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NOTE ON BOUNDEDNESS OF THE L-INDEX IN THE DIRECTION OF THE COMPOSITION OF SLICE ENTIRE FUNCTIONS

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We study a composition of two functions belonging to a class of slice holomorphic functions in the whole n-dimensional complex space. The slice holomorphy in the space means that for some fixed direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and for every point $z^0\in\mathbb{C}^n$ the function is holomorphic on its restriction on the slice $\{z^0+t\mathbf{b}\colon t\in\mathbb{C}\}$. An additional assumption on joint continuity for these functions allows to construct an analog of theory of entire functions having bounded index. The analog is applicable to study properties of slice holomorphic solutions of directional differential equations, to describe local behavior and value distribution. In particular, we find conditions providing boundedness of L-index in the direction $\mathbf{b}$ for a function $f(\underbrace{\Phi(z),\ldots,\Phi(z)}_{m\text{ times}})$, where $f\colon\mathbb{C}^n\to\mathbb{C}$, $\Phi\colon\mathbb{C}^n\to\mathbb{C}$ are a slice entire functions, $L\colon\mathbb{C}^n\to\mathbb{R}_+$ is a continuous function. The obtained results are also new in one-dimensional case, i.e. for $n=1$, $m=1$. They are deduced using new approach in this area analog of logarithmic criterion. For a class of nonvanishing outer functions in the composition the sufficient conditions obtained by logarithmic criterion are weaker than the conditions by the Hayman theorem.

1. Notations and definitions. Let us introduce some notations from [1] (see also [10, 11]). Let $\mathbb{R}_+=(0,+\infty)$, $\mathbb{R}_+^*=[0,+\infty)$, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{1}=(1,\ldots,1)$, and $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a given direction, $\mathbf{1}_j=(0,\ldots,0,\underbrace{1}_{j\text{-th place}},0,\ldots,0)\in\mathbb{C}^n$, $L\colon\mathbb{C}^n\to\mathbb{R}_+$ be a continuous function. The slice functions for fixed $z^0\in\mathbb{C}^n$ we will denote as $g_{z^0}(t)=F(z^0+t\mathbf{b})$ and $l_{z^0}(t)=L(z^0+t\mathbf{b})$ for $t\in\mathbb{C}$.

Let $\widetilde{\mathcal{H}}^n_{\mathbf{b}}$ be the class of functions which are holomorphic on every slices $\{z^0+t\mathbf{b}\colon t\in S_{z^0}\}$ for each $z^0\in\mathbb{C}^n$ and let $\mathcal{H}^n_{\mathbf{b}}$ be the class of functions from $\widetilde{\mathcal{H}}^n_{\mathbf{b}}$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for every $p\in\mathbb{N}$, $\partial^p_{\mathbf{b}}F(z)=g_z^{(p)}(0)$, where $g_z(t)=F(z+t\mathbf{b})$ is an analytic function of complex variable $t\in S_z$ for given $z\in\mathbb{C}^n$.

Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction do not imply holomorphy in whole n-dimensional unit ball. There were presented some examples to demonstrate it [2].

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A function $F \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ is said [2] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{C}^n$ inequality

$$\frac{|\partial^m_{\mathbf{b}} F(z)|}{m! L^m(z)} \leq \max_{0 \leq k \leq m_0} \frac{|\partial^k_{\mathbf{b}} F(z)|}{k! L^k(z)} \tag{1}$$

is true. The least such integer number m_0, obeying (1), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F, L, \mathbb{C}^n)$. For $n = 1$, $\mathbf{b} = 1$, $L(z) = l(z)$, $z \in \mathbb{C}$ inequality (1) defines a function of bounded l-index with the l-index $N(F, l) \equiv N_1(F, l, \mathbb{C})$ [19], and if in addition $l(z) \equiv 1$, then we obtain a definition of index boundedness with index $N(F) \equiv N_1(F, 1, \mathbb{C})$ [20, 21]. Similarly, an entire function $F \colon \mathbb{C}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if it satisfies (1) for all $z \in \mathbb{C}^n$.

We denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z \in \mathbb{C}^n} \sup_{t_1, t_2 \in \mathbb{C}} \left\{ \frac{L(z + t_1\mathbf{b})}{L(z + t_2\mathbf{b})} \colon |t_1 - t_2| \leq \frac{\eta}{\min\{L(z + t_1\mathbf{b}), L(z + t_2\mathbf{b})\}} \right\}.$$

The notation $Q^n_{\mathbf{b}}$ stands for a class of positive continuous functions $L \colon \mathbb{C}^n \to \mathbb{R}_+$, satisfying for every $\eta \in [0, \beta]$

$$\lambda_{\mathbf{b}}(\eta) < +\infty. \tag{2}$$

This class $Q^n_{\mathbf{b}}$ is an auxiliary class to study entire functions and slice entire functions having bounded L-index in the direction.

As in [28], $Q \equiv Q^1_1$ and $\lambda(\eta) \equiv \lambda_1(\eta)$ in the cases when $\mathbf{b} = 1$, $n = 1$, $L \equiv l$, where $l \in \mathbb{C} \to \mathbb{R}_+$ is a continuous functon.

2. Introduction and formulation of the problem. Despite many studies on composition of entire functions and slice entire functions and boundedness of L-index in direction and boundedness of $\mathbf{L}$-index in joint variables [4, 5, 9, 10, 19, 28], this topic is still interested because it allows to discover new properties of functions having bounded index and apply it to differential equations and value distribution theory. In particular, if an entire function has a bounded index in some sense, then multiplicities of its zeros are uniformly bounded. On the other hand, every entire function with uniformly bounded multiplicities of zeros has the bounded index in the same sense [8, 13]. However, in some cases all known results on composition (for example, see below Theorems 1 and 2) are not applicable even if we consider two entire functions of one complex variable with bounded multiplicities of zeros.

Among many papers on composition of entire functions and index boundedness [5, 9, 10, 19, 28] we should like to mention paper [4] because it is also devoted to slice entire functions. There was investigated some composition of slice entire functions by usage the analog of Hayman's Theorem for this class of function (see below Theorem 5).

Let $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ be a function, satisfying

$$|\partial^j_{\mathbf{b}} \Phi(z)| \leq K |\partial_{\mathbf{b}} \Phi(z)|^j, \quad K \equiv \text{const} > 0, \tag{3}$$

for all $z \in \mathbb{C}^n$ and for all $j \leq p$, where p is some positive integer number.

Theorem 1 ([4])**.** *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $f \in \widetilde{\mathcal{H}}_{\mathbf{1}}(\mathbb{C}^m)$, $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ be a function such that $\partial_{\mathbf{b}}\Phi(z) \neq 0$ for all $z \in \mathbb{C}^n$. Suppose that $l \in Q^m_{\mathbf{1}}$, $l(w) \geq 1$ $(w \in \mathbb{C}^m)$, $L \in Q_{\mathbf{b}}(\mathbb{C}^n)$, $L(z) = |\partial_{\mathbf{b}}\Phi(z)| l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$.*

If the function f has bounded l-index in the direction $\mathbf{1}$ *and the function* Φ *satisfies* (3) *with* $p = N_{\mathbf{1}}(f, l, \mathbb{C}^m)$, *then the function* $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ *has bounded L-index in the direction* $\mathbf{b}$.

And if the function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ *has bounded L-index in the direction* $\mathbf{b}$ *and the function* Φ *satisfies* (3) *with* $p = N_{\mathbf{b}}(F, L, \mathbb{C}^n)$, *then the function f has bounded l-index in the direction* $\mathbf{1}$.

Theorem 2 ([4])**.** *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $l \in Q^m_{\mathbf{1}}$, $l(w) \geq 1$ $(w \in \mathbb{C}^m)$, $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $f \in \widetilde{\mathcal{H}}_{\mathbf{1}}(\mathbb{C}^m)$ *be a function of bounded l-index in the direction* $\mathbf{1}$.

Suppose that $L \in Q_{\mathbf{b}}(\mathbb{C}^n)$ *with*

$$L(z) = \max\{1, |\partial_{\mathbf{b}}\Phi(z)|\}\, l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}). \tag{4}$$

and for all $z \in \mathbb{C}^n$ *and* $k \in \{1, 2, \ldots, N_{\mathbf{1}}(f, l) + 1\}$ *one has*

$$|\partial^k_{\mathbf{b}}\Phi(z)| \leq K(l(\Phi(z)))^{1/(N_{\mathbf{1}}(f,l)+1)}|\partial_{\mathbf{b}}\Phi(z)|^k, \tag{5}$$

where $K \geq 1$ *is a constant. Then the function* $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ *has bounded L-index in the direction* $\mathbf{b}$.

Let us consider the case $n = 1$, $m = 1$, $\mathbf{b} = 1$, i.e. for simplicity, we assume that the outer and the inner functions of the composition are entire functions. If we consider the functions $f(z) = e^{2z}$ and $\Phi(z) = z^3/3 + z$ then the function Φ also does not satisfy assumption of Theorem 1 because it vanishes at the point $z = 0$. Moreover, these functions do not satisfy conditions of Theorem 2. Indeed, the function f has bounded index with $l \equiv 1$ and its index equals 1 because $\frac{|f^{(p)}(z)|}{p!} = 2^p|e^{2z}|/p! \leq 2|e^{2z}| = \frac{|f'(z)|}{1!}$ for all $p \in \mathbb{N}$. Hence, for all $k \in \{1, 2\}$ and for all z the inequality $|\Phi^{(k)}(z)| \leq K|\Phi'(z)|^k$ must be satisfied. But for $k = 2$ one has $\Phi'(z) = z^2 + 1$, $\Phi''(z) = 2z$. Then $|\Phi''(i)| = 2$, $\Phi'(i) = 0$. This means that inequality (5) is false for these function, i.e. Theorem 2 is not applicable. Another example is $f(z) = z \cdot e^z$, $\Phi(z) = z^2$. As above, it can be proved that these functions do not obey conditions of Theorem 1, Theorem 2.

These examples lead to the following problem: *"to deduce sufficient conditions of index boundedness for holomorphic functions which are applicable to a wider class of functions (in particular, for our examples)."*

All known papers on composition in theory of functions having bounded index are devoted application of Hayman theorem analogs [18]. This theorem and its analogs for various classes of holomorphic function are very convenient to establish conditions providing index boundedness for the functions. Also, it is applicable to analytic solutions of differential equations and their systems. Hayman's theorem shows that we can prove validity of corresponding inequalities without factorials (compare inequalities (1) and (8)). Here we continue our investigations initialized in [1–3], i.e. our main object are slice holomorphic functions of bounded L-index in a direction. For these functions we examine their composition when the inner function is slice holomorphic in the unit ball and the outer function is slice holomorphic in whole n-dimensional complex space.

For such a composition we apply two theorem to establish sufficient conditions providing index boundedness in direction. They are Hayman's theorem and logarithmic criterion. The criterion describe behavior of the modulus of the logarithmic derivative outside some exceptional set and distrubution of zeros. These two theorems play important role in analytic theory of differential equations. They helped to find sufficient conditions by the coefficients of equations providing index boundedness of every analytic solution. For the composition we do not know results obtained by the logarithmic criterion (see [5,9,10,19,28]). In other words, the present paper is the first paper on application of logarithmic criterion to composition of holomorphic functions in the theory of bounded index. Moreover, we do not limit ourselves to establishing some new sufficient conditions by the approach. We give a qualitative characteristic of the conditions. In particular, we select a subclass of outer nonvanishing functions in the composition for which the conditions obtained by new approach (the logarithmic criterion) are weaker than the conditions obtained by old approach (the Hayman theorem). Therefore, our main goal in the paper is following: *to find weaker sufficient conditions of boundedness of L-index in direction for composition of two slice holomorphic functions with usage of new approach.*

Indeed, we will obtain much more than some analog of known results. Proposition 1 and Proposition 3 have no analogs in theory of bounded index even for composition of entire functions. Moreover, the conditions in Proposition 1 obtained by the logarithmic criterion are *significantly weaker* than in statements obtained by Hayman's theorem (Theorem 1 and Theorem 2) if an outer function of the composition does not vanish (see below Remark 1).

Proposition 3 firstly uses a connection between functions of bounded value distribution and functions of bounded index. W. Hayman [18] proved that an entire function has bounded value distribution if and only if its derivative has bounded index. Despite this fact we do no know other results on functions of bounded index with the usage the notion of bounded value distribution. Proposition 3 is the first result of such a type.

3. Auxiliary propositions We will use an analog of logarithmic criterion for function from the class $\widetilde{\mathcal{H}}^n_{\mathbf{b}}$. The one-dimensional analog of the criterion is efficient to investigate boundedness of l-index of infinite products [12, 29, 30]. As necessary conditions the criterion was obtained by G. H. Fricke [14, 15] for entire functions of one complex variable having bounded index.

Denote

$$G^{\mathbf{b}}_r(F) = G^{\mathbf{b}}_r(F; L) := \bigcup_{z\in\mathbb{C}^n\colon F(z)=0} \{z + t\mathbf{b}\colon |t| < r/L(z)\}.$$

If $n = 1$ and $\mathbf{b} = 1$ then we use the simplified notation $G_r(F; L) = G^{\mathbf{1}}_r(F; L)$ and $G_r(F) = G^{\mathbf{1}}_r(F; 1)$ (for $L \equiv 1$). By $n_{z^0}(r) = n_{\mathbf{b}}\big(r, z^0, 1/F\big) := \sum_{|a^0_k|\le r} 1$ we denote counting function of zeros a^0_k for the slice function $F(z^0 + t\mathbf{b})$ in the disc $\{t \in \mathbb{C}\colon |t| \le r\}$ for given $z^0 \in \mathbb{C}^n$. If for given $z^0 \in \mathbb{C}^n$ and for all $t \in S_z\colon \quad F(z^0 + t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z\in\mathbb{C}^n} n_z(r/L(z))$.

Theorem 3 ([3])**.** *Let $F \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $L \in Q^n_{\mathbf{b}}$. If the function F has bounded L-index in the direction $\mathbf{b}$, then*

1) for every $r > 0$ there exists $P = P(r) > 0$ that for each $z \in \mathbb{C}^n\setminus G^{\mathbf{b}}_r(F)$

$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le PL(z); \tag{6}$$

2) for every $r > 0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for each $z^0 \in \mathbb{C}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$

$$n_{\mathbf{b}}\left(r/L(z^0), z^0, 1/F\right) \le \widetilde{n}(r). \tag{7}$$

Theorem 4 ([3])**.** *Let $L \in Q^n_{\mathbf{b}}$, $F \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $\mathbb{C}^n \setminus G^{\mathbf{b}}_{\beta}(F) \ne \varnothing$. If the following conditions are satisfied*

1) *there exists $r_1 > 0$ such that $n(r_1) \in [-1; \infty)$;*
2) *there exist $r_2 > 0$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{C}^n \setminus G_{r_2}(F)$ inequality (6) is true;*

then the function F has bounded L-index in the direction $\mathbf{b}$.

One should observe that the sufficient conditions in Theorem 4 are weaker than the necessary conditions in Theorem 3. They are differed with existential and universal quantifiers, respectively. In other words, from validity of inequalities (6) and (7) for some radius r it follows their validity for all possible values of r belonging some interval.

Below we formulate a criterion that is analogous to Hayman's theorem [18] obtained for entire functions of single complex variable.

Theorem 5 ([3])**.** *Let $L \in Q_{\mathbf{b}}(\mathbb{C}^n)$. A function $F \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for every $z \in \mathbb{C}^n$ one has*

$$\frac{|\partial^{p+1}_{\mathbf{b}} F(z)|}{L^{p+1}(z)} \le C \max\left\{\frac{|\partial^{k}_{\mathbf{b}} F(z)|}{L^{k}(z)} : 0 \le k \le p\right\}. \tag{8}$$

Among many papers on composition of entire functions and index boundedness [5, 9, 10, 19, 28] we should like to mention paper [4] because it is also devoted slice entire functions. There was investigated some composition of slice entire functions by usage the analog of Hayman's Theorem for this class of function, i.e. Theorem 5.

4. Application of logarithmic criterion to composition In the theory of bounded index Hayman's theorem and the logarithmic criterion are most applicable in other problems. Indeed, there are many papers on their applications to study properties of analytic solutions of ordinary and partial differential equations and their systems. Moreover, Hayman's theorem also helps to deduce some conditions providing bounded L-index in direction for composition of entire and analytic functions. But we do not know results on composition obtained with application of logarithmic criterion. Here we will present such results. In this section we suppose that $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m$.

Proposition 1. *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $f \in \widetilde{\mathcal{H}}^m_{\mathbf{1}}$ be a function such that $f(w) \ne 0$ for all $w \in \mathbb{C}^m$.*

1) Suppose that $l \in Q^m_{\mathbf{1}}$, $L \in Q^n_{\mathbf{b}}$ and $L(z) \ge \left|\partial_{\mathbf{b}}\Phi(z)\right| l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ for all $z \in \mathbb{C}^n$. If the function f has bounded l-index in the direction $\mathbf{1}$, then the function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in the direction $\mathbf{b}$.

2) Suppose that $L \in Q^n_{\mathbf{b}}$, $\partial_{\mathbf{b}}\Phi(z) \ne 0$ and $l \in Q^m_{\mathbf{1}}$ be such that $l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}) \ge \frac{L(z)}{\left|\partial_{\mathbf{b}}\Phi(z)\right|}$ for all $z \in \mathbb{C}^n$. And if the function $F(z) = f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ has bounded L-index in the direction $\mathbf{b}$, then the function f has bounded l-index in the direction $\mathbf{1}$.

Proof. One should observe that

$$\partial_{\mathbf{b}}F(z) = \partial_{\mathbf{1}} f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}) \cdot \partial_{\mathbf{b}}\Phi(z). \tag{9}$$

Since $f(w) \neq 0$ for all $w \in \mathbb{C}^m$, the composite function $f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})$ does not vanish for all $z \in \mathbb{C}^n$ that is $G_r^{\mathbf{b}}(F) = \varnothing$. Therefore, we must check only condition 2) from Theorem 4. It is sufficient to prove inequality (6) for all $z \in \mathbb{C}^n$. Indeed, in view of (9) one has

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| = \left|\frac{\partial_{\mathbf{1}} f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})}{f(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}})}\right| \cdot |\partial_{\mathbf{b}}\Phi(z)| \tag{10}$$

Let f be of bounded l-index in the direction $\mathbf{1}$. Then by Theorem 3 for the class $\widetilde{\mathcal{H}}_{\mathbf{1}}^m$ inequality (6) is valid for the function f and for all $w \in \mathbb{C}^m$:

$$\frac{|\partial_{\mathbf{1}} f(w)|}{|f(w)|} \leq Pl(w) \tag{11}$$

Substituting $w = (\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}$ in (11) and applying this inequality to (10) we obtain

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| = Pl(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}) \cdot |\partial_{\mathbf{b}}\Phi(z)| \leq PL(z). \tag{12}$$

Remind that the function F also does not vanish as the function f. Therefore, inequality (12) yields that by Theorem 3 the function F has bounded L-index in the direction $\mathbf{b}$. The first part of Proposition 1 is proved.

The second part of the proposition can be proved by analogy to the first part. □

It is possible to consider more general composition of slice entire functions. But the conditions will be also stronger.

Proposition 2. *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $\Phi_j \in \widetilde{\mathcal{H}}_{\mathbf{b}}^n$, $j \in \{1, \ldots, m\}$, $f : \mathbb{C}^m \to \mathbb{C}$ be an entire function such that $f(w) \neq 0$ for all $w \in \mathbb{C}^m$. Suppose that $l \in Q_{\mathbf{1}_j}^m$ for each $j \in \{1, 2, \ldots, m\}$, $L \in Q_{\mathbf{b}}^n$ and*

$$L(z) \geq \sum_{j=1}^{m} \big|\partial_{\mathbf{b}}\Phi_j(z)\big| l(\Phi_1(z), \ldots, \Phi_m(z))$$

for all $z \in \mathbb{C}^n$. If the function f has bounded l-index in the each direction $\mathbf{1}_j$, $j \in \{1, \ldots, m\}$, then the function $F(z) = f(\Phi_1(z), \ldots, \Phi_m(z))$ has bounded L-index in the direction $\mathbf{b}$.

Proof. One should observe that

$$\partial_{\mathbf{b}}F(z) = \sum_{j=1}^{m} f'_{\Phi_j}(\Phi_1(z), \ldots, \Phi_m(z))\partial_{\mathbf{b}}\Phi_j(z). \tag{13}$$

Since $f(w)\neq 0$ for all $w\in\mathbb{C}^m$, the composite function $f(\Phi_1(z),\ldots,\Phi_m(z))$ does not vanish for all $z\in\mathbb{C}^n$ that is $G_r^{\mathbf{b}}(F)=\varnothing$. Therefore, we must check only condition 2) from Theorem 4. It is sufficient to prove inequality (6) for all $z\in\mathbb{C}^n$. Indeed, in view of (13) one has

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right|\leq\sum_{j=1}^{m}\left|\frac{f'_{\Phi_j}(\Phi_1(z),\ldots,\Phi_m(z))}{f(\Phi_1(z),\ldots,\Phi_m(z))}\right|\cdot|\partial_{\mathbf{b}}\Phi_j(z)| \tag{14}$$

Let f be of bounded l-index in the each direction $\mathbf{1}_j$. Then by analog of Theorem 3 for the class of entire functions of m complex variables (see [6]) inequality (6) is valid for the function f and for all $w\in\mathbb{C}^m$:

$$\frac{|\partial_{\mathbf{1}_j}f(w)|}{|f(w)|}\leq Pl(w) \tag{15}$$

Substituting $w=(\Phi_1(z),\ldots,\Phi_m(z))$ in (15) and applying this inequality to (14) we obtain

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right|=Pl(\Phi_1(z),\ldots,\Phi_m(z))\cdot\sum_{j=1}^{m}|\partial_{\mathbf{b}}\Phi_j(z)|\leq PL(z). \tag{16}$$

Remind that the function F also does not vanish as the function f. Therefore, inequality (16) yields that by Theorem 3 the function F has bounded L-index in the direction $\mathbf{b}$. $\square$

The condition $f(w)\neq 0$ can be replaced by some assumption on the function Φ.

Let us remind the definition of function having bounded value L-distribution in a direction.

Function $F\in\widetilde{\mathcal{H}}^n_{\mathbf{b}}$ is said [3] to be of bounded value L-distribution in a direction $\mathbf{b}$ if for some $p\in\mathbb{N}$ and for every $w\in\mathbb{C}$, $z_0\in\mathbb{C}^n$ such that $F(z^0+t\mathbf{b})\not\equiv w$, the inequality holds

$$n\Big(\frac{1}{L(z^0)},z^0,\frac{1}{F-w}\Big)\leq p,$$

i.e. the equation $F(z^0+t\mathbf{b})=w$ has at most p solutions in the disc $\{t\colon\ |t|\leq\frac{1}{L(z^0)}\}$. In other words, the function $F(z^0+t\mathbf{b})$ is p-valent in $\{t\colon\ |t|\leq\frac{1}{L(z^0)}\}$ for each fixed $z^0\in\mathbb{C}^n$. If $n=1$, $\mathbf{b}=1$ and $L\equiv 1$ then we obtain a definition of function of bounded value distribution [16, 17, 23–25]. Another approach to multivalence of bivariate function is considered in [22].

The following statement holds.

Proposition 3. *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, $\Phi\in\widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $f\colon\mathbb{C}\to\mathbb{C}$ be an entire function and $F(z)=f(\Phi(z))$.*

Suppose that $l\in Q$, $L\in Q^n_{\mathbf{b}}$ and $L(z)\geq\big|\partial_{\mathbf{b}}\Phi(z)\big|l(\Phi(z))$ for all $z\in\mathbb{C}^n$. If the following conditions are satisfied:

1) the function f has bounded l-index;

2) the function Φ has bounded value L-distribution in the direction $\mathbf{b}$,

3) for every $r_1>0$ there exists $r_2>0$ and $r_3>0$ such that

$$G_{r_2}(f;l)\subset\Phi(G^{\mathbf{b}}_{r_1}(F;L))\subset G_{r_3}(f;l),$$

then the function F has bounded L-index in the direction $\mathbf{b}$.

Proof. Taking into account the condition 3) inequality (12) can be proved by analogy to the proof of Proposition 1.

One should observe that

$$\partial_{\mathbf{b}}F(z) = f'(\Phi(z)) \cdot \partial_{\mathbf{b}}\Phi(z). \tag{17}$$

We choose $r_1 > 0$. Then in view of condition 3) there exists $r_2 > 0$ and $r_3 > 0$ such that $G_{r_2}(f;l) \subset \Phi(G^{\mathbf{b}}_{r_1}(F;L)) \subset G_{r_3}(f;l)$. Then $f(w) \neq 0$ for all $w \in \mathbb{C} \setminus G_{r_2}(f;l)$. Hence, the composite function $f(\Phi(z))$ does not vanish for all $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_{r_1}(F;L)$. Therefore, we must check only condition 2) from Theorem 4 for $r = r_1$. It is sufficient to prove inequality (6) for all $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_{r_1}(F;L)$. Indeed, in view of (17) one has

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| = \left|\frac{f'(\Phi(z))}{f(\Phi(z))}\right| \cdot |\partial_{\mathbf{b}}\Phi(z)| \tag{18}$$

Let f be of bounded l-index. Then by Theorem 3 for entire functions of single variable (see also [6, 27]) inequality (6) is valid for the function f and for all $w \in \mathbb{C} \setminus G_{r_2}(f;l)$ we obtain

$$\frac{|\partial_{\mathbf{1}}f(w)|}{|f(w)|} \leq P(r_2)l(w) \tag{19}$$

Substituting $w = \Phi(z)$ in (19) and applying this inequality to (18) we conclude that for all $z \in \mathbb{C}^n \setminus G^{\mathbf{b}}_{r_1}(F;L)$

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| = P(r_2)l(\underbrace{\Phi(z), \ldots, \Phi(z)}_{m \text{ times}}) \cdot |\partial_{\mathbf{b}}\Phi(z)| \leq P(r_2)L(z). \tag{20}$$

Inequality (7) holds for the function F because the equation $F(z^0 + t\mathbf{b}) = 0$ is equivalent the equation $\Phi(z^0 + t\mathbf{b}) = c_k$, where c_k are zeros of the function f, $k \in \mathbb{N}$. In view of condition 2), the equation $\Phi(z^0 + t\mathbf{b}) = c_k$ has at most $p(r_1)$ solutions for fixed k at the disc $\{t\colon\ |t| \leq \frac{r_1}{L(z^0)}\}$ for each $r_1 \in (0;\infty)$. In view of condition 3) the set $\{\Phi(z^0 + t\mathbf{b})\colon |t| \leq \frac{r_1}{L(z^0)}\}$ can contain at most $n(r_3)$ zeros of the function f. Therefore, the set $\{z^0 + t\mathbf{b}\colon\ |t| \leq \frac{r_1}{L(z^0)}\}$ has at most $p(r_1) \cdot n(r_3)$ zeros of the function F. Hence, it follows inequality (7). Then by Theorem 4 the function F has bounded L-index in the direction $\mathbf{b}$. □

Remark 1. Proposition 1, Proposition 2 and Proposition 3 contain new results even in one-dimensional case, i.e. for $n = 1$ and $m = 1$. Moreover, the restrictions by the inner function of the composition in Proposition 1 are significantly weaker than in Theorem 1 and Theorem 2. We do not require validity of (3) and (5). But the weakening is achieved by an additional assumption that the outer function of the composition does not vanish (Proposition 1). In Proposition 3, inequalities (3) and (5) for the inner function are replaced by the condition that inner function of the composition has bounded value L-distribution in the direction $\mathbf{b}$ and exceptional set of the resulting function containing its zeros is covered by corresponding exceptional set of the outer function for some radii and vice versa.

One should observe that Proposition 1 and Proposition 3 are new even if $n = 1$, $m = 1$, $\mathbf{b} = 1$ and $l \equiv 1$, i.e. in the case of entire functions of bounded index. Below we formulate the corresponding corollaries for a composition of entire functions.

Corollary 1. *Let Φ, $f\colon \mathbb{C}\to\mathbb{C}$ be entire functions.*

1) Suppose that $l\in Q$ and $l(z)\ge|\Phi'(z)|$ for all $z\in\mathbb{C}$. If the function f has bounded index, then the function $F(z)=f(\Phi(z))$ has bounded l-index.

2) Suppose that $l\in Q$, $\Phi'(z)\ne 0$ be such that $l(z)\le|\Phi'(z)|$ for all $z\in\mathbb{D}$. And if the function $F(z)=f(\Phi(z))$ has bounded l-index, then the function f has bounded index.

Corollary 2. *Let Φ, $f\colon\mathbb{C}\to\mathbb{C}$ be entire functions and $F(z)=f(\Phi(z))$. Suppose that $l\in Q$ and $l(z)\ge|\Phi'(z)|$ for all $z\in\mathbb{C}$. If the following conditions are satisfied:*

1) the function f has bounded index;

2) the function Φ has bounded value l-distribution,

3) for every $r_1>0$ there exists $r_2>0$ and $r_3>0$ such that

$$G_{r_2}(f)\subset\Phi(G_{r_1}(F;l))\subset G_{r_3}(f),$$

then the function F has bounded l-index.

If we choose $\Phi(z)=az+b$ $(a,b,z\in\mathbb{C},a\ne0)$ then $\Phi'(z)=a$. Putting $l(z)=|a|$, we conclude that by Corollary 1 the function $F(az+b)$ has bounded l-index. But for $l_2(z)=\theta l_1(z)$ $(\theta\in\mathbb{C})$ the entire function f has bounded l_1-index if and only if f is of bounded l_2-index [26]. Therefore, the function $F(az+b)$ has bounded index (with $l\equiv1$) i.e. the linear replacement z by $az+b$ does no change index boundedness. An analogous fact is also valid for $\Phi(z)=a_1z_1+\ldots+a_nz_n+b$.

Similarly, we can check assumptions of Proposition 3 for $\Phi(z)=az+b$ $(a,b,z\in\mathbb{C},\ a\ne0)$. As above, we put $l(z)=|a|$. The function Φ has bounded value l-distribution, because for any $w\in\mathbb{C}$ and for any $z_0\in\mathbb{C}$ the equation $az+b=w$ has at most one solution $z=(w-b)/a$ lying in the disc $|z-z_0|\le r/|a|$, $r>0$. Since $|a|r_1/l(z)=r_1$, the condition 3) of the corollary is also satisfied because we can choose r_2 and r_3 such that $r_2<r_1<r_3$. Thus, by Corollary 2 the function $F(az+b)$ has bounded l-index and, as above, F is of bounded index (with $l\equiv1$) i.e. the linear replacement z by $az+b$ again does no change index boundedness.

Moreover, in view of Proposition 2 an affine trasformation also does not change index boundedness in any direction $\mathbf{b}$. Indeed, let

$$\Phi_j(z)=c_j+\textstyle\sum_{k=1}^n a_{j,k}z_k,\text{ where } a_{j,k},c_j\in\mathbb{C},\ j\in\{1,\ldots,m\}.$$

Suppose that a fuction f has bounded index in the each direction $\mathbf{1}_j$, $j\in\{1,\ldots,m\}$, i.e. $l\equiv1$. Then we construct the function

$$L(z)=\textstyle\sum_{j=1}^m|\partial_{\mathbf{b}}\Phi_j(z)|=\sum_{j=1}^m|\partial_{\mathbf{b}}(c_j+\sum_{k=1}^n a_{j,k}z_k)|=\sum_{j=1}^m|a_{j,k}b_k|.$$

Then by Proposition 2 the function

$$f(c_1+\textstyle\sum_{k=1}^n a_{1,k}z_k, c_2+\sum_{k=1}^n a_{2,k}z_k,\ldots,c_m+\sum_{k=1}^n a_{m,k}z_k)$$

has the bounded L-index in the direction $\mathbf{b}$. But the functions L is a constant function. It is known that a constant does not change index boundedness in a direction (see [3]). Thus, the function $f(c_1+\sum_{k=1}^n a_{1,k}z_k, c_2+\sum_{k=1}^n a_{2,k}z_k,\ldots,c_m+\sum_{k=1}^n a_{m,k}z_k)$ has bounded index in the direction $\mathbf{b}$.

Note that Theorem 1, Theorem 2, Proposition 1, Proposition 3 differ only conditions by the outer and inner function of the composition. But they claims that the composite function has bounded L-index in the direction $\mathbf{b}$ with the same functions.

$$L(z)=|\partial_{\mathbf{b}}\Phi(z)|\cdot l(\underbrace{\Phi(z),\ldots,\Phi(z)}_{m\text{ times}})\text{ or } L(z)=\max\{1,|\partial_{\mathbf{b}}\Phi(z)|\}\cdot l(\underbrace{\Phi(z),\ldots,\Phi(z)}_{m\text{ times}}).$$

But there exists functions which does not satisfy simultaneously conditions of these all assertions.

For example, we put

$$f(z) = e^z,\ \Phi(z) = z^2.$$

Then the function Φ does not satisfy Theorem 1 because it vanishes at the point $z = 0$. But these functions satisfy conditions of Theorem 2. Indeed, the function f has bounded index with $l \equiv 1$ and its index equals zero $N(f, 1) = 0$ because $f' \equiv f$. Hence, inequality (5) is valid for $k = 1$. It transforms in the following obvious inequality $|\Phi'(z)| \le K|\Phi'(z)|$. Therefore, by Theorem 2 the function $f(\Phi(z)) = e^{z^2}$ has bounded L-index with $L(z) = \max\{1, 2|z|\}$.

In the introduction we show that the functions $f(z) = e^{2z}$ and $\Phi(z) = z^3/3 + z$ does not satisfy conditions of Theorem 1 and Theorem 2. Then we can apply Proposition 1 and conclude the function $f(\Phi(z)) = e^{2(z^3/3+z)}$ has bounded L-index with $L(z) = |z|^2 + 1$.

Finally, the functions $f(z) = z \cdot e^z$, $\Phi(z) = z^2$ do not obey conditions of Theorem 1, Theorem 2 and Proposition 1. But they satisfy conditions of Proposition 3 because the index of the f equals to 1 with $l \equiv 1$ and the function Φ as a polynomial has bounded value L-distribution with $L(z) \ge 2|z|$ and for each $r_1 > 0$ we can choose $r_2 < r_1^2 < r_3$ such that

$$G_{r_2}(f) \subset \Phi(G_{r_1}(F; L)) \subset G_{r_3}(f).$$

Then by Proposition 3 the function $f(\Phi(z)) = z^2 e^{z^2}$ has bounded L-index.

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