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APPROXIMATION OF CLASSES OF POISSON INTEGRALS BY FEJÉR MEANS

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The paper is devoted to the investigation of problem of approximation of continuous periodic functions by trigonometric polynomials, which are generated by linear methods of summation of Fourier series.

The simplest example of a linear approximation of periodic functions is the approximation of functions by partial sums of their Fourier series. However, the sequences of partial Fourier sums are not uniformly convergent over the class of continuous periodic functions. Therefore, many studies devoted to the research of the approximative properties of approximation methods, which are generated by transformations of the partial sums of Fourier series and allow us to construct sequences of trigonometrical polynomials that would be uniformly convergent for the whole class of continuous functions. Particularly, Fejér sums have been widely studied recently. One of the important problems in this area is the study of asymptotic behavior of the sharp upper bounds over a given class of functions of deviations of the trigonometric polynomials.

In the paper, we study upper asymptotic estimates for deviations between a function and the Fejér means for the Fourier series of the function. The asymptotic behavior is considered for the functions represented by the Poisson integrals of periodic functions of a real variable. The mentioned classes consist of analytic functions of a real variable. These functions can be regularly extended into the corresponding strip of the complex plane. An asymptotic equality for the upper bounds of Fejér means deviations on classes of Poisson integrals was obtained.

Let $\mathrm{L}(\mathbb{T})$, $\mathbb{T}=[-\pi;\pi]$ be the space of summable 2π-periodic functions and

$$S[f]=\frac{a_0[f]}{2}+\sum_{k=1}^{\infty}\left(a_k[f]\cos kx+b_k[f]\sin kx\right),$$

be the Fourier series of the function $f\in\mathrm{L}(\mathbb{T})$, where

$$a_0[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}}f(x)\,dx,\quad a_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}}f(x)\cos kx\,dx,\quad b_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}}f(x)\sin kx\,dx$$

$(k\in\mathbb{N})$ are the Fourier coefficients of the function f. Let us denote by

$$S_n[f](x)=\frac{a_0[f]}{2}+\sum_{k=1}^{n}(a_k[f]\cos kx+b_k[f]\sin kx)$$

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the partial sum of the Fourier series of the function f.

Let $\mathrm{C}(\mathbb{T})$ be the space of continuous 2π-periodic functions f with the norm

$$\|f\|_{\mathrm{C}} = \max_{x\in\mathbb{T}} |f(x)|.$$

We denote by $\mathrm{G}(q)$, $q \in (0;1)$ the class of continuous 2π-periodic functions given by the convolution

$$f(x) = A_0 + \frac{1}{\pi}\int_{\mathbb{T}} \varphi(x+t)P_q(t)\,dt,$$

where A_0 is a fixed constant, $P_q(t) = \sum_{k=1}^{\infty} q^k \cos kt$ is the Poisson kernel, the function φ satisfies the condition $|\varphi(t)| \le 1$ almost everywhere and $\int_{\mathbb{T}} \varphi(t)\,dt = 0$.

The set $\mathrm{G}(q)$ consist of 2π–periodic functions $f(x)$ that admit analytic extension to the functions $F(z) = F(x+iy)$ in the strip $|\operatorname{Im} z| < \ln\frac{1}{q}$ [1, p. 31].

Let $f \in \mathrm{C}(\mathbb{T})$. The polynomials given by the relation

$$\sigma_n[f](x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k[f](x)$$

are called Fejér means of function f. An asymptotic equality for upper bounds of deviations of Fejér means on classes $\mathrm{G}(q)$ was obtained in [2] (also [3]):

$$\mathcal{E}\left(\mathrm{G}(q);\sigma_n[f]\right) := \sup_{f\in\mathrm{G}(q)} \|f(\cdot) - \sigma_n[f](\cdot)\|_{\mathrm{C}} = \frac{4q}{\pi n(1+q^2)} + O(1)\frac{q^n}{n}, \quad q \in (0;q_0], \tag{1}$$

where $q_0 = \sqrt{2+\sqrt{5} - 2\sqrt{2+\sqrt{5}}} \approx 0.346$, $O(1)$ is a quantity uniformly bounded with respect to n. Also, paper [2] contains an overview of the literature on the topic.

For the case $q \in [q_0;1)$ the asymptotic formulas of approximation by the Fejér means $\sigma_n[f]$ have not been established. The goal of the article is to obtain an asymptotic equality for upper bounds of deviations of Fejér means taken over classes of Poisson integrals in the case $q \in [q_0;1)$. In general, the article uses the method presented in [2] and also the classical methods for the study of the upper bounds of the trigonometric polynomials deviations from periodic functions of a real variable. The mentioned methods were developed in the papers of B. Sz. Nagy, S. M. Nikolskii, O. I. Stepanets and other mathematicians. In the proof of the theorem, we overcome difficulties lying beyond the scope of [2, 3].

Our main result is contained in the following theorem.

Theorem 1. *Let $f \in \mathrm{G}(q)$. For $q \in [q_0;1)$ the equality*

$$\mathcal{E}\left(\mathrm{G}(q);\sigma_n[f]\right) = \frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)} + O(1)\frac{q^n}{n(1-q)^3}, \tag{2}$$

holds as $n\to\infty$, where $O(1)$ is uniformly bounded with respect to n, q.

Proof. In view of (4), (5) from [2] with $\mathrm{G}(q) = C^q_{0,\infty}$, $f(x) - \sigma_n[f](x) = \delta_n(f;x)$, we have

$$f(x) - \sigma_n[f](x) = \frac{q}{\pi n}\int_{\mathbb{T}} \varphi(x+t)\Gamma(t;q)\,dt + O(1)\frac{q^n}{n(1-q)^3},$$

where $\Gamma(t;q) = \frac{(1+q^2)\cos t - 2q}{(1-2q\cos t + q^2)^2}$.

For any function $f \in \mathrm{G}(q)$ and any constant I we can write

$$f(0) - \sigma_n[f](0) = \frac{q}{\pi n} \int_{\mathbb{T}} \varphi(t) \left(\Gamma(t;q) - I\right) \, dt + O(1) \frac{q^n}{n(1-q)^3}.$$

Since $\operatorname{ess\,sup}\{|\varphi(t)| \colon t \in \mathbb{T}\} \le 1$, we have

$$|f(0) - \sigma_n[f](0)| \le \frac{q}{\pi n} \int_{\mathbb{T}} |\Gamma(t;q) - I| \, dt + O(1) \frac{q^n}{n(1-q)^3}.$$

Then

$$\mathcal{E}\left(\mathrm{G}(q); \sigma_n[f]\right) = \frac{q}{\pi n} \int_{\mathbb{T}} |\Gamma(t;q) - I(q)| \, dt + O(1) \frac{q^n}{n(1-q)^3},$$

where the value $I(q)$ is such that

$$\operatorname{mes}\left(\Gamma(t;q) - I(q) \le 0\right) = \operatorname{mes}\left(\Gamma(t;q) - I(q) \ge 0\right), \quad t \in [-\pi; \pi].$$

We investigate the function $\Gamma(t;q)$, $t \in [0;\pi]$.

By elementary calculations we get $\Gamma'(t;q) \neq 0$, $t \in (0;\pi)$ for $q \in \left(0; 2-\sqrt{3}\right]$. For $q \in \left(2-\sqrt{3}; 1\right)$ the equation $\Gamma'(t;q) = 0$ has unique solution on interval $t \in (0;\pi)$. Therefore, for $q \in (0; 2-\sqrt{3}]$ the function $\Gamma(t;q)$ is decreasing on $(0;\pi)$. For $q \in (2-\sqrt{3}; 1)$ the function $\Gamma(t;q)$ has one extremum on $(0;\pi)$. It was also shown in [2,3].

Let $q \in (2-\sqrt{3}; 1)$. For arbitrary fixed t_q such that $0 < t_q \le \frac{\pi}{2}$ and

$$\Gamma(t_q; q) = \Gamma\left(t_q + \frac{\pi}{2}; q\right), \tag{3}$$

we have

$$\begin{aligned}
\mathcal{E}\left(\mathrm{G}(q); \sigma_n[f]\right) &= \frac{2q}{\pi n} \int\limits_0^{\pi} |\Gamma(t;q) - I(q)| \, dt + O(1) \frac{q^n}{n(1-q)^3} = \frac{2q}{\pi n} \Bigg(\int\limits_0^{t_q} \left(\Gamma(t;q) - I(q)\right) dt - \\
&- \int\limits_{t_q}^{t_q + \frac{\pi}{2}} \left(\Gamma(t;q) - I(q)\right) dt + \int\limits_{t_q + \frac{\pi}{2}}^{\pi} \left(\Gamma(t;q) - I(q)\right) dt \Bigg) + O(1) \frac{q^n}{n(1-q)^3} = \\
&= \frac{2q}{\pi n} \left(2J(t_q; q) - 2J\left(t_q + \frac{\pi}{2}; q\right) - J(0;q) - J(\pi;q) \right) + O(1) \frac{q^n}{n(1-q)^3} = \\
&= \frac{4q}{\pi n} \left(J(t_q; q) - J\left(t_q + \frac{\pi}{2}; q\right) \right) + O(1) \frac{q^n}{n(1-q)^3},
\end{aligned} \tag{4}$$

where

$$J(t;q) := \int_0^t \Gamma(u;q) \, du = \frac{\sin t}{1 - 2q\cos t + q^2}.$$

Let us consider the equation (3). We have

$$\frac{(1+q^2)\cos t_q - 2q}{(1 - 2q\cos t_q + q^2)^2} = \frac{-(1+q^2)\sin t_q - 2q}{(1 + 2q\sin t_q + q^2)^2}.$$

Then $(1+q^2) + 4q^2(1+q^2)\cos t_q \sin t_q + 8q^3(\cos t_q - \sin t_q) + q^4(1+q^2) - 8q^2 + {}$ $+2q^2(1+q^2) - 8q^4 = 0$.

Let us use the change of variable $\cos t_q - \sin t_q = a$, $\cos t_q \sin t_q = \frac{1-a^2}{2}$, $|a| \le 1$. We obtain

$$a^2 - \frac{4q}{1+q^2} - \frac{q^6 - 3q^4 - 3q^2 + 1}{2q^2(1+q^2)} = 0. \tag{5}$$

The unique solution of the equation (5) is the quantity $a = \frac{4q^2-\sqrt{2(1+q^4)}(1-q^2)}{2q(1+q^2)}$.

Note that condition $q \in [q_0; 1)$ follows from the inequality $|a| \leq 1$. We have

$$-1 \leq \frac{4q^2 - \sqrt{2(1+q^4)}(1-q^2)}{2q(1+q^2)} \leq 1.$$

Since $q \in (0;1)$, from the last inequality we deduce the inequality

$$(q+1)^2(q^2+1)(q^4-2q^3-2q^2-2q+1) \leq 0.$$

The solution of this inequality is the interval $[q_0; 1)$.

Further we calculate the quantity $J(t_q;q) - J\left(t_q + \frac{\pi}{2}; q\right)$ in the main addend of formula (4). We have

$$J(t_q;q) - J\left(t_q + \frac{\pi}{2}; q\right) = \frac{\sin t}{1-2q\cos t + q^2} - \frac{\cos t}{1+2q\sin t + q^2}.$$

Making elementary transformations, we obtain

$$J(t_q;q) - J\left(t_q + \frac{\pi}{2}; q\right) =$$

$$\frac{-(\cos t_q - \sin t_q) - q^2(\cos t_q - \sin t_q) + 2q}{(1+q^2)^2 - 2q(\cos t_q - \sin t_q) - 4q^2\cos t_q \sin t_q - 2q^3(\cos t_q - \sin t_q)} =$$

$$\frac{a(1+q^2)-2q}{2q(a(1+q^2)+q(1-a^2)) - (1+q^2)^2} = \frac{(1+q^2)^2}{2q(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}. \tag{6}$$

Combining relations (4), (6), we get the formula (2). □

Equality (2) is asymptotically exact without additional conditions. For $q = q_0$ formulas (1) and (2) match between them.

Note that Theorem 1 agrees with Corollary 10 of Theorem 3 in [4].

The results of the work, as well as the method of obtaining them, can be used for study of open problems in the theory of approximation of functions and computational mathematics. The results of the work can also have practical application in such fields as fast algorithms, image processing, optics, partial differential equations, spectral estimation, speech processing, etc.

REFERENCES

1. A.I. Stepanets, Classification and approximation of periodic functions, Nauk. Dumka, Kyiv, 1987.
2. O.O. Novikov, O.G. Rovenska, *Approximation of periodic analytic functions by Fejér sums*, Mat. Stud., **47** (2017), №2, 196–201. https://doi.org/10.15330/ms.47.2.196-201
3. O.O. Novikov, O.G. Rovenska, Yu.A. Kozachenko, *Approximation of classes of Poisson integrals by Fejér sums*, Visn. V.N. Karazin Kharkiv Nat. Univer., Ser. Math., Appl. Math., Mech., **87** (2018), 4–12.
4. V.V. Savchuk, S.O. Chaichenko, M.V. Savchuk, *Approximation of bounded holomorphic and harmonic functions by Fejér means*, Ukr. Math. J., **71** (2019), 589–618. https://doi.org/10.1007/s11253-019-01665-0

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