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A DOMAIN FREE OF THE ZEROS OF THE PARTIAL THETA FUNCTION

V. P. Kostov. *A domain free of the zeros of the partial theta function*, Mat. Stud. **58** (2022), 142–158.

The partial theta function is the sum of the series

$$\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j,$$

where q is a real or complex parameter ($|q| < 1$). Its name is due to similarities with the formula for the Jacobi theta function $\Theta(q,x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$. The function θ has been considered in Ramanujan's lost notebook. It finds applications in several domains, such as Ramanujan type q-series, the theory of (mock) modular forms, asymptotic analysis, statistical physics, combinatorics and most recently in the study of section-hyperbolic polynomials, i. e. real polynomials with all coefficients positive, with all roots real negative and all whose sections (i. e. truncations) are also real-rooted. For each q fixed, θ is an entire function of order 0 in the variable x. When q is real and $q \in (0, 0.3092\ldots)$, $\theta(q,.)$ is a function of the Laguerre-Pólya class $\mathcal{L}-\mathcal{P}I$. More generally, for $q \in (0,1)$, the function $\theta(q,.)$ is the product of a real polynomial without real zeros and a function of the class $\mathcal{L}-\mathcal{P}I$. Thus it is an entire function with infinitely-many negative, with no positive and with finitely-many complex conjugate zeros. The latter are known to belong to an explicitly defined compact domain containing 0 and independent of q while the negative zeros tend to infinity as a geometric progression with ratio $1/q$. A similar result is true for $q \in (-1,0)$ when there are also infinitely-many positive zeros. We consider the question how close to the origin the zeros of the function θ can be. In the general case when q is complex it is true that their moduli are always larger than $1/2|q|$. We consider the case when q is real and prove that for any $q \in (0,1)$, the function $\theta(q,.)$ has no zeros on the set

$$\{x \in \mathbb{C}\colon |x| \le 3\} \cap \{x \in \mathbb{C}\colon \mathrm{Re}x \le 0\} \cap \{x \in \mathbb{C}\colon |\mathrm{Im}x| \le 3/\sqrt{2}\}$$

which contains the closure left unit half-disk and is more than 7 times larger than it. It is unlikely this result to hold true for the whole of the left half-disk of radius 3. Similar domains do not exist for $q \in (0,1)$, $\mathrm{Re}x \ge 0$, for $q \in (-1,0)$, $\mathrm{Re}x \ge 0$ and for $q \in (-1,0)$, $\mathrm{Re}x \le 0$. We show also that for $q \in (0,1)$, the function $\theta(q,.)$ has no real zeros ≥ -5 (but one can find zeros larger than -7.51).

1. Introduction. The present paper deals with analytic properties of the *partial theta function*

$$\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j. \tag{1}$$

2010 *Mathematics Subject Classification:* 26A06.
Keywords: partial theta function; Jacobi theta function; Jacobi triple product.
doi:10.30970/ms.58.2.142-158

It owes its name to the resemblance between the function $\theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2} x^j$ and the *Jacobi theta function* $\Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$; "partial" refers to the fact that summation in the case of θ takes place only from 0 to ∞.

We consider the situation when the variable x and the parameter q are real, more precisely, when $(q, x) \in (0, 1) \times \mathbb{R}$. This function has been studied also for $(q, x) \in (-1, 0) \times \mathbb{R}$ and $(q, x) \in \mathbb{D}_1 \times \mathbb{C}$; here $\mathbb{D}_1$ stands for the open unit disk. For any fixed non-zero value of the parameter q ($|q| < 1$), the function $\theta(q, .)$ is an entire function in x of order 0.

The partial theta function finds various applications – from Ramanujan type q-series ([22]) to the theory of (mock) modular forms ([4]), from asymptotic analysis ([2]) to statistical physics and combinatorics ([21]). How θ can be applied to problems dealing with asymptotics and modularity of partial and false theta functions and their relationship to representation theory and conformal field theory is made clear in [5] and [3]. The place which this function finds in Ramanujan's lost notebook is explained in [1] and [22]. Its Padé approximants are studied in [18].

A recent interest in the partial theta function is connected with the study of section-hyperbolic polynomials, i. e. real polynomials with positive coefficients, with all roots real negative and all whose finite sections (i.e. truncations) have also this property, see [17], [8] and [19]; the cited papers use results of Hardy, Petrovitch and Hutchinson (see [6], [20] and [7]). Various analytic properties of the partial theta function are proved in [10]–[16] and other papers of the author.

The analytic properties of θ known up to now, in particular, the behaviour of its zeros, are discussed in the next section. One of them is the fact that for any $q \in (0, 1)$, all complex conjugate pairs of zeros of $\theta(q, .)$ remain within the domain

$$\{x \in \mathbb{C} \colon \text{Re } x \in (-5792.7, 0), |\text{Im } x| < 132\} \cup \{|x| < 18\}.$$

For $q \in (-1, 0)$, this is true for the domain $\{x \in \mathbb{C} \colon |\text{Re } x| < 364.2,\ |\text{Im } x| < 132\}$, see [14] and [13]. In this sense the complex conjugate zeros of θ never go too far from the origin. It is also true that they never enter into the unit disk, see [16] (but this property is false if q and x are complex, see the next section). In the present paper we exhibit a convex domain which contains the left unit half-disk, which is more than 7 times larger than the latter and which is free of zeros of θ for any $q \in (0, 1)$:

Theorem 1. *For any fixed $q \in (0, 1)$, the partial theta function has no zeros in the domain $\mathcal{D} := \{x \in \mathbb{C} \colon |x| \leq 3,\ \text{Re } x \leq 0,\ |\text{Im } x| \leq 3/\sqrt{2}\}$ (with $3/\sqrt{2} = 2.121320344\ldots$).*

When only the real zeros of θ are dealt with, one can improve the above theorem:

Proposition 2. *For any $q \in (0, 1)$ fixed, the function $\theta(q, .)$ has no real zeros ≥ -5.*

Before giving comments on these results in the next section we explain the structure of the paper. Section 3 reminds certain analytic properties of θ. Proposition 2 is proved in Section 4. In Section 5 we prove some lemmas which are used to prove Theorem 1; their proofs can be skipped at first reading. Section 6 contains a plan of the proof of Theorem 1. The proofs of the proposition and lemmas formulated in Section can be found in Section 7.

2. Comments. Throughout the paper we use the following notation:

Notation 3. We define four arcs of the circle centered at 0 and of radius 3:

$$C_k := \{x \in \mathbb{C} \colon |x| = 3, \arg x \in [\pi/2 + (k-1)\pi/4, \pi/2 + k\pi/4]\}, \quad k \in \{1, 2, 3, 4\}.$$

We set $w := 3/\sqrt{2} = 2.121320344\ldots$. The border $\partial\mathcal{D}$ of the domain $\mathcal{D}$ defined in Theorem 1 consists of the arc $C_2 \cup C_3$, the horizontal segments $S_\pm := [-w \pm wi, \pm wi]$ and the vertical segment $S_v := [-wi, wi]$. We parametrise the segment S_+ by setting $x := -t + wi$, $t \in [0, w]$.

One can make the following observations with regard to Theorem 1 and Proposition 1:

(1) It is not clear whether Theorem 1 should hold true for the whole of the left half-disk of radius 3, because $|\theta(0.71, e^{0.51884511144\pi i})| = 0.0141\ldots$, i. e. one obtains a very small value of $|\theta|$ for a point of the arc C_1. This might mean that a zero of θ crosses the arc C_1 for q close to 0.71.

(2) The difficulty to prove results as the ones of Theorem 1 and Proposition 2 resides in the fact that the rate of convergence of the series of θ decreases as q tends to 1^-, and for $q = 1$, one obtains as limit of θ the rational (not entire) function $1/(1-x)$. It is true that the series of θ converges to the function $1/(1-x)$ (which has no zeros at all) on a domain larger than the unit disk and containing the domain $\mathcal{D}$, see [9]. Yet one disposes of no concrete estimations about this convergence, so one cannot deduce from it the absence of zeros of θ in the domain $\mathcal{D}$ for all $q \in (0, 1)$.

(3) The domain $\mathcal{D}$ contains the left half-disk of radius $3/\sqrt{2} > 2$. The ratio of the surfaces of $\mathcal{D}$ and of the left unit half-disk is $(\pi 3^2/4 + (3/\sqrt{2})^2)/(\pi/2) = 7.364788974\ldots$.

(4) One knows that for $q = 0.3092\ldots$, the function $\theta(q, .)$ has a double real zero $-7.5032\ldots$, see [17]. Pictures of the zero set of the function θ (see [15]) suggest that for certain values of $q \in (0, 1)$, it has a zero in the interval $(-7, -6)$, so Proposition 2 cannot be made much stronger.

We explain by examples why analogs of the property of the zeros of θ to avoid the domain $\mathcal{D}$ cannot be found in cases other than $q \in (0, 1)$, $x \leq 0$:

(i) If q is complex, then some of the zeros of θ can be of modulus < 1. Indeed, for $q = \rho e^{3\pi i/4}$, where $\rho \in (0, 1)$ is close to 1, the function θ has a zero close to $0.33\ldots + 0.44\ldots i$ whose modulus is $0.56\ldots < 1$. Similar examples can be given for any q of the form $\rho e^{k\pi i/\ell}$, k, $\ell \in \mathbb{Z}^*$, see [16]. It is true however that θ has no zeros for $|x| \leq 1/2|q|$, see Proposition 7 in [10].

(ii) If $q \in (0, 1)$, the function θ has no positive zeros, but $\theta(0.98, .)$ is likely to have a zero close to $1.209\ldots + 0.511\ldots i$ (i. e. of modulus $1.312\ldots$), see [16]. Conjecture: *As $q \to 1^-$, one can find complex zeros of $\theta(q, .)$ as close to 1 as possible.* One can check numerically that for q close to 0.726475, θ has a complex conjugate couple of zeros close to $\pm 2.9083\ldots i$ (which by the way corroborates the idea that the statement of Theorem 1 cannot be extended to the whole of the left half-disk of radius 3). Thus a convex domain free of zeros of θ should belong to the rectangle $\{\text{Re}x \in (0, 1), |\text{Im}x| < 2.9083\ldots\}$.

(iii) For $q \in (-1, 0)$, it is true that the leftmost of the positive zeros of θ tends to 1^+ as q tends to -1^+, see part (2) of Theorem 3 in [15]. The function $\theta(-0.96, .)$ is supposed to have a couple of conjugate zeros close to the zeros $z_\pm := 0.824\ldots \pm 1.226\ldots i$ (of modulus $1.478\ldots$) of its truncation $\theta^\bullet_{100}(-0.96, .)$; when truncating, the first two skipped terms are of modulus $6.57\ldots \times 10^{-75}$ and $1.51\ldots \times 10^{-76}$. As $q \to -1^+$, the limit of θ equals $(1-x)/(1+x^2)$. One can suppose that the zeros, which equal $z_\pm$ for $q = -0.96$, tend to $\pm i$ as $q \to -1^+$. One knows that for $q \in (-1, 0)$, complex zeros do not cross the imaginary axis, see Theorem 8 in [15]. Hence these zeros of θ should remain in the right half-plane and close to $\pm i$. This means that it is hard to imagine a convex domain in the right half-plane much larger than the right unit half-disk and free of zeros of θ.

As for the left half-plane, the truncation $\theta^{\bullet}_{100}(-0.96,.)$ of $\theta(-0.96,.)$ has conjugate zeros $0.769\ldots \pm 1.255\ldots i$ (of modulus $1.473\ldots$) about which, as about $z_{\pm}$ above, one can suggest that they tend to $\pm i$ as $q \to -1^+$. This could make one think that if one wants to find a domain in the left half-plane containing the left unit half-disk and free of zeros of θ, then in this domain the modulus of the imaginary part should not be larger than 1. On the other hand $\theta(-0.7,.)$ has a zero close to $w_0 := -2.69998\ldots$ so the width of the desired domain should be $< |w_0|$.

3. Known properties of the partial theta function. In this section we remind first that the Jacobi theta function satisfies the *Jacobi triple product*

$$\sum_{j=-\infty}^{\infty} q^{j^2} x^{2j} = \Theta(q, x^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + x^2 q^{2m-1})(1 + x^{-2} q^{2m-1})$$

from which we deduce the equalities

$$\Theta^*(q, x) := \Theta(\sqrt{q}, \sqrt{q}x) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} x^j = \prod_{m=1}^{\infty} (1 - q^m)(1 + xq^m)(1 + q^{m-1}/x) =$$

$$= (1 + 1/x) \prod_{m=1}^{\infty} ((1 - q^m)(1 + xq^m)(1 + q^m/x)). \quad (2)$$

It is clear that

$$\theta = \Theta^* - G \quad \text{with} \quad G(q, x) := \sum_{j=-\infty}^{-1} q^{j(j+1)/2} x^j. \quad (3)$$

Notation 4. (1) When treating the function G we often change the variable x to $X := 1/x$. To distinguish the truncations of the function θ in the variable x from the ones in the variable t (see Notation 3) we write $\theta = \theta^{\bullet}_k + \theta^{\bullet}_*$, where $\theta^{\bullet}_k := \sum_{j=0}^{k} q^{j(j+1)/2} x^j$ and $\theta^{\bullet}_* := \sum_{j=k+1}^{\infty} q^{j(j+1)/2} x^j$, i. e. we use the superscript "bullet" when in the variable x (no index k is added to $\theta^{\bullet}_*$). No superscript is used for the truncations of $\theta(q, -t + wi)$ and of G.

(2) We set $\lambda := 3e^{3\pi i/4}$, $R(q, x) := \prod_{m=1}^{\infty} (1 + q^{m-1}/x)$, $M := |(1 + qx)(1 + q/x)|$, $M_0 := (1 - q)M$ and $M_1(q, t) := M_0(q, -t + wi)$.

Remark 5. In the proofs we use the convergence of the series (1) when the parameter q belongs to an interval of the form $[0, a]$, $a \in (0, 1)$. When we need to deal with intervals of the form $[a, 1]$, we use the equalities (3) in which the modulus of the term Θ^* tends to 0 as q tends to 1^- while the series of G converges uniformly for $|x| \in [c, \infty)$ for any fixed $c > 1$. When in the proof of a lemma or a proposition we use the fact that a certain function in one variable (mainly a polynomial) is increasing or decreasing, we do not give a detailed proof of this, because in all such cases the proof can be given using elementary methods (computation of derivatives and numerical computation of their real roots). We do not give details when proving the absence of critical points of polynomials in two variables in given rectangles. In this text their degree is never too high and the necessary computations are easily performed using MAPLE.

For $q \in (0, 1)$, the real zeros of θ (which are all negative) and of any of its derivatives w.r.t. the variable x form a sequence tending to $-\infty$ and behaving asymptotically as a geometric progression with ratio $1/q$, see Theorem 4 in [10].

There exists an increasing and tending to 1^- sequence of *spectral values* $\tilde{q}_j$ of q such that $\theta(\tilde{q}_j,.)$ has a multiple (more exactly double) real zero, see [17].

The 6-digit truncations of the first 6 spectral values are:

$$0.309249, \quad 0.516959, \quad 0.630628, \quad 0.701265, \quad 0.749269, \quad 0.783984.$$

When q passes from $\tilde{q}_j^-$ to $\tilde{q}_j^+$, the rightmost two of the real zeros of θ coalesce and then form a complex conjugate pair. All other real zeros of θ remain negative and distinct, see Theorem 1 in [10]. The inverse (complex couples becoming double and then two distinct real zeros) never happens. No zeros are born at ∞.

Thus for q fixed, the function θ belongs to the Laguerre-Pólya class $\mathcal{L}-\mathcal{P}I$ exactly if $q \in (0, \tilde{q}_1]$. For $q \in (\tilde{q}_j, \tilde{q}_{j+1}]$, the function θ is the product of a real polynomial of degree $2j$ without real zeros and a function of the class $\mathcal{L}-\mathcal{P}I$. See the details in [11].

Spectral values exist also for $q \in (-1, 0)$, see [12]. The existence of spectral values for complex values of q is proved in [12], see Proposition 8 therein.

At the end of this section we mention the fact that the function θ satisfies the two conditions

$$\theta(q,x) = 1 + qx\theta(q,qx) \quad \text{and} \quad 2q\partial\theta/\partial q = 2x\partial\theta/\partial x + x^2\partial^2\theta/\partial x^2.$$

4. Proof of Proposition 2. For $q \leq 0.1$, all zeros of $\theta(q,.)$ are real negative and smaller than $-1/q$, see Proposition 7 in [10]. Hence they are smaller than -5 and one has $\theta(q,x) > 0$ for $x \in [-1/q, 0] \supset [-5, 0]$. As q increases, its zeros depend continuously on q. For a spectral value of q, certain zeros coalesce to form then a complex conjugate pair, but new real zeros are not born, see the previous section. Therefore it suffices to show that for $q \in (0,1)$, one has $\theta(q, -5) > 0$.

For $q \in (0, 0.8]$, one finds numerically that $\theta(q,-5)$ is larger than 0.04. To this end one can consider

$$\theta^\bullet_{15}(q,-5) := \sum_{j=0}^{15} q^{j(j+1)/2}(-5)^j$$

which is a polynomial in q and show that for $q \in (0, 0.5]$ (resp. for $q \in [0.5, 0.8]$), it is larger than 0.05 (resp. larger than 0.16). The sum of the moduli of all skipped terms is smaller than 1.8×10^{-30} and 0.07 respectively.

To prove that $\theta(q,-5) > 0$ for $q \in [0.8, 1)$, we first show that $-G(q,-5) > 4/25$. This is true, because $-G(q,-5)$ is a Leibniz series with first two terms $1/5$ and $-q/25$, so its sum is

$$\geq 1/5 - q/25 > 1/5 - 1/25 = 4/25.$$

Now we estimate $|\Theta^*|$. We set

$$K(q) := (1+qx)(1+q/x)|_{x=-5} = (1-5q)(1-q/5) = 1 - 26q/5 + q^2.$$

Hence,

$$\Theta^*(q,-5) = (4/5)\prod_{m=1}^{\infty}(1-q^m)\prod_{m=1}^{\infty}K(q^m),$$

see (2). The function $|K|$ is decreasing on $[0, u']$ and increasing on $[u', 1]$, where $u' = 0.2$. One has $|K(0)| = |K(u'')| = 1$, $u'' = 0.4182575771\ldots$, and $|K(1)| = 3.2$. We denote by

$$0.1357556939\ldots =: t' < t'' := 0.2660119966\ldots$$

the values of q for which $|K| = 1/3.2$, by

$$0.09800079936\ldots =: d' < d'' := 0.3065310118\ldots$$

its values where $|K(q)| = 1/2$ and by $s := 0.6609280570\ldots$ the value of q such that $|K(q)| = 2$.

For $a \in (0,1]$, set $\phi(a) := \ln(1/a)/\ln(1/q)$, so $\phi(a) \geq 0$ and $\phi(a_1) - \phi(a_2) = \phi(a_1/a_2)$. We remind that

$$\text{for} \quad b, c \in \mathbb{R}, \quad [b-c] = [b] - [c] - \chi(b,c), \quad \text{where} \quad \chi(b,c) = 0 \quad \text{or} \quad 1\ ; \tag{4}$$

here [.] stands for the integer part. Denote by ℓ_1, $\ell_2 \in \mathbb{N}$ the maximal indices ℓ for which one has $q^\ell \geq u''$ and $q^\ell \geq s$ respectively

$$\ell_1 = [\phi(u'')] \quad \text{and} \quad \ell_2 = [\phi(s)].$$

These are the numbers of terms of the sequence $\{q^m\}$ belonging to the intervals $[u'', 1)$ and $[s, 1)$. For the intervals $[t', t'')$ and $[d', d'')$, these numbers are equal to

$$m_1 := [\phi(t')] - [\phi(t'')] \quad \text{and} \quad m_2 := [\phi(d')] - [\phi(d'')].$$

One computes directly that

$$\ln(t''/t') = 0.672\ldots > 0.414\ldots = \ln(1/s).$$

Thus,

$$\ln(t''/t') - \ln(1/s) > 0.25.$$

Moreover, as $q \in [0.8, 1)$, one has $1/\ln(q) \geq 1/\ln(0.8) = 4.48\ldots > 4$, so

$$\ln(t''/t')/\ln(1/q) - \ln(1/s)/\ln(1/q) > 1.$$

Hence, $[\phi(t'/t'')] > [\phi(s)]$ and

$$m_1 = [\phi(t')] - [\phi(t'')] = [\phi(t'/t'')] + \chi(\phi(t'), \phi(t'')) > [\phi(s)] = \ell_2.$$

This means that out of the factors $|K(q^m)|$ which are present in $|\Theta^*(q, -5)|$, the ones which are in the interval $[2, 3.2)$ are less than the ones which are in $[0, 1/3.2]$. We denote their sets by $\Sigma_{[2,3.2)}$ and $\Sigma_{[0,1/3.2]}$ and we write $\sharp(\Sigma_{[2,3.2)}) < \sharp(\Sigma_{[0,1/3.2]})$. The latter inequality implies

$$\prod_{|K(q^m)| \in \Sigma_{[2,3.2)} \cup \Sigma_{[0,1/3.2]}} |K(q^m)| < 1.$$

Using the above notation and (4) one can write

$$\begin{aligned} \sharp(\Sigma_{[1,2)}) &=: \ell_3 = [\phi(u'')] - [\phi(s)] \leq [\phi(u''/s)] + 1 \quad \text{and} \\ \sharp(\Sigma_{(1/3.2,1/2]}) &=: m_3 \geq [\phi(d')] - [\phi(t')] + [\phi(t'')] - [\phi(d'')] - 1. \end{aligned} \tag{5}$$

To prove the latter inequality one has to observe that the numbers q^m corresponding to factors $|K(q^m)|$ from the set $\Sigma_{(1/3.2,1/2]}$ belong to the union $[d', t') \cup (t'', d'']$. The numbers q^m which belong to the interval $[d', t')$ are exactly $[\phi(d')] - [\phi(t')]$. The ones that are in $(t'', d'']$ are not less than $[\phi(t'')] - [\phi(d'')] - 1$ (at most one of them equals t'' and there are exactly $[\phi(t'')] - [\phi(d'')]$ numbers q^m in $[t'', d'')$). One finds that

$$\phi(u''/s) = 0.457\ldots < \phi(d'/t') + \phi(t''/d'') = 0.467\ldots,$$

so

$$[\phi(u''/s)] \leq [\phi(d'/t')] + [\phi(t''/d'')] + 2 \leq m_3 + 5$$

(see the equalities and inequalities (5) and (4)) and one concludes that $\ell_3 \leq m_3 + 6$. The factors $|K(q^m)|$ which have not been mentioned up to now are all of modulus < 1; the corresponding numbers q^m belong to the intervals $(0, d')$ and (t'', u'').

Thus,

$$\prod_{m=1}^{\infty}|K(q^m)|<2^6.$$

At the same time

$$\prod_{m=1}^{\infty}(1-q^m)\leq\prod_{m=1}^{\infty}(1-0.8^m)<7\times 10^{-6}.$$

This shows that $|\Theta^*(q,-5)|<10^{-4}<4/25<|-G(q,-5)|$ from which the proposition follows.

5. Some technical lemmas. In this section we formulate and prove several lemmas.

Lemma 6. *For $q\in[0,1]$ and $|x|=a>2$, one has $|G|\geq(a-2)/a(a-1)$. In particular, for $a=3$, $|G|\geq 1/6$.*

Proof. Indeed,

$$|G|\geq(1/|x|)(1-\sum_{j=-\infty}^{-2}q^{j(j+1)/2}|x^{j+1}|)\geq(1/|x|)(1-\sum_{j=-\infty}^{-2}|x^{j+1}|)=$$

$$=(1/a)(1-\sum_{j=-\infty}^{-2}a^{j+1})=(a-2)/a(a-1).$$

□

We set $X:=1/x$ and we represent the function G in the form

$$G=G_5+G_*,\quad G_5:=X+qX^2+q^3X^3+q^6X^4+q^{10}X^5,\quad G_*:=\sum_{j=5}^{\infty}q^{j(j+1)/2}X^{j+1}.$$

Lemma 7. *For $(q,t)\in[0.6,1]\times[0,w]$, one has $|G_*|<0.0208$ and $|G_5|>0.147$.*

Proof. For $(q,t)\in[0.6,1]\times[0,w]$, it is true that $|X|=1/|-t+wi|\leq 1/w$ and

$$|G_*|\leq\sum_{j=5}^{\infty}|q^{j(j+1)/2}X^{j+1}|\leq\sum_{j=5}^{\infty}|X^{j+1}|\leq\sum_{j=5}^{\infty}|w^{-j-1}|=0.02076055760\ldots<0.0208$$

which proves the first claim of the lemma. To prove the second one we represent the function $G_I:=\mathrm{Im}(G_5(q,1/(-t+wi)))$ in the form

$$G_I=(3\sqrt{2}/(2t^2+9)^5)G_I^{\flat},\quad \text{where}\quad G_I^{\flat}=g_0+g_1q+g_3q^3+g_6q^6+g_{10}q^{10},\quad\text{with}$$

$$\begin{array}{rclrcl} g_0 & := & -16t^8-288t^6-1944t^4-5832t^2-6561, & g_6 & := & 64t^5-1296t,\\ g_1 & := & 32t^7+432t^5+1944t^3+2916t, & & & \\ g_3 & := & -48t^6-360t^4-324t^2+1458, & g_{10} & := & -80t^4+720t^2-324.\end{array}$$

Our first goal is to give an upper bound for G_I for $t\in[0,1]$ (hence an upper bound for $G_I^{\flat}$).

We use the evident equalities and inequalities

$$\begin{array}{llll} 288=256+32, & 1944=1512+432, & 5832=3168+1944+720, & 6561=2916+1458+2187, \\ 32t^6 \geq 32t^7q, & 432t^4 \geq 432t^5q, & 1944t^2 \geq 1944t^3q, & 2916 \geq 2916tq, \\ 64tq^6 \geq 64t^5q^6, & 720t^2 \geq 720t^2q^{10}, & 1296 = 64 + 1232 & 1458 \geq 1458q^3 \end{array}$$

to obtain an upper bound G^u for $G_I^\flat$ in which all coefficients are negative:

$$G^u := -16t^8 - 256t^6 - 1512t^4 - 3168t^2 - 2187- \\ -(48t^6 + 360t^4 + 324t^2)q^3 - 1232tq^6 - (80t^4 + 324)q^{10}.$$

For $t \in [0, 1]$ fixed, the upper bound of the product $(3\sqrt{2}/(2t^2 + 9)^5)G^u$ is attained for $q = 0.6$. The function $(3\sqrt{2}/(2t^2 + 9)^5)G^u|_{q=0.6}$ is decreasing in t and its value for $t = 0$ is $v_1 := -0.1572756008\ldots$, so v_1 is the upper bound of G_I for $t \in [0, 1]$, $q \in [0.6, 1]$.

Suppose now that $t \in [1, w]$ and $q \in [0.6, 1]$. We observe first that the function $G_I|_{q=1}$ is increasing and $(G_I|_{q=1})|_{t=w} = -0.1478254790\ldots =: v_2$. Next, we prove that

$$\partial G_I^\flat/\partial q = g_1 + 3qg_3 + 6q^5g_6 + 10q^9g_{10} > 0 \quad \text{hence} \quad \partial G_I/\partial q > 0.$$

The quantities g_1, g_6 and g_{10} do not change sign for $t \in [1, w]$: $g_1 > 0$, $g_6 \leq 0$, with equality only for $t = w$, while $g_{10} > 0$. The polynomial g_3 is negative for $t > t_1 := 1.224744871\ldots$ and positive for $t \in [1, t_1)$; it vanishes for $t = t_1$. Thus for $t \geq t_1$,

$$\partial G_I^\flat/\partial q \geq g_1 + 3g_3 + 6g_6 + 10 \times 0.6^{10}g_{10} =: G^\ddagger.$$

The polynomial $(G^\ddagger)'$ has a single real root $t_2 := 1.144295977\ldots$. The function $G^\ddagger$ is increasing for $t \geq t_2$, so for $t \geq t_1 > t_2$, one has $G^\ddagger(t) > G^\ddagger(t_2) = 9.468005\ldots > 0$.

For $t \in [1, t_1]$, we minorize the function $\partial G_I^\flat/\partial q$ in each of the four cases $q \in [0.6, 0.7]$, $q \in [0.7, 0.8]$, $q \in [0.8, 0.9]$ and $q \in [0.9, 1]$. Denote any of these four intervals by $[a, b]$. The minoration is looked for in the form

$$G_{a,b} := g_1 + 3a^2g_3 + 6b^5g_6 + 10a^9g_{10}.$$

Each of the four functions $G_{a,b}$ turns out to be monotone increasing on $[1, t_1]$, so the four minima are attained for $t = 1$. They equal $4897.5\ldots$, $4096.5\ldots$, $2777.1\ldots$ and $920.4\ldots$ respectively. Hence for $t \in [1, w]$, $\partial G_I^\flat/\partial q > 0$ and the function G_I is maximal for $q = 1$. Hence it is $\leq v_2$. For $t \in [0, 1]$, it is $\leq v_1$, so it is $\leq v_2 < -0.147$ for $(q, t) \in [0.6, 1] \times [0, w]$ and $|G_I| > 0.147$. □

Lemma 8. *Consider the factors $|1 + q^mx|$ and $|1 + q^{m-1}/x|$, $m = 1, 2, \ldots$.*

(1) For $q \in (0, 1)$ fixed and for $x \in C_1 \cup C_2$, these quantities are decreasing functions in $\varphi := \arg x$.

(2) For $x = 3e^{3\pi i/4}$, each factor $|1+q^{m-1}/x|$, $m \geq 2$, is a decreasing function in $q \in (0, 1)$.

(3) For $q \in [0.5, 1]$ and for $x = 3e^{3\pi i/4}$, the factor $|1 + qx|$ is an increasing function in q.

Proof. The first claim of the lemma follows from the cosine theorem. Indeed, recall that $1/x = \bar{x}/|x|^2$, so $\arg(1/x) = -\arg x = -\varphi$ and $\cos(\arg(1/x)) = \cos\varphi$. Hence,

$$\begin{aligned} |q^m x - (-1)|^2 = q^{2m}|x|^2 + (-1)^2 - 2(-1)q^m|x|\cos\varphi = \\ = q^{2m}|x|^2 + 1 + 2q^m|x|\cos\varphi, \\ |(q^{m-1}/x) - (-1)|^2 = q^{2m-2}/|x|^2 + (-1)^2 - 2(-1)(q^{m-1}/|x|)\cos\varphi = \\ = q^{2m-2}/|x|^2 + 1 + 2(q^{m-1}/|x|)\cos\varphi. \end{aligned}$$

For q fixed, these quantities are decreasing in φ, because such is $\cos\varphi$. Set $\cos\varphi := -\sqrt{2}/2$, $|x| := 3$. The displayed formulas show that

$$d(|1 + q^{m-1}/x|)/dq = ((m-1)q^{m-2}/|x|^2)(2q^{m-1} - \sqrt{2}|x|) < 0$$

and

$$d(|1 + qx|)/dq = m|x|(2q|x| - \sqrt{2}) > 0$$

from which one deduces the last two claims of the lemma. □

In the proofs we need some properties of the functions $M := |(1 + qx)(1 + q/x)|$ and $M_0 := (1 - q)M$. We remind that we set $x = -t + wi$, $t \in [0, w]$, $w = 3/\sqrt{2}$.

Lemma 9. *For $t \in [0, w]$ and for $q \in [0.6, 1]$ fixed, the quantities M and M_0 are maximal for $t = 0$. For $q \in [0.6, 0.75]$ fixed and for $t \in [1, w]$, they are maximal for $t = 1$.*

Proof. It suffices to prove the claims of the lemma about the function M. One checks directly for the square of M that

$$M^2 = (2q^2t^2 + 9q^2 - 4qt + 2)(2q^2 - 4qt + 2t^2 + 9)/(2t^2 + 9).$$

One verifies straightforwardly that

$$M^2 - M^2|_{t=0} = -2qtP/(2t^2 + 9),$$

where

$$P := 36q^2t^2 - 18qt^3 + 198q^2 - 149qt + 36t^2 + 198.$$

The discriminant of the trinomial $36(qt)^2 - 149qt + 198$ is negative, so this trinomial is positive-valued. For the remaining terms of P, for $t \in [0, w]$ (hence $t^2 \leq 9/2$), one obtains

$$-18qt^3 + 198q^2 + 36t^2 \geq -81qt + 198q^2 + 36t^2$$

which is again a trinomial with negative discriminant. Thus $P > 0$ and $M^2 - M^2|_{t=0} \leq 0$ with equality only for $t = 0$ which proves the first claim of the lemma. To prove its second statement we consider the difference

$$M^2 - M^2|_{t=1} = -2q(t-1)(V_2q^2 + V_1q + V_0)/11(2t^2 + 9),$$

where

$$V_2 = V_0 := 44t^2 - 8t + 234 \quad \text{and} \quad V_1 := -(t+1)(22t^2 + 167).$$

The polynomial $V_2q^2 + V_1q + V_0$ has no crfitical points for $(q, t) \in [0.6, 0.75] \times [1, w]$. Its restrictions to each of the sides of this rectangle (i. e. its restrictions obtained for $q = 0.6$, $q = 0.75$, $t = 1$ and $t = w$) are positive-valued. Hence the difference $M^2 - M^2|_{t=1}$ is negative in the given rectangle which proves the second claim of the lemma. □

Remark 10. For $x = -t + wi$, we consider the function

$$M_1(q,t) := M_0(q,-t+wi) := (1-q)|(1+qx)(1+q/x)| =$$
$$= (1-q)(2q^2t^2+9q^2-4tq+2)^{1/2}(2q^2-4tq+2t^2+9)^{1/2}/(2(2t^2+9))^{1/2}.$$

The two functions in the variable q

$$M_1(q,0) = (1-q)(9q^2+2)^{1/2}(2q^2+9)^{1/2}/3\sqrt{2}$$

and

$$M_1(q,1) = (1-q)(11q^2-4q+2)^{1/2}(2q^2-4q+11)^{1/2}/\sqrt{22}$$

are decreasing on $[0,1]$.

6. Plan of the proof of Theorem 1. The zeros of θ depend continuously on q and no zeros are born at ∞. We prove that for $q \in (0,1)$, there is no zero of θ on the border $\partial\mathcal{D}$ of the domain $\mathcal{D}$. For $q \in (0,0.5]$, this follows from the proposition below. We remind that (see Notation 3)

$$\partial\mathcal{D} = C_2 \cup C_3 \cup S_+ \cup S_- \cup S_v.$$

Proposition 11. *For $q \in (0,0.5]$, the function $\theta(q,.)$ has no zeros in the closed rectangle $\Delta := \{-3 \leq \mathrm{Re}x \leq 0,\ -3 \leq \mathrm{Im}x \leq 3\}$.*

The proof of this proposition and of all lemmas formulated in this section are given in Section 7. The rectangle Δ contains the domain $\mathcal{D}$, so for $q \in (0,0.5]$, there are no zeros of θ on $\partial\mathcal{D}$. One can observe that for $q \in (0,\tilde{q}_1]$, $\tilde{q}_1 = 0.3092\ldots$, there are no complex conjugate pairs of θ (see Section 3), and for $q \in (\tilde{q}_1, 0.5]$, there is exactly one such pair.

From now on we assume that $q \in [0.5,1)$. The next lemma explains why no zeros of θ can be found on the arc C_2 hence none on the arc C_3 either.

Lemma 12. *For $q \in [0.5,1)$ and $x \in C_2$, one has $|G| > |\Theta^*|$ hence $|\theta| > 0$.*

The next lemma states that the function θ has no zeros on the segment S_v:

Lemma 13. *For $q \in (0,1)$, the function $\theta(q,.)$ has no purely imaginary zeros of modulus ≤ 2.2 hence no such zeros of modulus $\leq 3/\sqrt{2} = 2.1\ldots$.*

It remains to show that there are no zeros of θ on the segments $S_\pm$. It suffices to deal with the segment S_+. We consider the restrictions to $[0.3,0.6]\times[0,w]$ of the functions $\theta_5(q,t) := \sum_{j=0}^{5} q^{j(j+1)/2}(-t+wi)^j$ and $\theta_*(q,t) = \sum_{j=6}^{\infty} q^{j(j+1)/2}(-t+wi)^j$.

Lemma 14. *For $(q,t) \in [0.3,0.6]\times[0,w]$, one has $|\theta_*(q,t)| \leq 0.018$ and $\theta_I :=\mathrm{Im}(\theta_5(q,t)) > 0.13$. Hence for $(q,t) \in [0.3,0.6]\times[0,w]$, the function θ has no zeros.*

Next we settle the case $q \in [0.75,1)$.

Lemma 15. *For $(q,t) \in [0.75,1)\times[0,w]$, the function θ has no zeros.*

The remaining case $q \in [0.6,0.75]$ will be subdivided in two cases:

Lemma 16. *For $(q,t) \in [0.6,0.75]\times[1,w]$, the function θ has no zeros.*

Lemma 17. *For $(q,t) \in [0.6,0.75]\times[0,1]$, the function θ has no zeros.*

7. Proofs.

Proof of Proposition 11. A) For $q \in [0, 0.3]$, all zeros of $\theta(q, .)$ are real, negative and distinct, see Section 2. All these zeros are $< -5 < -3$, see Proposition 2.

B) We set $\theta = \theta_4^\bullet + \theta_*^\bullet$,
where

$$\theta_4^\bullet := 1 + qx + q^3x^2 + q^6x^4 + q^{10}x^4 \quad \text{and} \quad \theta_*^\bullet := \sum_{j=5}^{\infty} q^{j(j+1)/2}x^j.$$

For $x \in \Delta$, one has

$$|x| \leq 3\sqrt{2} = 4.24\ldots < 4.25,$$

so for $(q, x) \in [0, 0.5] \times \Delta$, one obtains the majoration

$$|\theta_*^\bullet(q, x)| \leq \sum_{j=5}^{\infty} 0.5^{j(j+1)/2} 4.25^j = 0.045\ldots < 0.046. \tag{6}$$

C) We denote the border of the rectangle Δ by $\partial\Delta$ and we set $I_0 := [0, 0.5]$. We show that for each $q \in I_0$ fixed, one has

$$|\theta_4^\bullet(q, x)| > |\theta_*^\bullet(q, x)| > 0 \tag{7}$$

for any $x \in \partial\Delta$. For $q \in [0, 0.01]$, there is no zero of $\theta_4^\bullet$ in Δ. Indeed, one obtains

$$|\theta_4^\bullet| \geq 1 - 0.01 \times 4.25 - 0.01^3 \times 4.25^2 - 0.01^6 \times 4.25^3 - 0.01^{10} \times 4.25^4 = 0.95\ldots > 0.$$

The condition (7) is fulfilled for $x \in \partial\Delta$, so it implies that no zero of $\theta_4^\bullet$ may enter Δ as q increases from 0.01 to 0.5. Hence $\theta_4^\bullet$ has no zeros in Δ for $q \in I_0$.

Again from condition (7) and from the Rouché theorem follows that θ has no zeros for $(q, x) \in I_0 \times \Delta$. So our aim is to show that condition (7) holds true.

D) When proving condition (7) we deal only with the part of $\partial\Delta$ with $\mathrm{Im}x \geq 0$. For $\mathrm{Re}x = -3$, we set $x := -3 + it$, $t \in [0, 3]$. Then

$$G_R(q, t) := \mathrm{Re}(\theta_4^\bullet) = q^{10}t^4 - 54q^{10}t^2 + 81q^{10} + 9q^6t^2 - 27q^6 - q^3t^2 + 9q^3 - 3q + 1,$$

$$G_I(q, t) := \mathrm{Im}(\theta_4^\bullet) = 12q^{10}t^3 - 108q^{10}t - q^6t^3 + 27q^6t - 6q^3t + qt.$$

We use the fact that

$$|\theta_4^\bullet| \geq \max(|\mathrm{Re}(\theta_4^\bullet)|, |\mathrm{Im}(\theta_4^\bullet)|) =: \mu.$$

Neither of the functions G_R and G_I has a critical point with $q \in I_0$, so G_R (resp. G_I) attains its maximal and its minimal value when one of the following conditions takes place: $t = 0$, $t = 3$, $q = 0$ or $q = 0.5$.

For $q = 0$, one has $G_R \equiv 1$, so $\mu \geq 1 > 0.046$. For $q = 0.5$, one gets

$$G_R = 0.0009765625t^4 - 0.037109375t^2 + 0.2822265625, \quad G_I = -0.00390625t^3 + 0.06640625t$$

and one checks directly that for $t \in [0, 1]$ and $t \in [1, 3]$, one has $G_R > 0.05 > 0.046$ and $G_I > 0.05 > 0.046$ respectively.

For $t = 0$, one obtains

$$G_R = 81q^{10} - 27q^6 + 9q^3 - 3q + 1$$

which is $> 0.2 > 0.046$ for $q \in I_0$. For $t = 3$, it is clear that

$$G_R = -324q^{10} + 54q^6 - 3q + 1 \quad \text{and} \quad G_I = 54q^6 - 18q^3 + 3q,$$

with $G_R > 0.2 > 0.046$ for $q \in [0, 0.2]$ and with $G_I > 0.05 > 0.046$ for $q \in [0.2, 0.5]$ respectively.

E) For Re$x = 0$, one sets $x := i\tau$ to obtain

$$U_R(q,\tau) := \mathrm{Re}(\theta_4^\bullet) = q^{10}\tau^4 - q^3\tau^2 + 1 \quad \text{and} \quad U_I(q,\tau) := \mathrm{Im}(\theta_4^\bullet) = -q\tau(q^5\tau^2 - 1).$$

Neither of the functions U_R and U_I has a critical point with $q \in I_0$, so their maximal and minimal values are attained for $\tau = 0$, $\tau = 3$, $q = 0$ or $q = 0.5$. In each of the cases $\tau = 0$ and $q = 0$ one has $U_R \equiv 1 > 0.046$.

Suppose that $\tau = 3$. Then

$$U_R > 0.05 > 0.046$$

for $q \in [0, 0.3]$ and

$$U_I > 0.05 > 0.046$$

for $q \in [0.3, 0.5]$. Finally, if $q = 0.5$, then

$$U_R = 0.0009765625\tau^4 - 0.125\tau^2 + 1 \quad \text{and} \quad U_I = -0.5\tau(0.03125\tau^2 - 1),$$

with $U_R > 0.05$ for $\tau \in [0, 2]$ and with $U_I > 0.05$ for $\tau \in [2, 3]$.

F) Suppose that Im$x = 3$. Then we set $x := u + 3i$, $u \in [-3, 0]$. Then

$$S_R(q,u) := \mathrm{Re}(\theta_4^\bullet) = q^{10}u^4 - 54q^{10}u^2 + 81q^{10} + q^6u^3 - 27q^6u + q^3u^2 - 9q^3 + qu + 1,$$

$$S_I(q,u) := \mathrm{Im}(\theta_4^\bullet) = 12q^{10}u^3 - 108q^{10}u + 9q^6u^2 - 27q^6 + 6q^3u + 3q.$$

The functions S_R and S_I have no critical points for (q, u) inside the rectangle $I_0 \times [-3, 0]$, so their maximal and minimal values are attained on its border. Obviously $S_R|_{q=0} \equiv 1 > 0.046$, $S_R|_{u=0} = 81q^{10} + 1 \geq 1 > 0.046$ and

$$S_I|_{q=0.5} = 0.01171875u^3 + 0.140625u^2 + 0.64453125u + 1.078125$$

which is $> 0.05 > 0.046$ for $u \in [-3, 0]$. For $u = -3$, one obtains

$$S_R = -324q^{10} + 54q^6 - 3q + 1 \quad \text{and} \quad S_I = 54q^6 - 18q^3 + 3q,$$

with $S_R > 0.05 > 0.046$ for $q \in [0, 0.2]$ and with $S_I > 0.05 > 0.046$ for $q \in [0.2, 0.5]$. □

Proof of Lemma 12. It suffices to show that $|\Theta^*| < 1/6$, see Lemma 6 with $a = 3$. By part (1) of Lemma 8, it is sufficient to prove this for $x = \lambda := 3e^{3\pi i/4}$. The modulus

$$|R| := \prod_{m=1}^{\infty} |1 + q^{m-1}/x|$$

is maximal (see Notation 4 and part (2) of Lemma 8) when $q = 0.5$ in which case one gets

$$|1 + q^{m-1}/x| = r_m := |1 + 0.5^{m-1}(-\sqrt{2} - \sqrt{2}i)/6| = ((1 - 0.5^{m-1}\sqrt{2}/6)^2 + 0.5^{2m-2}/18)^{0.5}$$

and one finds numerically that

$$|R| \leq \prod_{m=1}^{11} r_m = 0.6329437509\ldots < 0.633.$$

Next, for $x = \lambda$, the points representing the complex numbers $u_m := 1 + xq^m$ lie on the straight line passing through the points 1 and i; they lie above the abscissa-axis. We denote by $m_0 \in \mathbb{N}$ the index m for which $\mathrm{Re}(u_m) \leq 0$ (hence $|u_m| \geq 1$) and $\mathrm{Re}(u_{m+1}) > 0$ (hence $|u_{m+1}| < 1$). One has $m_0 \geq 1$. Indeed, for $q \in [0.5, 1]$,

$$|1 + q\lambda| \geq |1 + 0.5\lambda| = 1.062393362\ldots > 1$$

(see part (3) of Lemma 8). A numerical check shows that one has $m_0 \geq 2$ exactly if $q \geq 0.6865890479\ldots > 0.68 =: q^{\dagger}$.

For $m \leq m_0$, one has $|u_m| \leq |v_m|$, where $v_m := q^m + \lambda q^m = q^m(1+\lambda)$ (with equality only for $q = 1$). One finds numerically that

$$|1 + \lambda| = 2.399449794\ldots < 2.4,$$

so for $m \leq m_0$, $|u_m| < 2.4$. For each product $p_m := (1 - q^m)u_m$ one can write

$$|p_m| \leq |(1 - q^m)v_m| = |(1 - q^m)q^m||1 + \lambda| \leq (1/4) \times 2.4 = 0.6.$$

Suppose that $m_0 \geq 3$. Then

$$\begin{array}{rcl} |\Theta^*| & \leq & (\prod_{m=1}^{m_0} |p_m|) \times 0.633 \times (\prod_{m=m_0+1}^{\infty} |u_m|) \times \prod_{m=m_0+1}^{\infty}(1 - q^m) \leq \\ & \leq & 0.6^{m_0} \times 0.633 \leq 0.6^3 \times 0.633 = 0.136728 < 1/6. \end{array}$$

Suppose that $m_0 = 2$. The maximal value of the function

$$q(1 - q)q^2(1 - q^2) = q^3(1 - q)^2(1 + q)$$

for $q \in [0, 1]$ is $0.05579835315\ldots < 0.056$; it is attained for $q = 0.6286669788\ldots$. Thus,

$$|p_1||p_2| \leq 0.056 \times 2.4^2, \qquad \prod_{m=3}^{\infty}(1 - q^m) < 0.78$$

and

$$\begin{array}{l} |\Theta^*| \leq |p_1||p_2| \times 0.633 \times \left(\prod_{m=3}^{\infty} |u_m|\right) \times \prod_{m=3}^{\infty}(1 - q^m) \leq \\ \leq 0.056 \times 2.4^2 \times 0.633 \times 0.78 < 0.16 < 1/6. \end{array}$$

Suppose that $m_0 = 1$. Then $0.5 \leq q < 0.69$. One finds that

$$\prod_{m=2}^{\infty}(1 - q^m) < \prod_{m=2}^{100}(1 - 0.5^m) = 0.5775\ldots < 0.5776$$

and

$$\prod_{m=2}^{\infty} |u_m| < \prod_{m=2}^{30} |u_m| =: g(q).$$

The function g is decreasing for $q \in [0.5, 0.69]$ and $g(0.5) = 0.4254\ldots < 0.4255$.

Therefore,

$$|\Theta^*| \leq |p_1| \times 0.633 \times g(q) \times 0.5776 <$$
$$< 0.6 \times 0.633 \times 0.4255 \times 0.5776 = 0.093\ldots < 1/6.$$

□

Proof of Lemma 13. Indeed, set $x := iy$, $y \in \mathbb{R}$. Hence

$$\theta(q, iy) = \theta(q^4, -y^2/q) + iqy\theta(q^4, -qy^2).$$

The first and the second summand represent the real and the imaginary part of θ when restricted to the imaginary axis. If iy_0 is a zero of $\theta(q, .)$, $y_0 \in \mathbb{R}$, then

$$\theta(q^4, -y_0^2/q) = \theta(q^4, -qy_0^2) = 0.$$

By Proposition 2, $-y_0^2/q < -5$ and $-qy_0^2 < -5$, so $|y_0| > \sqrt{5} > 2.2$. □

Proof of Lemma 14. For $t \in [0, w]$, one has $|-t + wi| \leq 3$, with equality only for $t = w$. Thus,

$$|\theta_*(q,t)| \leq \sum_{j=6}^{\infty} q^{j(j+1)/2} 3^j \leq \sum_{j=6}^{\infty} 0.6^{j(j+1)/2} 3^j = 0.017\ldots < 0.018.$$

One finds by direct computation that

$$\theta_I = (3\sqrt{2}q/8)(20q^{14}t^4 - 180q^{14}t^2 + 81q^{14} - 16q^9t^3 + 72q^9t + 12q^5t^2 - 18q^5 - 8q^2t + 4).$$

By lowercase indices t or q we denote derivations w.r.t. these variables. We show first that $(\theta_I)_t < 0$. Thus for $q \in [0.3, 0.6]$ fixed, θ_I is maximal for $t = 0$ and minimal for $t = w$. One finds that

$$(\theta_I)_t = 3\sqrt{2}(10q^{12}t^3 - 45q^{12}t - 6q^7t^2 + 9q^7 + 3q^3t - 1)q^3,$$

$$(\theta_I)_{tt} = 9\sqrt{2}q^6(10q^9t^2 - 15q^9 - 4q^4t + 1), \quad (\theta_I)_{ttt} = 36\sqrt{2}q^{10}(5q^5t - 1).$$

For $(q,t) \in [0.3, 0.6] \times [0, w]$, one has $(\theta_I)_{ttt} < 0$. Hence $(\theta_I)_{tt}$ is minimal for $t = w$; in this case it equals

$$9\sqrt{2}q^6(30q^9 - 6q^4\sqrt{2} + 1)$$

which is positive for $q > 0$. Therefore $(\theta_I)_t$ is maximal for $t = w$ when it equals

$$3\sqrt{2}(-18q^7 + (9\sqrt{2}q^3)/2 - 1)q^3$$

which is negative for $q > 0$. So θ_I is minimal for $t = w$ and

$$\theta_I(q, w) = -(3\sqrt{2}q(81q^{14} - 9q^5 + 3\sqrt{2}q^2 - 1))/2.$$

The derivative $(\theta_I(q, w))_q$ is negative for $q \geq 0.3$, this means that θ_I is minimal for $(q, t) = (0.6, w)$. One has $\theta_I(0.6, w) = 0.1387526518\ldots > 0.018$, so for $q \in (0, 0.6]$, θ has no zeros for $t \in [0, w]$. □

Proof of Lemma 15. For $q \in [0.75, 1)$ fixed and for $t \in [0, w]$, the quantity

$$M_1 := (1-q)|(1+qx)(1+q/x)||_{x=-t+wi} \tag{8}$$

is maximal for $t = 0$, see Lemma 9. The quantity $M_1|_{t=0}$ is maximal for $q = 0.75$, see Remark 10. In this case

$$\prod_{m=1}^{\infty} M_1(q^m, t) \leq \prod_{m=1}^{\infty} M_1(0.75^m, 0) < \prod_{m=1}^{40} M_1(0.75^m, 0) = 0.1103687051\ldots =: h_1,$$

$$|1+1/x| = |1+1/(-t+wi)| = |(1-t)+wi|/|-t+wi| =$$
$$= f(t) := (((1-t)^2+w^2)/(t^2+w^2))^{1/2} \leq (11/9)^{1/2} = 1.105541597\ldots =: h_2$$

and

$$|\Theta^*| \leq h_1 h_2 = 0.1220171945\ldots < 0.123,$$

see (3). (It is easy to show that the function f is decreasing on $[0, w]$, so $f(t) \leq f(0) = (11/9)^{1/2}$.) On the other hand Lemma 7 implies

$$|\theta| \geq |G| - |\Theta^*| \geq |G_5| - |G_*| - |\Theta^*| \geq 0.147 - 0.0208 - 0.123 = 0.0032 > 0.$$

□

Proof of Lemma 16. For $q \in [0.6, 0.75]$ fixed and for $t \in [1, w]$, the quantity M_1 (see (8)) is maximal for $t = 1$ (see Lemma 9). The quantity $M_1|_{t=1}$ is maximal for $q = 0.6$, see Remark 10. Therefore, it is true that

$$\prod_{m=1}^{\infty} M_1(q^m, t) \leq \prod_{m=1}^{\infty} M_1(0.6^m, 1) < \prod_{m=1}^{40} M_1(0.6^m, 1) = 0.1048026086\ldots =: h_3.$$

The function f defined in the proof of Lemma 15 is decreasing and takes its maximal value $h_4 := (9/11)^{1/2}$ for $t = 1$. Thus,

$$|\Theta^*| \leq h_3 h_4 = 0.09479752467\ldots < 0.095.$$

Using Lemma 7, one deduces that

$$|\theta| \geq |G_5| - |G_*| - |\theta^*| \geq 0.147 - 0.0208 - 0.095 = 0.0312 > 0.$$

□

Proof of Lemma 17. We set

$$\theta(q, x) = \theta_7^{\bullet}(q, x) + \theta_*^{\bullet}(q, x),$$

where

$$\theta_7^{\bullet} := \sum_{j=0}^{7} q^{j(j+1)/2} x^j, \qquad \theta_*^{\bullet} := \sum_{j=8}^{\infty} q^{j(j+1)/2} x^j.$$

The maximal possible modulus $|x|$ for $t \in [1, w]$ is obtained for $t = 1$. This gives $x = -1 + 3i/\sqrt{2}$ and in this case $|x| = 2.345207880\ldots < 2.346$. This means that

$$|\theta_*^{\bullet}| \leq \sum_{j=8}^{\infty} |q|^{j(j+1)/2} |x|^j \leq \sum_{j=8}^{\infty} 0.75^{j(j+1)/2} 2.346^j < 0.036.$$

On the other hand when setting $x := -t + iw$, $t \in [0, 1]$, and $T_I := \mathrm{Im}(\theta_7^\bullet(q, -t + iw))$, one obtains that

$$T_I = (3\sqrt{2}q/16)(56q^{27}t^6 - 1260q^{27}t^4 + 3402q^{27}t^2 - 729q^{27} - 48q^{20}t^5 + 720q^{20}t^3 - 972q^{20}t+$$
$$+40q^{14}t^4 - 360q^{14}t^2 + 162q^{14} - 32q^9t^3 + 144q^9t + 24q^5t^2 - 36q^5 - 16q^2t + 8).$$

We set $T_{I,k} := \partial^k T_I/\partial t^k|_{t=0}$. These functions are equal respectively to

$$T_{I,0} = -(3\sqrt{2}/16)q(81q^{18} + 18q^9 - 18q^5 + 4)(9q^9 - 2), \quad T_{I,4} = -90\sqrt{2}q^{15}(63q^{13} - 2),$$
$$T_{I,1} = -(3\sqrt{2}/4)q^3(243q^{18} - 36q^7 + 4), \quad T_{I,2} = (9\sqrt{2}/4)q^6(567q^{22} - 60q^9 + 4),$$
$$T_{I,3} = 18\sqrt{2}q^{10}(45q^{11} - 2), \quad T_{I,5} = -1080\sqrt{2}q^{21}, \quad T_{I,6} = 7560\sqrt{2}q^{28}.$$

These derivatives do not change sign on the interval $[0.6, 0.75]$, with $\mathrm{sgn}(T_{I,k}) = (-1)^k$. The Taylor series of T_I at $t = 0$ reads

$$T_I = (T_{I,0} + tT_{I,1}) + (t^2/2)(T_{I,2} + tT_{I,3}/3) + (t^4/24)(T_{I,4} + tT_{I,5}/5) + T_{I,6}t^6/6!.$$

One verifies directly that the following inequalities hold true:

$$T_{I,0} > |T_{I,1}| + 0.02, \quad T_{I,2} > |T_{I,3}|/3 \quad \text{and} \quad T_{I,4} > |T_{I,5}|/5.$$

Hence the Taylor series of T_I takes only values > 0.02 for $t \in [0, 1]$. We need, however, a stronger result. It is to be checked directly that:
(i) $T_{I,0} - |T_{I,1}|/2 > 0.2$, so for $t \in [0, 1/2]$, one has $T_I > 0.2$, $\theta_7^\bullet > 0.2 > 0.038 > |\theta_*^\bullet|$ and thus $|\theta| > 0$;
(ii) $T_{I,2}/2 - |T_{I,3}|/6 > 0.1$, so for $t \in [1/2, 1]$, one has

$$T_I > (T_{I,0} - |T_{I,1}|) + (1/4)(T_{I,2}/2 - |T_{I,3}|/6) > 0.02 + 0.025 > 0.038$$

and again $|\theta| \geq \theta_7^\bullet - |\theta_*^\bullet| > 0$. □

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Received 07.11.2022
Revised 26.12.2022