inf_{u\,\in\,W_0(E)}\int\limits_A|\nabla u|^p\;dm(x)\,, \tag{7}$$

where $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$, $W_0(E)=W_0\,(A,\,C)$ is a family of all nonnegative absolutely continuous on lines (ACL) functions $u\colon A\to\mathbb{R}$ with compact support in $A$ and such that $u(x)\geqslant 1$ on $C$, is called *$p$-capacity* of the condenser $E$.

The main results of the paper are the following.

**Theorem 1.** *Let $f\colon D\to\mathbb{C}$ be a homeomorphism with a finite distortion and let $1<\alpha\leqslant 2$. Assume that $K_{I,\alpha}(z,f)\in L^1_{\rm loc}(D)$. Then $f$ satisfies the relation (5) at any point $z_0\in\overline{D}\setminus\{\infty\}$ with $Q(z)=K_{I,\alpha}(z,f)$.*

For a mapping $f\colon D\to\mathbb{R}^n$, a set $E\subset D$, and $y\in\mathbb{R}^n$, we define the *multiplicity function* $N(y,f,E)$ to be the number of preimages of $y$ in $E$, i.e.,

$$N(y,f,E)=\operatorname{card}\{x\in E:\ f(x)=y\}\,,\quad N(f,E)=\sup_{y\in\mathbb{R}^n}N(y,f,E). \tag{8}$$

Let $X$ and $Y$ be metric spaces. A mapping $f\colon X\to Y$ is *discrete* if $f^{-1}(y)$ is discrete for all $y\in Y$ and $f$ is *open* if $f$ maps open sets onto open sets. A mapping $f\colon X\to Y$ is called *closed* if $f(A)$ is closed in $f(X)$ whenever $A$ is closed in $X$. For mappings with a branching, we have the following.

**Theorem 2.** *Let $f\colon D\to\mathbb{C}$ be an open and discrete bounded mapping with a finite distortion such that $N(f,D)<\infty$. Let $1<\alpha\leqslant 2$ and let $z_0\in D$. If $K_{I,\alpha}(z,f)$ is integrable over $S(x_0,r)$ for almost all $r>0$, then $f$ satisfies the relation*

$$\operatorname{cap}_\alpha f(\mathcal{E})\leqslant\int\limits_A N^{\alpha-1}(f,D)K_{I,\alpha}(z,f)\cdot\eta^\alpha(|z-z_0|)\,dm(z) \tag{9}$$

*holds for $\mathcal{E}=(B(z_0,r_2),\overline{B(z_0,r_1)})$, $A=A(z_0,r_1,r_2)$, $0<r_1<r_2<\varepsilon_0:=\operatorname{dist}(z_0,\partial D)$, and $\eta\colon(r_1,r_2)\to[0,\infty]$ is arbitrary nonnegative Lebesgue measurable function satisfying the relation (6). In particular, Theorem 2 holds if $K_{I,\alpha}(z,f)\in L^1_{\rm loc}(D)$.*

Note that, the relation (9) is given in terms of a capacity, but not a modulus. For the modulus, we have the following.

**Theorem 3.** *Let $f\colon D\to\mathbb{C}$ be an open, discrete and closed bounded mapping of a finite distortion and let $1<\alpha\leqslant 2$. Assume that $K_{I,\alpha}(z,f)\in L^1_{\rm loc}(D)$. Then, for any $z_0\in\partial D$, any $\varepsilon_0<d_0:=\sup\limits_{z\in D}|z-z_0|$ and any compactum $C_2\subset D\setminus B(z_0,\varepsilon_0)$ there is $\varepsilon_1$, $0<\varepsilon_1<\varepsilon_0$, such that the relation*

$$M_\alpha(f(\Gamma(C_1,C_2,D)))\leqslant\int\limits_{A(z_0,\varepsilon,\varepsilon_1)\cap D}N^{\alpha-1}(f,D)K_{I,\alpha}(z,f)\eta^\alpha(|z-z_0|)\,dm(z) \tag{10}$$

*holds for any $\varepsilon\in(0,\varepsilon_1)$ and any $C_1\subset\overline{B(z_0,\varepsilon)}\cap D$, where $A(z_0,\varepsilon,\varepsilon_1)$ is defined in (3), and $\eta\colon(\varepsilon,\varepsilon_1)\to[0,\infty]$ is an arbitrary Lebesgue measurable function satisfying the relation (6).*

**Remark 1.** It should be noted that, the modulus and capacity inequalities for quasiconformal and quasiregular mappings (mappings with finite distortion) have been known for a long time. In this case, the relations (5), (9) or (10) hold for $\alpha=2$ (see, for example, [12], [19, Ch. 8] and [22, Ch. 2]). The fulfillment of modulus inequalities in Sobolev and Orlicz-Sobolev classes is a separate topic. Relevant results related to the search of "minimal conditions" under which these inequalities hold. As a rule, these conditions do not contain $N$- and $N^{-1}$-Luzin properties with respect to the Lebesgue measure. On the other hand, the Sobolev classes have $N$-property on almost all circles, and this is already needed to prove these relations. The same applies to the Orlicz-Sobolev classes in space, which have $N$-Luzin property on almost all spheres under the Calderon condition.

In particular, estimates similar to (5), (9) or (10) were established in [15] for homeomorphisms with $\alpha=2$ in (5), $Q(z)=K_{I,2}(z,f)$. The papers [27] and [30] deal with Orlicz-Sobolev classes in space (the plane case is not studied here). The article [28] refer to the case when the order of the modulus is equal to the dimension of the space. The case when mappings: **1**) are defined in some planar domain, **2**) are Sobolev mappings with a finite distortion, and **3**) the order of the modulus $\alpha$ not necessarily equal to 2, is considered for the first time in this manuscript.

**2. On lower estimates of the modulus.** Let us give some important information concerning the relationship between the moduli of the families of paths joining the sets and the moduli of the families of the sets separating these sets. Mostly this information can be found in Ziemer's publication, see [37]. Let $G$ be a bounded domain in $\mathbb{R}^n$, and $C_0, C_1$ are disjoint compact sets in $\overline{G}$. Put $R = G \setminus (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. For a number $p > 1$ we define a *$p$-capacity of the pair $C_0, C_1$ relative to the closure $G$* by the equality

$$C_p[G, C_0, C_1] = \inf \int\limits_R |\nabla u|^p \, dm(x),$$

where the exact lower bound is taken for all functions $u$, continuous in $R^*$, $u \in ACL(R)$, such that $u = 1$ on $C_1$ and $u = 0$ on $C_0$. These functions are called *admissible* for $C_p[G, C_0, C_1]$. We say that a set $\sigma \subset \mathbb{R}^n$ *separates* $C_0$ and $C_1$ in $R^*$, if $\sigma \cap R$ is closed in $R$ and there are disjoint sets $A$ and $B$, open relative $R^* \setminus \sigma$, such that $R^* \setminus \sigma = A \cup B$, $C_0 \subset A$ and $C_1 \subset B$. Let $\Sigma$ denote the class of all sets that separate $C_0$ and $C_1$ in $R^*$. For the number $p' = p/(p-1)$ we define the quantity

$$\widetilde{M_{p'}}(\Sigma) = \inf_{\rho \in \widetilde{\mathrm{adm}}\Sigma} \int\limits_{\mathbb{R}^n} \rho^{p'} dm(x) \tag{11}$$

where the notation $\rho \in \widetilde{\mathrm{adm}}\,\Sigma$ means that $\rho$ is nonnegative Borel function in $\mathbb{R}^n$ such that

$$\int\limits_{\sigma \cap R} \rho \, d\mathcal{H}^{n-1} \geqslant 1 \quad \forall \sigma \in \Sigma\,. \tag{12}$$

Note that, according to the result of Ziemer,

$$\widetilde{M_{p'}}(\Sigma) = C_p[G, C_0, C_1]^{-1/(p-1)}\,, \tag{13}$$

see [37, Theorem 3.13] for $p = n$ and [38, p. 50] for $1 < p < \infty$, in addition, by the Hesse result

$$M_p(\Gamma(E, F, D)) = C_p[D, E, F]\,, \tag{14}$$

where $(E \cup F) \cap \partial D = \varnothing$ (see [9, Theorem 5.5]). Shlyk has proved that the requirement $(E \cup F) \cap \partial D = \varnothing$ can be omitted. In other words, the equality (14) holds for any disjoint non-empty sets $E, F \subset \overline{D}$ (see [33, Theorem 1]).

Let $S$ be a *$k$-dimensional surface,* in other words, $S\colon D_s \to \mathbb{R}^n$ be a continuous mapping of an open set $D_s \subset \mathbb{R}^k$. We put $N(y, S) = \mathrm{card}\, S^{-1}(y) = \mathrm{card}\{x \in D_s\colon S(x) = y\}$ and recall this function a *multiplicity function* of the surface $S$ with respect to a point $y \in \mathbb{R}^n$. Given a Borel set $B \subset \mathbb{R}^n$, its $k$-measured Hausdorff area associated with the surface $S$ is determined by the formula

$$\mathcal{A}_S(B) = \mathcal{A}_S^k(B) = \int\limits_B N(y, S) \, d\mathcal{H}^k y\,, \tag{15}$$

see [6, item 3.2.1]. For a Borel function $\rho\colon \mathbb{R}^n \to [0, \infty]$ its integral over the $k$-dimensional surface $S$ is determined by the formula $\int\limits_S \rho \, d\mathcal{A} = \int\limits_{\mathbb{R}^n} \rho(y) N(y, S) \, d\mathcal{H}^k y$. Surfaces of dimension

$k = n-1$ we will simply call "surfaces" without the prefix $(n-1)$. In what follows, $J_k f(x)$ denotes the *$k$-dimensional Jacobian* of the mapping $f$ at a point $x$ (see [6, § 3.2, Ch. 3]).

Let $n \geqslant 2$, and let $\Gamma$ be a family of surfaces $S$. A Borel function $\rho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is called *an admissible* for $\Gamma$, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_S \rho^{n-1}\, d\mathcal{A} \geqslant 1 \tag{16}$$

for any $S \in \Gamma$. Given $p \in (0,\infty)$, a *$p$-modulus* of $\Gamma$ is called the quantity

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)\,.$$

We also set $M(\Gamma) := M_n(\Gamma)$. Let $p \geqslant 1$. We say that some property $P$ holds for *$p$-almost all surfaces* of the domain $D$, if this property holds for all surfaces in $D$, except, maybe be, some of their subfamily, $p$ -modulus of which is zero. If we are talking about the conformal modulus $M(\Gamma) := M_n(\Gamma)$, the prefix "$n$" in the expression "$n$-almost all", as a rule, is omitted. We say that a Lebesgue measurable function $\rho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is *$p$-extensively admissible* for the family $\Gamma$ of surfaces $S$ in $\mathbb{R}^n$, abbr. $\rho \in \operatorname{ext}_p \operatorname{adm}\Gamma$, if the relation (16) is satisfied for $p$-almost all surfaces $S$ of the family $\Gamma$.

The next class of mappings is a generalization of quasiconformal mappings in the sense of Gehring's ring definition (see [7]; it is the subject of a separate study, see, e.g., [19, Chapter 9]). Let $D$ and $D'$ be domains in $\mathbb{R}^n$ with $n \geqslant 2$. Suppose that $x_0 \in \overline{D} \setminus \{\infty\}$ and $Q\colon D \to (0,\infty)$ is a Lebesgue measurable function. A function $f\colon D \to D'$ is called a *lower $Q$-mapping at a point $x_0$ relative to the $p$-modulus* if

$$M_p(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho \in \operatorname{ext}_p \operatorname{adm}\Sigma_\varepsilon} \int\limits_{D \cap A(x_0,\varepsilon,r_0)} \frac{\rho^p(x)}{Q(x)}\, dm(x) \tag{17}$$

for every spherical ring $A(x_0,\varepsilon,r_0) = \{x \in \mathbb{R}^n : \varepsilon < |x-x_0| < r_0\}$, $r_0 \in (0,d_0)$, $d_0 = \sup_{x\in D}|x-x_0|$, where $\Sigma_\varepsilon$ is the family of all intersections of the spheres $S(x_0,r)$ with the domain $D$, $r \in (\varepsilon,r_0)$. If $p = n$, we say that $f$ is a lower $Q$-mapping at $x_0$. We say that $f$ is a lower $Q$-mapping relative to the $p$-modulus in $A \subset \overline{D}$ if (17) is true for all $x_0 \in A$.

The following statement may be proved similarly to Theorem 9.2 in [19], so we omit the arguments. We note that the statement of Theorem 9.2 in [19] refers to the case $p = n$, and the statement given below related to a more general case $p > n-1$.

**Lemma 1.** *Let $D$, $D' \subset \overline{\mathbb{R}^n}$, let $x_0 \in \overline{D} \setminus \{\infty\}$, and let $Q$ be a Lebesgue measurable function. A mapping $f\colon D \to D'$ is a lower $Q$-mapping relative to the $p$-modulus at a point $x_0$, $p > n-1$, if and only if*

$$M_p(f(\Sigma_\varepsilon)) \geqslant \int\limits_\varepsilon^{r_0} \frac{dr}{\|Q\|_s(r)}$$

*for all $\varepsilon \in (0,r_0)$, $r_0 \in (0,d_0)$, $d_0 = \sup_{x\in D}|x-x_0|$, $s = \frac{n-1}{p-n+1}$, where, as above, $\Sigma_\varepsilon$ denotes the family of all intersections of the spheres $S(x_0,r)$ with $D$, $r \in (\varepsilon,r_0)$, $\|Q\|_s(r) = (\int_{D(x_0,r)} Q^s(x)\, d\mathcal{A})^{\frac{1}{s}}$ is the $L_s$-norm of $Q$ over the set $D(x_0,r) = \{x \in D : |x-x_0| = r\} = D \cap S(x_0,r)$.*

The following statement may be found in [19, Theorem 9.1].

**Proposition 1.** *Let $k = 1, \dots, n-1$, $p \in [k, \infty)$, and let $E$ be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. Then $E$ is measurable by Lebesgue in $\mathbb{R}^n$ if and only if $E$ is measurable with respect to area on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$. Moreover, $m(E) = 0$ if and only if $\mathcal{A}_S(E) = 0$ on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$, where $\mathcal{A}_S(E)$ is defined in (15).*

The following statement is established in Lemma 4.1 in [10].

**Proposition 2.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $p \in [n-1, \infty)$ and $x_0 \in D$. The following statements are equivalent:*

*(1) a property $\mathcal{P}$ holds for $p$-a.e. surfaces $D(x_0, r) := S(x_0, r) \cap D$ in the sense of $p$-modulus, whenever the set $E = \{r \in \mathbb{R} \colon P \text{ holds for } D(x_0, r)\}$ is Lebesgue measurable;*

*(2) $\mathcal{P}$ holds for a.e. $D(x_0, r)$ with respect to the parameter $r \in \mathbb{R}$.*

The definition of quasiisometry, which is used below, can be found, for example, in [20, section 1.1.7]. The following assertion holds (see, e.g., [20, Theorem, Section 1.1.7]).

**Proposition 3.** *Sobolev classes are invariant under superposition with intrinsic quasiisometries. In other words, if $f \in W^{1,1}_{\rm loc}(D)$, $D \subset \mathbb{R}^n$, $n \geqslant 2$, and $g$ is a quasiisometric mapping of some domain $G \subset \mathbb{R}^n$ onto $D$, then $v = f \circ g \in W^{1,1}_{\rm loc}(G)$.*

The following statement can be found in [20, Theorems 1 and 2, item I.1.1.3], cf. [22, Proposition I.1.2].

**Proposition 4.** *Given $p \geqslant 1$, we denote $ACL^p(D)$ the class of all mappings $f \colon D \to \mathbb{R}^n$, $n \geqslant 2$, which belongs to the class $ACL$ in $D$, the partitional derivatives of which are locally integrable in $D$ in the degree $p$. Then $ACL^p(D) = W^{1,p}_{\rm loc}(D)$.*

**Proposition 5.** *If $f \colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitzian and $m \leqslant n$, then*

$$\int\limits_A g(f(x))|J(x,f)|\,dm(x) = \int\limits_{\mathbb{R}^n} g(y)N(y,f,A)\,d\mathcal{H}^m(y)$$

*whenever $A$ is a Lebesgue measurable set, and $g \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is measurable with a respect to $m$-measured Hausdorff measure $\mathcal{H}^m$. Here $J(x,f) = \det f'(x)$ and $N(y,f,A)$ is defined in (8).*

The following statement holds, cf. [13, Theorem 2.1], [27, Lemma 2.3] and [29, Lemma 2].

**Theorem 4.** *Let $p > 1$ and let $f \colon D \to \mathbb{C}$ be an open discrete mapping of a finite distortion such that $N(f, D) < \infty$. Then $f$ satisfies the relation (17) at any $z_0 \in \overline{D}$ for $Q(z) = N(f,D) \cdot K^{p-1}_{I,\alpha}(z,f)$, where $\alpha := \frac{p}{p-1}$, $K_{I,\alpha}(z,f)$ is defined by (1) and $N(f,D)$ is defined in (8).*

*Proof.* In many ways, the proof of this lemma uses the scheme outlined in [28, Theorem 4]. Observe that, $f = \varphi \circ g$, where $g$ is some homeomorphism and $\varphi$ is an analytic function, see ([34, 5.III.V]). Thus, $f$ is differentiable almost everywhere (see, e.g., [14, Theorem III.3.1]). Let $B$ be a Borel set of all points $z \in D$, where $f$ has a total differential $f'(z)$ and $J(z,f) \neq 0$. Observe that, $B$ may be represented as a countable union of Borel sets $B_l$, $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ are bi-lipschitzian homeomorphisms (see [6, items 3.2.2, 3.1.4 and 3.1.8]). Without loss of generality, we may assume that the sets $B_l$ are pairwise disjoint. Denote by $B_*$ the set of all points $z \in D$ in which $f$ has a total differential, however, $f'(z) = 0$.

Since $f$ is of finite distortion, $f'(z)=0$ for almost all $z$, where $J(z,f)=0$. Thus, by the construction the set $B_0:=D\setminus(B\cup B_*)$ has zero Lebesgue measure. Thus, by Proposition 1, $\mathcal{H}^1(B_0\cap S_r)=0$ for almost all circles $S_r:=S(z_0,r)$ centered at $z_0\in\overline{D}$, where, as usually, $\mathcal{H}^1$ denotes the linear Hausdorff measure, and "almost all" means in the sense of $p$-modulus. Observe that, a function $\psi(r):=\mathcal{H}^1(B_0\cap S_r)$ is Lebesgue measurable by the Fubini theorem, thus, the set $E=\{r\in\mathbb{R}\colon \mathcal{H}^1(B_0\cap S_r)=0\}$ is Lebesgue measurable. Now, by Proposition 2 we obtain that $\mathcal{H}^1(B_0\cap S_r)=0$ for almost any $r\in\mathbb{R}$.

Denote by $D_*:=B(z_0,\varepsilon_0)\cap D\setminus\{z_0\}$, $0<\varepsilon_0<d_0=\sup_{z\in D}|z-z_0|$, and consider the division of $D_*$ by the ring segments $A_k$, $k=1,2,\ldots$. Let $\varphi_k$ be an auxiliary quasiisometry which maps $A_k$ onto rectangular $\widetilde{A_k}$ such that arcs of circles map onto line segments. (For instance, we may take $\varphi_k(\omega)=\log(\omega-z_0)$, $\omega\in A_k$). Consider a family of mappings $g_k=f\circ\varphi_k^{-1}$, $g_k\colon\widetilde{A_k}\to\mathbb{C}$. Observe that, $g_k\in W^{1,1}_{\rm loc}$ (see Proposition 3). Thus, $g_k\in ACL$ due to Proposition 4. Since the absolute continuity on a fixed segment implies $N$-property with a respect to the Lebesgue measure (see [6, Section 2.10.13]), we obtain that $\mathcal{H}^1((g_k\circ\varphi_k)(B_0\cap A_k\cap S_r))=\mathcal{H}^1(f(B_0\cap A_k\cap S_r))=0$. Therefore, by the subadditivity of the Hausdorff measure, we obtain that $\mathcal{H}^1(f(B_0\cap S_r))=0$ for almost all $r\in\mathbb{R}$.

Let us show that $\mathcal{H}^1(f(B_*\cap S_r))=0$ for almost any $r\in\mathbb{R}$. Indeed, let $\varphi_k$, $g_k$ and $A_k$ be such as above $A_k=\{z\in\mathbb{C}\colon z-z_0=re^{i\varphi}, r\in(r_{k-1},r_k),\varphi\in(\psi_{k-1},\psi_k)\}$, and let $S_k(r)$ be a part of the sphere $S(z_0,r)$ which belongs to the segment $A_k$, i.e. $S_k(r)=\{z\in\mathbb{C}\colon z-z_0=re^{i\varphi},\varphi\in(\psi_{k-1},\psi_k)\}$. By the construction, $\varphi_k$ maps $S_k(r)$ onto the segment $I(k,r)=\{z\in\mathbb{C}\colon z=\log r+it,\ t\in(\psi_{k-1},\psi_k)\}$. Applying Proposition 5, we obtain that

$$\mathcal{H}^1(g_k(\varphi_k(B_*\cap S_k(r))))\leqslant\int\limits_{g_k(\varphi_k(B_*\cap S_k(r)))}N(y,g_k,\varphi_k(B_*\cap S_k(r)))\,d\mathcal{H}^1y\leqslant$$

$$\leqslant\int\limits_{\varphi_k(B_*\cap S_k(r))}|g_k'(\log r+it)|dt=0$$

for almost all $r\in(r_{k-1},r_k)$. Thus, $\mathcal{H}^1(f(B_*\cap S_k(r)))=0$ for almost all $r\in(r_{k-1},r_k)$. By subadditivity of the Hausdorff measure we obtain that $\mathcal{H}^1(f(B_*\cap S_r))=0$ for almost all $r\in\mathbb{R}$, as required.

Let $\Gamma$ be a family of all intersections of circles $S_r$, $r\in(\varepsilon,\varepsilon_0)$, $\varepsilon_0<d_0=\sup\limits_{z\in D}|z-z_0|$, with a domain $D$. Given an admissible function $\rho_*\in{\rm adm}\,f(\Gamma)$, $\rho_*\equiv 0$ outside of $f(D)$, we set $\rho\equiv0$ outside of $D$ and on $B_0$, and

$$\rho(z)\colon=\rho_*(f(z))\left(\frac{|f_z|^2-|f_{\overline{z}}|^2}{|f_z|-|f_{\overline{z}}|}\right)\qquad\text{for } z\in D\setminus B_0\,.$$

Given a fixed domain $D_r^*\in f(\Gamma)$, $D_r^*=f(S_r\cap D)$, observe that

$$D_r^*=\bigcup\limits_{j=0}^{\infty}f(S_r\cap B_j)\cup f(S_r\cap B_*)\,,$$

and, consequently, for almost all $r\in(0,\varepsilon_0)$, we obtain that

$$1\leqslant\int\limits_{D_r^*}\rho_*(y)d\mathcal{A}_*=\sum\limits_{j=0}^{\infty}\int\limits_{f(S_r\cap B_j)}N(y,S_r\cap B_j)\rho_*(y)d\mathcal{H}^1y+\int\limits_{f(S_r\cap B_*)}N(y,S_r\cap B_*)\rho_*(y)d\mathcal{H}^1y\,. \tag{18}$$

Due to the above, by (18) we obtain that

$$1 \leqslant \int\limits_{D_r^*} \rho_*(y) d\mathcal{A}_* = \sum_{j=1}^{\infty} \int\limits_{f(S_r \cap B_j)} N(y, S_r \cap B_j) \rho_*(y) d\mathcal{H}^1 y \tag{19}$$

for almost all $r \in (0, \varepsilon_0)$. Observe that, $l(f'(z)) := \min_{|h|=1} |f'(z)h| = |f_z| - |f_{\overline{z}}|$, see [1, relation (10).A.I]. Now, arguing on each $B_j$, $j = 1, 2, \ldots$, by [6, item 1.7.6] and Proposition 5 and taking into account that $\frac{|f_z|^2 - |f_{\overline{z}}|^2}{|f_z| - |f_{\overline{z}}|} = \|f'(z)\| \geqslant \frac{d\mathcal{A}_*}{d\mathcal{A}}$ we obtain that

$$\int\limits_{B_j \cap S_r} \rho \, d\mathcal{A} = \int\limits_{B_j \cap S_r} \rho_*(f(z)) \left( \frac{|f_z|^2 - |f_{\overline{z}}|^2}{|f_z| - |f_{\overline{z}}|} \right) d\mathcal{A} =$$

$$= \int\limits_{B_j \cap S_r} \rho_*(f(z)) \cdot \left( \frac{|f_z|^2 - |f_{\overline{z}}|^2}{|f_z| - |f_{\overline{z}}|} \right) \cdot \frac{1}{\frac{d\mathcal{A}_*}{d\mathcal{A}}} \cdot \frac{d\mathcal{A}_*}{d\mathcal{A}} \, d\mathcal{A} \geqslant \int\limits_{B_j \cap S_r} \rho_*(f(z)) \cdot \frac{d\mathcal{A}_*}{d\mathcal{A}} \, d\mathcal{A} =$$

$$= \int\limits_{f(B_j \cap S_r)} \rho_* \, d\mathcal{A}_* \tag{20}$$

for almost any $r \in (0, \varepsilon_0)$. By (19) and (20) together with Proposition 2 we obtain that $\rho \in \mathrm{ext}_p \, \mathrm{adm} \, \Gamma$.

Using the change of variables on $B_l$, $l = 1, 2, \ldots$ (see Proposition 5), by the countable additivity of the Lebesgue integral we obtain that

$$\int\limits_D \frac{\rho^p(z)}{K_{I, \frac{p}{p-1}}^{p-1}(z)} \, dm(z) = \sum_{l=1}^{\infty} \int\limits_{B_l} \frac{\rho^p(z)(|f_z| - |f_{\overline{z}}|)^p}{|J(z, f)|^{p-1}} \, dm(z) =$$

$$= \sum_{l=1}^{\infty} \int\limits_{B_l} \frac{(|f_z| - |f_{\overline{z}}|)^p}{|J(z, f)|^{p-1}} \cdot \rho_*^p(f(z)) \left( \frac{(|f_z|^2 - |f_{\overline{z}}|^2)^p}{(|f_z| - |f_{\overline{z}}|)^p} \right) \, dm(z) =$$

$$= \sum_{l=1}^{\infty} \int\limits_{B_l} \rho_*^p(f(z)) |J(z, f)| \, dm(z) = \sum_{l=1}^{\infty} \int\limits_{f(B_l)} \rho_*^p(y) \, dm(y) \leqslant \int\limits_{f(D)} N(f, D) \rho_*^p(y) \, dm(y) \,,$$

as required. □

We also need the following statement given in [22, Proposition 10.2, Ch. II].

**Proposition 6.** *Let $E = (A, C)$ be a condenser in $\mathbb{R}^n$ and let $\Gamma_E$ be the family of all paths of the form $\gamma\colon [a, b) \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \varnothing$ for every compact $F \subset A$. Then $\mathrm{cap}_q E = M_q(\Gamma_E)$.*

The following statement may be found in [25, Lemma 2].

**Proposition 7.** *Let $x_0 \in \mathbb{R}^n$, $p \geqslant 1$, $n \geqslant 2$, $0 < r_1 < r_2 < \infty$, and let $Q\colon \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. Set $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n \colon r_1 < |x - x_0| < r_2\}$,*

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)} \,,$$

*where*

$$I = I(x_0,r_1,r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\,, q_{x_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{S(x_0,r)} Q(x)\,d\mathcal{H}^{n-1}$$

*and* $\omega_{n-1}$ *denotes the area of the unit sphere* $\mathbb{S}^{n-1} := S(0,1)$ *in* $\mathbb{R}^n$. *Then*

$$\frac{\omega_{n-1}}{I^{p-1}} = \int\limits_A Q(x)\cdot\eta_0^p(|x-x_0|)\,dm(x) \leqslant \int\limits_A Q(x)\cdot\eta^p(|x-x_0|)\,dm(x)$$

*for any Lebesgue measurable function* $\eta\colon (r_1,r_2)\to[0,\infty]$ *such that* $\int\limits_{r_1}^{r_2}\eta(r)dr = 1$.

An analogue of the following assertion has been proved several times earlier under slightly different conditions, see [28, Lemma 5] and [31, Lemma 4.2]. In the formulation given below, this result is proved for the first time.

**Lemma 2.** *Let* $D$ *be a domain in* $\mathbb{R}^n$, $n\geqslant 2$, *let* $p>n-1$, *let* $x_0\in D$ *and let* $f\colon D\to\mathbb{R}^n$ *be an open and discrete mapping satisfying the relation (17) at the point* $x_0$. *Assume that* $Q\colon D\to[0,\infty]$ *is a Lebesgue measurable function such that the function* $Q^*(x) = Q^{\frac{n-1}{p-n+1}}(x)$ *has a finite integral over spheres* $S(x_0,r)$ *for almost any* $r>0$. *Then the relation*

$$\mathrm{cap}_\alpha f(\mathcal{E}) \leqslant \int\limits_A Q^*(x)\cdot\eta^\alpha(|x-x_0|)\,dm(x) \tag{21}$$

*holds for* $\alpha = \frac{p}{p-n+1}$ *and where* $\mathcal{E} = (B(x_0,r_2),\overline{B(x_0,r_1)})$, $A = A(x_0,r_1,r_2)$, $0<r_1<r_2<\varepsilon_0 := \mathrm{dist}\,(x_0,\partial D)$, *and* $\eta\colon(r_1,r_2)\to[0,\infty]$ *is arbitrary nonnegative Lebesgue measurable function satisfying the relation (6). In particular, Lemma 2 holds if* $Q$ *is locally integrable in the degree* $s := \frac{n-1}{p-n+1}$ *in* $D$.

*Proof.* Observe that $s=\alpha-1$. Due to Proposition 7, it is sufficient to prove that

$$\mathrm{cap}_\alpha f(\mathcal{E}) \leqslant \frac{\omega_{n-1}}{I^{*\,\alpha-1}}\,,$$

where $\mathcal{E}$ is a condenser $\mathcal{E} = (B(x_0,r_2),\overline{B(x_0,r_1)})$, and $q^*_{x_0}(r)$ denotes the integral mean of $Q^{\alpha-1}(x)$ under $S(x_0,r)$,

$$q^*_{x_0}(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{S(x_0,r)} Q^{\alpha-1}(x)\,d\mathcal{H}^{n-1}\,, \tag{22}$$

$\omega_{n-1}$ denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and

$$I^* = I^*(x_0,r_1,r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{\alpha-1}} q_{x_0}^{*\,\frac{1}{\alpha-1}}(r)}\,.$$

Let $\varepsilon\in(r_1,r_2)$. We put $C_0 = \partial f(B(x_0,r_2))$, $C_1 = f(\overline{B(x_0,r_1)})$, $\sigma = \partial f(B(x_0,\varepsilon))$. Since $f$ is continuous in $D$, the set $f(B(x_0,r_2))$ is bounded.

Since $f$ is continuous, $\overline{f(B(x_0,r_1))}$ is a compact subset of $f(B(x_0,\varepsilon))$, and $\overline{f(B(x_0,\varepsilon))}$ is a compact subset of $f(B(x_0,r_2))$. In particular, $\overline{f(B(x_0,r_1))}\cap\partial f(B(x_0,\varepsilon))=\varnothing$.

Let, as above, $R=G\setminus(C_0\cup C_1)$, $G:=f(D)$, and $R^*=R\cup C_0\cup C_1$. Then $R^*$. Observe that, $\sigma$ separates $C_0$ from $C_1$ in $R^*=G$. Indeed, the set $\sigma\cap R$ is closed in $R$, besides that, if $A:=G\setminus\overline{f(B(x_0,\varepsilon))}$ and $B=f(B(x_0,\varepsilon))$, then $A$ and $B$ are open in $G\setminus\sigma$, $C_0\subset A$, $C_1\subset B$ and $G\setminus\sigma=A\cup B$.

Let $\Sigma$ be a family of all sets, which separate $C_0$ from $C_1$ in $G$. Below by $\bigcup_{r_1<r<r_2}\partial f(B(x_0,r))$ or $\bigcup_{r_1<r<r_2} f(S(x_0,r))$ we mean the union of all Borel sets into a family, but not in the theoretical-set sense (see [37, item 3, p. 464]). Let $\rho^{n-1}\in\widetilde{\rm adm}\bigcup_{r_1<r<r_2}\partial f(B(x_0,r))$ in the sense of the relation (12). Then $\rho\in{\rm adm}\bigcup_{r_1<r<r_2}\partial f(B(x_0,r))$ in the sense of (16). By the openness of the mapping $f$ we obtain that $\partial f(B(x_0,r))\subset f(S(x_0,r))$, therefore, $\rho\in{\rm adm}\bigcup_{r_1<r<r_2} f(S(x_0,r))$ and, consequently, by (11)

$$\widetilde{M_{\frac{p}{n-1}}}(\Sigma)\geqslant\widetilde{M_{\frac{p}{n-1}}}\left(\bigcup_{r_1<r<r_2}\partial f(B(x_0,r))\right)\geqslant$$
$$\geqslant\widetilde{M_{\frac{p}{n-1}}}\left(\bigcup_{r_1<r<r_2} f(S(x_0,r))\right)\geqslant M_p\left(\bigcup_{r_1<r<r_2} f(S(x_0,r))\right). \tag{23}$$

However, by (13) and (14) we obtain that

$$(M_\alpha(\Gamma(C_0,C_1,G)))^{1/(1-\alpha)}=\widetilde{M_{\frac{p}{n-1}}}(\Sigma)\,. \tag{24}$$

Let $\Gamma_{f(\mathcal{E})}$ be the family of all paths which corresponds to the condenser $f(\mathcal{E})$ in the sense of Proposition 6, and let $\Gamma^*_{f(\mathcal{E})}$ be family of all rectifiable paths of $\Gamma_{f(\mathcal{E})}$. Now, observe that, the families $\Gamma_{f(\mathcal{E})}$ and $\Gamma(C_0,C_1,G)$ have the same families of admissible functions $\rho$. Thus,

$$M_\alpha(\Gamma_{f(\mathcal{E})})=M_\alpha(\Gamma(C_0,C_1,G))\,.$$

By Proposition 6, we obtain that $M_\alpha(\Gamma_{f(\mathcal{E})})={\rm cap}_\alpha f(\mathcal{E})$. By (24) we obtain that

$$\left(\widetilde{M_{\frac{p}{n-1}}}(\Sigma)\right)^{\alpha-1}=\frac{1}{{\rm cap}_\alpha f(\mathcal{E})}\,. \tag{25}$$

Finally, by (23) and (25) we obtain that

$${\rm cap}_\alpha f(\mathcal{E})\leqslant\left(M_p\left(\bigcup_{r_1<r<r_2} f(S(x_0,r))\right)\right)^{1-\alpha}.$$

By Lemma 1, we obtain that

$${\rm cap}_\alpha f(\mathcal{E})\leqslant\left(\int\limits_{r_1}^{r_2}\frac{dr}{\Vert Q\Vert_s(r)}\right)^{-s}=\frac{\omega_{n-1}}{I^{*\,\alpha-1}}\,,$$

as required. □

**3. Proof of the main results.** The following result is proved in [30, Theorem 5].

**Proposition 8.** *Let $x_0 \in \partial D$, let $f\colon D \to \mathbb{R}^n$ be an open, discrete and closed bounded lower $Q$-mapping with respect to $p$-modulus in $D \subset \mathbb{R}^n$, $Q \in L_{loc}^{\frac{n-1}{p-n+1}}(\mathbb{R}^n)$, $n-1<p$, and $\alpha := \frac{p}{p-n+1}$. Then for any $\varepsilon_0 < d_0 := \sup_{x\in D}|x-x_0|$ and any compactum $C_2 \subset D\setminus B(x_0,\varepsilon_0)$ there is $\varepsilon_1$, $0<\varepsilon_1<\varepsilon_0$, such that the inequality*

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant \int\limits_{A(x_0,\varepsilon,\varepsilon_1)} Q^{\frac{n-1}{p-n+1}}(x)\eta^\alpha(|x-x_0|)\,dm(x)\,, \tag{26}$$

*holds for any $\varepsilon \in (0,\varepsilon_1)$ and any compactum $C_1 \subset \overline{B(x_0,\varepsilon)} \cap D$, where $A(x_0,\varepsilon,\varepsilon_1) = \{x \in \mathbb{R}^n\colon \varepsilon < |x-x_0| < \varepsilon_1\}$ and $\eta\colon (\varepsilon,\varepsilon_1) \to [0,\infty]$ is any nonnegative Lebesgue measurable function such that*

$$\int\limits_{\varepsilon}^{\varepsilon_1} \eta(r)\,dr = 1\,. \tag{27}$$

**Remark 2.** Note that, if (5) holds for any function $\eta$ with condition (27), then the same relationship holds for any function $\eta$ with condition (6). Indeed, let $\eta$ be a nonnegative Lebesgue function that satisfies the condition (6). If $J := \int_{r_1}^{r_2}\eta(t)\,dt < \infty$, then we put $\eta_0 := \eta/J$. Obviously, the function $\eta_0$ satisfies condition (27). Then relation (5) gives that

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant$$
$$\leqslant \frac{1}{J^\alpha}\int\limits_A Q(x)\cdot\eta^\alpha(|x-x_0|)\,dm(x) \leqslant \int\limits_A Q(x)\cdot\eta^\alpha(|x-x_0|)\,dm(x) \tag{28}$$

because $J \geqslant 1$. Let now $J=\infty$. Then, by [26, Theorem I.7.4], a function $\eta$ is a limit of a nondecreasing nonnegative sequence of simple functions $\eta_m$, $m=1,2,\ldots$. Set $J_m := \int_{r_1}^{r_2}\eta_m(t)\,dt < \infty$ and $w_m(t) := \eta_m(t)/J_m$. Then, similarly to (28) we obtain that

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant$$
$$\leqslant \frac{1}{J_m^\alpha}\int\limits_A Q(x)\cdot\eta_m^\alpha(|x-x_0|)\,dm(x) \leqslant \int\limits_A Q(x)\cdot\eta_m^\alpha(|x-x_0|)\,dm(x)\,, \tag{29}$$

because $J_m \to J = \infty$ as $m\to\infty$ (see [26, Lemma I.11.6]). Thus, $J_m \geqslant 1$ for sufficiently large $m \in \mathbb{N}$. Observe that, a functional sequence $\varphi_m(x) = Q(x)\cdot\eta_m^\alpha(|x-x_0|)$, $m=1,2\ldots$, is nonnegative, monotone increasing and converges to the function $\varphi(x) := Q(x)\cdot\eta^\alpha(|x-x_0|)$ almost everywhere. By the Lebesgue theorem on the monotone convergence (see [26, Theorem I.12.6]), it is possible to pass to the limit on the right-hand side of the inequality (29), which gives us the desired inequality (5).

The following result is proved in [29, Theorem 6].

**Proposition 9.** *Let $x_0 \in \partial D$, and let $f\colon D \to \mathbb{R}^n$ be a bounded lower $Q$-homeomorphism with respect to $p$-modulus in a domain $D \subset \mathbb{R}^n$, $Q \in L_{\rm loc}^{\frac{n-1}{p-n+1}}(\mathbb{R}^n)$, $p>n-1$ and $\alpha := \frac{p}{p-n+1}$. Then $f$ is a ring $Q^{\frac{n-1}{p-n+1}}$-homeomorphism with respect to $\alpha$-modulus at this point, where $\alpha := \frac{p}{p-n+1}$.*

*Proof of Theorem 1.* Fix $z_0\in \overline{D}$. Two situations are possible: when $z_0\in D$, and when $z_0\in \partial D$. Let $z_0\in D$. Set $p:=\frac{\alpha}{\alpha-1}$. Due to Theorem 4, $f$ satisfies the relation (17) with $Q(z):=K^{p-1}_{I,\alpha}(z,f)$. Now, $f$ satisfies the relation (21) with $Q^*(z):=K_{I,\alpha}(z,f)$. Observe that the relation

$$M_{\alpha}(f(\Gamma(C_1,C_2,D)))\leqslant \mathrm{cap}_{\alpha}(f(B(z_0,r_2)),f(\overline{B(z_0,r_1)}))\tag{30}$$

holds for all $0<r_1<r_2<d_0:=\mathrm{dist}\,(z_0,\partial D)$ and for any continua $C_1\subset \overline{B(z_0,r_1)}$, $C_2\subset D\setminus B(z_0,r_2)$. Indeed, $f(\Gamma(C_1,C_2,D))>\Gamma_{f(E)}$, where $E:=(f(B(z_0,r_2)),f(\overline{B(z_0,r_1)}))$, and $\Gamma_{f(E)}$ is a family from Proposition 6 for the condenser $f(E)$. The relation (30) finishes the proof for the case $z_0\in D$. Let now $z_0\in\partial D$. Again, by Theorem 4, $f$ satisfies the relation (17) with $Q(z):=K^{p-1}_{I,\alpha}(z,f)$. Now $f$ satisfies the relation (5) with $Q^*(z):=K_{I,\alpha}(z,f)$ by Proposition 9. □

Proof of Theorem 2 directly follows by Theorem 4 and Lemma 2.

Proof of Theorem 3 directly follows by Theorem 4 and Proposition 8.

**4. Examples.**

**Example 1.** Let $Q(z)=\log\frac{e}{|z|}$, $z\in\mathbb{D}$, let $q_0(r):=\log\frac{e}{r}$, and let $1<\alpha<2$. Observe that $\int_{\varepsilon}^{\varepsilon_0}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}<\infty$ for any $\varepsilon_0\in(0,1)$ and any $\varepsilon\in(0,\varepsilon_0)$, in addition,

$$\int\limits_0^{\varepsilon_0}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}=\infty\,,\tag{31}$$

where $q_0(t)$ is defined by (22). Set $f(z)=\frac{z}{|z|}\rho(|z|),\ z\in\mathbb{D}\setminus\{0\},\ f(0):=0$, where

$$\rho(|z|)=\left(1+\frac{2-\alpha}{\alpha-1}\int\limits_{|z|}^{1}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}\right)^{\frac{\alpha-1}{\alpha-2}}.$$

Observe that, $f\in ACL$ and, besides that, $f$ is differentiable in $\mathbb{D}$ almost everywhere. By the technique used in [19, Proposition 6.3]

$$|J(z,f)|=\delta_{\tau}(z)\cdot\delta_r(z),\qquad \|f^{\prime}(z)\|=\max\{\delta_{\tau}(z),\delta_r(z)\}\,,\quad l(f^{\prime}(z))=\min\{\delta_{\tau}(z),\delta_r(z)\}\,,$$

where $l(f^{\prime}(z))=|f_z|-|f_{\overline{z}}|$, $\|f^{\prime}(z)\|=|f_z|+|f_{\overline{z}}|$, $\delta_{\tau}(z)=\frac{|f(z)|}{|z|}$, $\delta_r(z)=\frac{|\partial|f(z)||}{\partial|z|}$. Thus,

$$\|f^{\prime}(z)\|=\left(1+\frac{2-\alpha}{\alpha-1}\int\limits_{|z|}^{1}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}\right)^{\frac{\alpha-1}{\alpha-2}}\frac{1}{|z|}\,,$$

$$l(f^{\prime}(z))=\left(1+\frac{2-\alpha}{\alpha-1}\int\limits_{|z|}^{1}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}\right)^{\frac{1}{\alpha-2}}\frac{1}{|z|^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(|z|)},$$

$$|J(z,f)|=\left(1+\frac{2-\alpha}{\alpha-1}\int\limits_{|z|}^{1}\frac{dt}{t^{\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(t)}\right)^{\frac{\alpha}{\alpha-2}}\frac{1}{|z|^{1+\frac{1}{\alpha-1}}q_0^{\frac{1}{\alpha-1}}(|z|)}\,.$$

Observe that, $f \in W^{1,\alpha}_{\rm loc}$ for $\alpha > 1$. Indeed, $\|f'(z)\|$ is bounded outside of some neighborhood of the origin, in addition, for sufficiently small $r > 0$ and some $C > 0$, we obtain that $\|f'(z)\| \leqslant C/|z|$. Besides that, by the Fubini theorem $\int_{B(0,r)} \|f'(z)\|^{\alpha}\, dm(z) \leqslant C^{\alpha}2\pi \cdot \int_0^r r^{1-\alpha}dr < \infty$. Let $K$ be any compact set in $\mathbb{D}$. Then there is $R > 0$ such that $K \subset B(0, R)$. Due to the above, we obtain that

$$\int\limits_K \|f'(z)\|^{\alpha}\, dm(z) \leqslant \int\limits_{B(0,r)} \|f'(z)\|^{\alpha}\, dm(z) + \int\limits_{B(0,R)\setminus B(0,r)} \|f'(z)\|^{\alpha}\, dm(z) < +\infty\,,$$

therefore, $f \in W^{1,\alpha}_{\rm loc}$. Observe that $K_{I,\alpha}(z, f) = q_0(|z|)$. By Theorem 1, $f$ satisfies the inequality (5) with $Q(z) = q_0(|z|)$ at any point $z_0 \in \overline{\mathbb{D}}$.

**Example 2.** Let $f(z) = \frac{z}{|z|} \cdot \frac{1}{\log \frac{1}{|z|}}$ , $z \in \mathbb{D}\setminus\{0\}$, $f(0) = 0$. Observe that $f$ is a homeomorphism of the unit disk. Arguing similarly to Example 1, we obtain that $K_{I,\alpha}(z, f) = |z|^{\alpha-2}\log^{2\alpha-3}\frac{1}{|z|}$. Obviously, $K_{I,\alpha}(z, f)$ is integrable in $\mathbb{D}$, thus, by Theorem 1 $f$ satisfies the relation (5) at any point $z_0 \in \overline{\mathbb{D}}$ for $Q(z) = K_{I,\alpha}(z, f)$. Observe also that $f$ is not quasiconformal mapping because the «usual» dilatation $K_{I,2}(z, f) = \log \frac{1}{|z|}$ is unbounded.

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