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ON A SEMITOPOLOGICAL SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ WHEN A FAMILY $\mathscr{F}$ CONSISTS OF INDUCTIVE NON-EMPTY SUBSETS OF ω

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Let $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be the bicyclic semigroup extension for the family $\mathscr{F}$ of ω-closed subsets of ω which is introduced in [19]. We study topologizations of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}$ of inductive ω-closed subsets of ω. We generalize Eberhart-Selden and Bertman-West results about topologizations of the bicyclic semigroup [6,12] and show that every Hausdorff shift-continuous topology on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is discrete and if a Hausdorff semitopological semigroup S contains $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ as a proper dense subsemigroup then $S\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an ideal of S. Also, we prove the following dichotomy: every Hausdorff locally compact shift-continuous topology on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero is either compact or discrete. As a consequence of the last result we obtain that every Hausdorff locally compact semigroup topology on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero is discrete and every Hausdorff locally compact shift-continuous topology on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}\sqcup I$ with an adjoined compact ideal I is either compact or the ideal I is open, which extent many results about locally compact topologizations of some classes of semigroups onto extensions of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

1. Introduction. We shall follow the terminology of [7,10,11,13,30]. By ω we denote the set of all non-negative integers. Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F\in\mathscr{P}(\omega)$ and integer n we put $nF=\{n+k\colon k\in F\}$.

A family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is called *ω-closed* if $F_1\cap(-n+F_2)\in\mathscr{F}$ for all $n\in\omega$ and $F_1,F_2\in\mathscr{F}$.

A semigroup S is called *inverse* if for any element $x\in S$ there exists a unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s\preccurlyeq t$ iff there exists $e\in E(S)$ such that $s=te$, for $s,t\in S$. This order is called the *natural partial order* on S [31].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q and the condition $pq=1$. Thus each element of $\mathscr{C}(p,q))$ equals q^mp^n for some $m,n\in\omega$ and the semigroup operation on $\mathscr{C}(p,q)$ can be described as follows

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}},$$

for each $k,l,m,n\in\omega$. It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [10].

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On the other hand, we can we define the semigroup operation "$\cdot$" on the set $\boldsymbol{B}_{\omega}=\omega\times\omega$ in the following way

$$(i_1,j_1)\cdot(i_2,j_2)=\begin{cases}(i_1-j_1+i_2,j_2), & \text{if } j_1\leqslant i_2,\\ (i_1,j_1-i_2+j_2), & \text{if } j_1\geqslant i_2;\end{cases}$$

for each $i_1,i_2,j_1,j_2\in\omega$. It is well-known that the semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the bicyclic monoid by the mapping $\mathfrak{h}\colon \mathscr{C}(p,q)\to\boldsymbol{B}_{\omega}$, $q^kp^l\mapsto(k,l)$ (see: [10, Section 1.12] or [29, Exercise IV.1.11(ii)]).

A *topological* (*semitopological*) *semigroup* is a topological space endowed with a continuous (separately continuous) semigroup operation. If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ a *semigroup topology* on S, and if τ is a topology on S such that (S,τ) is a semitopological semigroup, then we shall call τ a *shift-continuous topology* on S.

The well-known A. Weil Theorem states that *every locally compact monothetic topological group G* (i.e., G contains a cyclic dense subgroup) *is either compact or discrete* (see [32]). A semitopological semigroup S is called *monothetic* if it contains a cyclic dense subsemigroup. Locally compact and compact monothetic topological semigroups were studied by Hewitt [21], Hofmann [22], Koch [24], Numakura [28] and others (for more related information see the books [8] and [23]). Koch in [24] posed the following problem: "*If S is a locally compact monothetic semigroup and S has an identity, must S be compact?*" From the other hand, Zelenyuk in [33] constructed a countable monothetic locally compact topological semigroup without an identity which is neither compact nor discrete and in [34] he constructed a monothetic locally compact topological monoid with the same property. The topological properties of monothetic locally compact (semi)topological semigroups are studied in [2, 14, 35, 36]. In the paper [15] it is proved that every Hausdorff locally compact shift-continuous topology on the bicyclic monoid with an adjoined zero is either compact or discrete. This result was extended by Bardyla to the polycyclic monoid [3] and graph inverse semigroups [4], and by Mokrytskyi to the monoid of order isomorphisms between principal filters of $\mathbb{N}^n$ with an adjoined zero [27]. Also, in [18] it is proved that the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}^0$ with an adjoined zero admits continuum many different shift-continuous topologies, however every Hausdorff locally compact semigroup topology on $\mathscr{C}_{\mathbb{Z}}^0$ is discrete. In [5] Bardyla proved that a Hausdorff locally compact semitopological McAlister semigroup $\mathcal{M}_1$ is either compact or discrete. However, this dichotomy does not hold for the McAlister semigroup $\mathcal{M}_2$ and moreover, $\mathcal{M}_2$ admits continuum many different Hausdorff locally compact inverse semigroup topologies [5]. Also, different locally compact semitopological semigroups with zero were studied in [16, 17, 26].

Next we shall describe the construction which is introduced in [19].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. We can we define the semigroup operation "$\cdot$" on the set $\boldsymbol{B}_{\omega}\times\mathscr{F}$ in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\begin{cases}(i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)), & \text{if } j_1\geqslant i_2\end{cases}$$

for each $i_1,i_2,j_1,j_2\in\omega$. In [19] is proved that $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$ is an inverse semigroup. Moreover, if a family $\mathscr{F}$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}}=\begin{cases}(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{cases}$$

is defined in [19]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. In [19] it is provd that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and when $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particularly in [19] is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega{\times}\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set.

A subset F of ω is called *inductive* if $n+1\in F$ provided $n\in F$. It is obvious that the empty set $\varnothing$ is inductive. Also, a set $F\subseteq\omega$ is inductive if and only if $F\subseteq -1+F$ (see [19, Lemma 6]). If $\mathscr{F}$ is a family in $\mathscr{P}(\omega)$ of non-empty inductive subsets then the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a simple monoid (see [19, Corollary 2 and Theorem 4]). Group congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its homomorphic retracts when the family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [20]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also, all non-trivial homomorphic retracts of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are described.

By Proposition 1 of [20] for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0)\in\mathscr{F}^*$ and the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}$ contains the set $[0)$. Also, ω-closeness of $\mathscr{F}$ implies that if $[k)\in\mathscr{F}$ for some $k\in\omega$ then $[l)\in\mathscr{F}$ for all $l\leqslant k$ with $l\in\omega$.

In this paper we extend the results of [6, 12, 15] onto the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for an ω-closed family $\mathscr{F}$ of inductive nonempty subsets of ω. In particular we show that every Hausdorff shift-continuous topology on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is discrete and if a Hausdorff semitopological semigroup S contains $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ as a proper dense subsemigroup then $S\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an ideal of S. Also, we prove that every Hausdorff locally compact shift-continuous topology on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero is either compact or discrete.

2. On a topologization and a closure of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ Later we shall need the following proposition from [19], which describes the natural partial order on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the general case of $\mathscr{F}$.

Proposition 1 ([19, Proposition 6]). *Let (i_1,j_1,F_1) and (i_2,j_2,F_2) be non-zero elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Then $(i_1,j_1,F_1)\preccurlyeq(i_2,j_2,F_2)$ if and only if $F_1\subseteq -k+F_2$ and $i_1-i_2=j_1-j_2=k$ for some $k\in\omega$.*

Proposition 2. *For every non-zero elements (i_1,j_1,F_1) and (i_2,j_2,F_2) of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, both sets*

$$\big\{(i,j,F)\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\colon (i_1,j_1,F_1)\cdot(i,j,F)=(i_2,j_2,F_2)\big\},$$
$$\big\{(i,j,F)\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\colon (i,j,F)\cdot(i_1,j_1,F_1)=(i_2,j_2,F_2)\big\}$$

are finite.

Proof. It is obvious that the set $A=\big\{(i,j,F)\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\colon (i_1,j_1,F_1)\cdot(i,j,F)=(i_2,j_2,F_2)\big\}$ is a subset of

$$B=\big\{(i,j,F)\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\colon (i_1,j_1,F_1)^{-1}\cdot(i_1,j_1,F_1)\cdot(i,j,F)=(i_1,j_1,F_1)^{-1}\cdot(i_2,j_2,F_2)\big\}.$$

By Lemmas 2 and 3 from [19], $(i_1,j_1,F_1)^{-1}\cdot(i_1,j_1,F_1)=(j_1,j_1,F_1)$ is an idempotent of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, and hence Lemma 1.4.6 of [25] implies that

$$B=\big\{(i,j,F)\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\colon (i_1,j_1,F_1)^{-1}\cdot(i_2,j_2,F_2)\preccurlyeq(i,j,F)\big\}.$$

Proposition 1 implies that there exist finitely many $i, j \in \omega$ and $F \in \mathscr{F}$ such that $(i_1, j_1, F_1)^{-1} \cdot (i_2, j_2, F_2) \preccurlyeq (i, j, F)$, and hence the set A is finite. $\square$

The following theorem generalizes the results on the topologizabily of the bicyclic monoid obtained in [12] and [6].

Theorem 1. *Let $\mathscr{F}$ be a family of non-empty inductive subsets of ω. Then every Hausdorff shift-continuous topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is discrete.*

Proof. For any element $(i, j, [k))$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ we have that

$$(0,0,[1))\cdot(i,j,[k)) = (i,j,(-i+[1))\cap[k)) = \begin{cases} (0,j,[1)), & \text{if } \ i=0 \ \text{ and } \ k\leqslant 1;\\ (0,j,[k)), & \text{if } \ i=0 \ \text{ and } \ k\geqslant 2;\\ (i,j,[k)), & \text{if } \ i\geqslant 1 \end{cases}$$

and

$$(i,j,[k))\cdot(0,0,[1)) = (i,j,[k)\cap(-j+[1))) = \begin{cases} (i,0,[1)), & \text{if } \ j=0 \ \text{ and } \ k\leqslant 1;\\ (i,0,[k)), & \text{if } \ j=0 \ \text{ and } \ k\geqslant 2;\\ (i,j,[k)), & \text{if } \ j\geqslant 1. \end{cases}$$

So

$$\big((0,0,[1))\cdot\boldsymbol{B}_{\omega}^{\mathscr{F}}\big)\cup\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}\cdot(0,0,[1))\big) = \boldsymbol{B}_{\omega}^{\mathscr{F}}\setminus\{(0,0,[0))\}. \tag{1}$$

Since τ is Hausdorff, every retract of $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$ is its closed subset. It is obvious that $(0,0,[1))\cdot\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}}\cdot(0,0,[1))$ are retracts of the topological space $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$, because $(0,0,[1))$ is an idempotent of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Since any retract of Hausdorff space is closed (see: [13, Ex. 1.5.c]), equality (1) implies that the point $(0,0,[0))$ is an isolated point of the space $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$. By Corollary 2 of [19], $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a simple semigroup. This implies that for any $(i,j,[k))\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ there exist $(i_1,j_1,[k_1)), (i_2,j_2,[k_2))\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that $(i_1,j_1,[k_1))\cdot(i,j,[k))\cdot(i_2,j_2,[k_2)) = (0,0,[0))$, and moreover by Proposition 2 the equation $(i_1,j_1,[k_1))\cdot\chi\cdot(i_2,j_2,[k_2)) = (0,0,[0))$ has finitely many solutions in the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Since $(0,0,[0))$ is an isolated point of $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$, the separate continuity of the semigroup operation in $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$ and the above arguments imply that $\big(\boldsymbol{B}_{\omega}^{\mathscr{F}}, \tau\big)$ is the discrete space. $\square$

The following proposition generalizes the results obtained for the bicyclic monoid in [12] and [15].

Proposition 3. *Let $\mathscr{F}$ be a family of non-empty inductive subsets of ω and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be a proper dense subsemigroup of a Hausdorff semitopological semigroup S. Then $I = S\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a closed ideal of S.*

Proof. By Theorem 1, $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a dense discrete subspace of S, and hence $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an open subspace of S.

Fix an arbitrary element $y\in I$. If $xy = z\notin I$ for some $x\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ then there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{x\}\cdot U(y) = \{z\}\subset\boldsymbol{B}_{\omega}^{\mathscr{F}}$. The neighbourhood $U(y)$ contains infinitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which contradicts Proposition 2. The obtained contradiction implies that $xy = z\in I$ for all $x\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $y\in I$. The proof of the statement that $yx\in I$ for all $x\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $y\in I$ is similar.

Suppose to the contrary that $xy = z\notin I$ for some $x, y\in I$. Then $z\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and the separate continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S, respectively, such that $\{x\}\cdot U(y) = \{z\}$ and

$U(x)\cdot\{y\} = \{z\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, any of equalities $\{x\} \cdot U(y) = \{z\}$ and $U(x) \cdot \{y\} = \{z\}$ contradicts mentioned above Proposition 2. The obtained contradiction implies that $xy \in I$. □

3. On a semitopological locally compact monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero. In this section we assume that $S = \boldsymbol{B}_{\omega}^{\mathscr{F}} \sqcup \{\mathbf{0}\}$, i.e., S the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero $\mathbf{0}$. We observe that the semigroup S is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^0}$, where the family $\mathscr{F}^0$ consists of elements of $\mathscr{F}$ and the empty set $\varnothing$ (see [19, Lemma 1]). Later in the following series of lemmas we assume that S is a Hausdorff locally compact semitopological semigroup with the nonisolated zero $\mathbf{0}$ and the family $\mathscr{F}$ is ω-closed and consists of nonempty inductive subsets of ω.

By Theorem 1, $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a discrete subspace of S. This implies the following lemma.

Lemma 1. *Let $U(\mathbf{0})$ and $V(\mathbf{0})$ be any compact-and-open neighbourhoods of $\mathbf{0}$ in S. Then the set $U(\mathbf{0}) \setminus V(\mathbf{0})$ is finite.*

Since $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a discrete subspace of S without loss of generality we consider only compact-and-open neighbourhoods of zero in S.

In the general case if the family $\mathscr{F}$ contains an inductive set F then

$$\boldsymbol{B}_{\omega}^{\{F\}} = \{(i, j, F) \colon i, j \in \omega\}$$

is an inverse subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Moreover, if F is non-empty then by Proposition 3 of [19], $\boldsymbol{B}_{\omega}^{\{F\}}$ is isomorphic to the bicyclic semigroup.

Lemma 2. *For any neighbourhood $U(\mathbf{0})$ of $\mathbf{0}$ in S there exists $F \in \mathscr{F}$ such that the set $U(\mathbf{0}) \cap \boldsymbol{B}_{\omega}^{\{F\}}$ is infinite.*

Proof. The statement of the lemma is obvious when the family $\mathscr{F}$ is finite. Hence we assume that $\mathscr{F}$ is infinite.

Suppose to the contrary that the set $U(\mathbf{0}) \cap \boldsymbol{B}_{\omega}^{\{F\}}$ is finite for any $F \in \mathscr{F}$. By the separate continuity of the semigroup operation in S there exists a neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of $\mathbf{0}$ in S such that $V(\mathbf{0}) \cdot (0, 1, [0)) \subseteq U(\mathbf{0})$. Since

$$(i, j, F) \cdot (0, 1, [0)) = (i, j + 1, F)$$

for all $i, j \in \omega$ and any $F \in \mathscr{F}$, we obtain that $U(\mathbf{0}) \setminus V(\mathbf{0})$ is an infinite set, which contradicts Lemma 1. The obtained contradiction implies the statement of the lemma. □

Lemma 3. *For any neighbourhood $U(\mathbf{0})$ of $\mathbf{0}$ in S there exists $F \in \mathscr{F}$ such that the set $\boldsymbol{B}_{\omega}^{\{F\}} \setminus U(\mathbf{0})$ is finite.*

Proof. By Lemma 2 there exists $F \in \mathscr{F}$ such that the set $U(\mathbf{0}) \cap \boldsymbol{B}_{\omega}^{\{F\}}$ is infinite. By Theorem 1 all non-zero elements of the semigroup S are isolated points in S, and hence $\boldsymbol{B}_{\omega}^{\{F\}} \cup \{\mathbf{0}\}$ is a closed subset of S, which by Corollary 3.3.10 of [13] is locally compact. It obvious that $\boldsymbol{B}_{\omega}^{\{F\}} \cup \{\mathbf{0}\}$ is a subsemigroup of S, which by Proposition 3 of [19] is algebraically isomorphic to the bicyclic monoid with an adjoined zero. By Theorem 1 of [15], $\boldsymbol{B}_{\omega}^{\{F\}} \cup \{\mathbf{0}\}$ is compact, which implies the statement of the lemma. □

Lemma 4. *For any neighbourhood $U(\mathbf{0})$ of $\mathbf{0}$ in S and any $F \in \mathscr{F}$ the set $\boldsymbol{B}_{\omega}^{\{F\}} \setminus U(\mathbf{0})$ is finite.*

Proof. If the family $\mathscr{F}$ is a singleton then the statement of the lemma follows from Theorem 1 of [15]. Hence we assume that $\mathscr{F}$ is not a singleton.

We shall prove the statement of the lemma by induction.

By Lemma 3 there exists $F_0 \in \mathscr{F}$ such that the set $\boldsymbol{B}_{\omega}^{\{F_0\}} \setminus U(\mathbf{0})$ is finite. Since F_0 is inductive there exists $k_0 \in \omega$ such that $F = [k_0)$. This proves that the base of induction holds.

Next we shall show the inductive step. We consider two cases:

(1) if $[k), [k+1) \in \mathscr{F}$ then the statement that the set $\boldsymbol{B}_{\omega}^{\{[k)\}} \setminus U(\mathbf{0})$ is finite implies that the set $\boldsymbol{B}_{\omega}^{\{[k+1)\}} \setminus U(\mathbf{0})$ is finite, too;

(2) if $[k), [k-1) \in \mathscr{F}$ then the statement that the set $\boldsymbol{B}_{\omega}^{\{[k)\}} \setminus U(\mathbf{0})$ is finite implies that the set $\boldsymbol{B}_{\omega}^{\{[k-1)\}} \setminus U(\mathbf{0})$ is finite, too.

(1) The neighbourhood $U(\mathbf{0})$ contains almost all elements of the form $(i, j, [k))$, $i, j \in \omega$. The separate continuity of the semigroup operation in S implies that there exists a neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of $\mathbf{0}$ in S such that $(1, 1, [k+1)) \cdot V(\mathbf{0}) \subseteq U(\mathbf{0})$ and the set $U(\mathbf{0}) \setminus V(\mathbf{0})$ is finite. This implies that $V(\mathbf{0})$ contains almost all elements of the semigroup $\boldsymbol{B}_{\omega}^{\{[k)\}}$. Then the equalities

$$(1, 1, [k+1)) \cdot (0, p, [k)) = (1, 1+p, [k+1) \cap (-1+[k))) = (1, 1+p, [k+1)), \quad p \in \omega,$$

implies that the neighbourhood $U(\mathbf{0})$ contains infinitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\{[k+1)\}}$. By Corollary 3.3.10 of [13] and Proposition 3 of [19], $\boldsymbol{B}_{\omega}^{\{[k)\}} \cup \{\mathbf{0}\}$ is a locally compact semitopological semigroup which is algebraically isomorphic to the bicyclic monoid with an adjoined zero. By Theorem 1 of [15] the set $\boldsymbol{B}_{\omega}^{\{[k+1)\}} \setminus U(\mathbf{0})$ is finite.

(2) Since the neighbourhood $U(\mathbf{0})$ contains almost all elements of the form $(i, j, [k))$, $i, j \in \omega$, the separate continuity of the semigroup operation in S implies that there exists a neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of $\mathbf{0}$ in S such that $(1, 1, [0)) \cdot V(\mathbf{0}) \subseteq U(\mathbf{0})$ and the set $U(\mathbf{0}) \setminus V(\mathbf{0})$ is finite. Since $V(\mathbf{0})$ contains almost all elements of the semigroup $\boldsymbol{B}_{\omega}^{\{[k)\}}$ and the set $U(\mathbf{0}) \setminus V(\mathbf{0})$ is finite, the equalities

$$(1, 1, [0)) \cdot (0, p, [k)) = (1, 1+p, [0) \cap (-1+[k)) = (1, 1+p, [k-1)), \quad p \in \omega,$$

imply that the neighbourhood $U(\mathbf{0})$ contains infinitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\{[k-1)\}}$. By Corollary 3.3.10 of [13] and Proposition 3 of [19], $\boldsymbol{B}_{\omega}^{\{[k-1)\}} \cup \{\mathbf{0}\}$ is a locally compact semitopological semigroup which is algebraically isomorphic to the bicyclic monoid with an adjoined zero. By Theorem 1 of [15] the set $\boldsymbol{B}_{\omega}^{\{[k_0-1)\}} \setminus U(\mathbf{0})$ is finite. □

Lemma 5. *For any neighbourhood $U(\mathbf{0})$ of $\mathbf{0}$ in S the set $S \setminus U(\mathbf{0})$ is finite.*

Proof. In the case when the family $\mathscr{F}$ is finite the statement of the lemma follows from Lemma 4, and hence later we assume that $\mathscr{F}$ is infinite.

Suppose to the contrary that there exists a neighbourhood $U(\mathbf{0})$ of $\mathbf{0}$ in S such that the set $S \setminus U(\mathbf{0})$ is infinite. By Lemma 4 there exists a sequence $\{(m_i, n_i, [k_i))\}_{i\in\omega} \subseteq S \setminus \{\mathbf{0}\}$ such that $k_i = k_j$ if and only if $i = j$ and $(m_i, n_i, [k_i)) \notin U(\mathbf{0})$ and $(m_i+1, n_i+1, [k_i)) \in U(\mathbf{0})$ for all $i \in \omega$. By the separate continuity of the semigroup operation in S there exists a neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of $\mathbf{0}$ in S such that $(0, 1, [0)) \cdot V(\mathbf{0}) \cdot (1, 0, [0)) \subseteq U(\mathbf{0})$. Then we have that

$$\begin{aligned}(0, 1, [0)) \cdot (m_i+1, n_i+1, [k_i)) \cdot (1, 0, [0)) &= (m_i, n_i+1, (-1+[0)) \cap [k_i)) \cdot (1, 0, [0)) = \\ &= (m_i, n_i+1, [k_i)) \cdot (1, 0, [0)) = (m_i, n_i, [k_i) \cap (-1+[0))) = (m_i, n_i, [k_i)),\end{aligned}$$

which contradicts that $U(\mathbf{0})\setminus V(\mathbf{0})$ is an infinite set, a contradiction. The obtained contradiction implies the statement of the lemma. $\square$

Definition 1 ([9])**.** We shall say that a semigroup S has the F-*property* if for every $a,b,c,d\in S^1$ the sets $\{x\in S \mid a\cdot x=b\}$ and $\{x\in S \mid x\cdot c=d\}$ are finite.

Lemma 6 was proved in [17] and it shows that on the semigroup T with the F-property there exists a Hausdorff compact shift-continuous topology τ_{Ac}.

Lemma 6 ([17])**.** *Let T be a semigroup with the F-property and T^0 be the semigroup T with an adjoined zero. Let τ_{Ac} be the topology on T^0 such that*

- (i) *every element of T is an isolated point in the space (T^0,τ_{Ac});*
- (ii) *the family $\mathscr{B}(0)=\{U\subseteq T^0 \colon U\ni 0 \text{ and } T^0\setminus U \text{ is finite}\}$ determines a base of the topology τ_{Ac} at zero $0\in T^0$.*

Then (T^0,τ_{Ac}) is a Hausdorff compact semitopological semigroup.

Remark 1. By Theorem 1 the discrete topology is a unique Hausdorff shift-continuous topology on a semigroup T. So τ_{Ac} is the unique compact shift-continuous topology on T.

Theorem 2. *Let $\mathscr{F}$ be a family of inductive non-empty subsets of ω and S be the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero. Then every Hausdorff locally compact shift-continuous topology on S is either compact or discrete.*

Proof. In the case when zero of S is an isolated point of S the statement of the theorem follows from Theorem 1. If zero of S is a non-isolated point of S then we apply Lemma 5. $\square$

Since the bicyclic monoid embeds into no Hausdorff compact topological semigroup [1] and by Proposition 3 of [19] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ contains an isomorphic copy of the bicyclic monoid, Theorem 2 implies the following theorem.

Theorem 3. *Let $\mathscr{F}$ be a family of inductive non-empty subsets of ω and S be the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with an adjoined zero. Then every Hausdorff locally compact semigroup topology on S is discrete.*

Remark 2. On the other hand, in [15] is constructed the Čech-complete non-discrete metrizable semigroup topology on the bicyclic semigroup with the adjoined zero.

We need the following simple lemma, which is implied from separate continuity of the semigroup operation in semitopological semigroups.

Lemma 7. *Let X be a Hausdorff semitopological semigroup and I be a compact ideal in X. Then the Rees-quotient semigroup X/I with the quotient topology is a Hausdorff semitopological semigroup.*

The proof of the following lemma is simple (see [17]).

Lemma 8. *Let X be a Hausdorff locally compact space and I be a compact subset of X. Then there exists an open neighbourhood $U(I)$ of I with the compact closure $\overline{U(I)}$.*

Theorem 4. *Let $\mathscr{F}$ be a family of inductive non-empty subsets of ω. Let (S_I,τ) be a Hausdorff locally compact semitopological semigroup, where $S_I=\boldsymbol{B}_{\omega}^{\mathscr{F}}\sqcup I$ and I is a compact ideal of S_I. Then either (S_I,τ) is a compact semitopological semigroup or the ideal I is open.*

Proof. Suppose that I is not open. By Lemma 7 the Rees-quotient semigroup S_I/I with the quotient topology τ_{q} is a semitopological semigroup. Let $\pi\colon S_I \to S_I/I$ be the natural homomorphism which is a quotient map. Since the Rees-quotient semigroup S_I/I is naturally isomorphic to the semigroup S, without loss of generality we can assume that $\pi(S_I) = S$ and the image $\pi(I)$ is the zero $\mathbf{0}$ of S.

By Lemma 8 there exists an open neighbourhood $U(I)$ of I with the compact closure $\overline{U(I)}$. Since by Theorem 1 every point of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isolated in (S_I, τ) we have that $\overline{U(I)} = U(I)$ and its image $\pi(U(I))$ is compact-and-open neighbourhood of zero in S. Since for any open neighbourhood $V(I)$ of I in (S_I, τ) the set $\overline{U(I) \cap V(I)}$ is compact, Theorem 2 implies that $S \setminus \pi(U(I))$ is finite for any compact-and-open neighbourhood $U(I)$ of I in (S_I, τ). Then compactness of I implies that (S_I, τ) is compact as well. $\square$

Since the bicyclic monoid embeds into no Hausdorff compact topological semigroup [1] and by Proposition 3 of [19] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ contains an isomorphic copy of the bicyclic monoid, Theorem 4 implies

Theorem 5. *Let $\mathscr{F}$ be an ω-closed family of inductive non-empty subsets of ω. Let (S_I, τ) be a Hausdorff locally compact topological semigroup, where $S_I = \boldsymbol{B}_{\omega}^{\mathscr{F}} \sqcup I$ and I is a compact ideal of S_I. Then the ideal I is open.*

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