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ON GENERALIZED HOMODERIVATIONS OF PRIME RINGS

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Let $\mathscr{A}$ be a ring with its center $\mathscr{Z}(\mathscr{A})$. An additive mapping $\xi\colon \mathscr{A}\to\mathscr{A}$ is called a homoderivation on $\mathscr{A}$ if

$$\forall\ a,b\in\mathscr{A}:\quad \xi(ab)=\xi(a)\xi(b)+\xi(a)b+a\xi(b).$$

An additive map $\psi\colon \mathscr{A}\to\mathscr{A}$ is called a generalized homoderivation with associated homoderivation ξ on $\mathscr{A}$ if

$$\forall\ a,b\in\mathscr{A}:\quad \psi(ab)=\psi(a)\psi(b)+\psi(a)b+a\xi(b).$$

This study examines whether a prime ring $\mathscr{A}$ with a generalized homoderivation ψ that fulfils specific algebraic identities is commutative. Precisely, we discuss the following identities:

$$\psi(a)\psi(b)+ab\in\mathscr{Z}(\mathscr{A}),\quad \psi(a)\psi(b)-ab\in\mathscr{Z}(\mathscr{A}),\quad \psi(a)\psi(b)+ab\in\mathscr{Z}(\mathscr{A}),$$
$$\psi(a)\psi(b)-ab\in\mathscr{Z}(\mathscr{A}),\quad \psi(ab)+ab\in\mathscr{Z}(\mathscr{A}),\quad \psi(ab)-ab\in\mathscr{Z}(\mathscr{A}),$$
$$\psi(ab)+ba\in\mathscr{Z}(\mathscr{A}),\quad \psi(ab)-ba\in\mathscr{Z}(\mathscr{A})\quad (\forall\ a,b\in\mathscr{A}).$$

Furthermore, examples are given to prove that the restrictions imposed on the hypothesis of the various theorems were not superfluous.

1. Introduction. Throughout, $\mathscr{A}$ denotes a ring with the center $\mathscr{Z}(\mathscr{A})$. A ring $\mathscr{A}$ is called prime if $a\mathscr{A}b=\{0\}$ $(\forall\ a,b\in\mathscr{A})$ implies $a=0$ or $b=0$. The non-zero central elements of a prime ring are not zero divisors. The symbol $[a,b]$ where $a,b\in\mathscr{A}$ stands for the commutator $ab-ba$. An additive map d from $\mathscr{A}$ into itself is called a derivation if $d(ab)=d(a)b+ad(b)$ $(\forall\ a,b\in\mathscr{A})$. An additive map $\mathscr{F}$ from $\mathscr{A}$ into itself is called a generalized derivation with the associated derivation d if $\mathscr{F}(ab)=\mathscr{F}(a)b+ad(b)$ $(\forall\ a,b\in\mathscr{A})$. An additive map $\mathscr{T}$ from $\mathscr{A}$ into itself is called left centralizer (or left multiplier) if $\mathscr{T}(ab)=\mathscr{T}(a)b$ $(\forall\ a,b\in\mathscr{A})$. El Sofy (2000) [4] introduced the concept of a homoderivations as an additive map ξ from $\mathscr{A}$ into itself such that $\xi(ab)=\xi(a)\xi(b)+\xi(a)b+a\xi(b)$ $(\forall\ a,b\in\mathscr{A})$. An example of such map is to let $\xi(a)=f(a)-a$ for all $(\forall\ a\in\mathscr{A})$, where f is an endomorphism on $\mathscr{A}$. It is clear that a homoderivation ξ is also a derivation if $\xi(a)\xi(b)=0$ $(\forall\ a,b\in\mathscr{A})$. In this case, $\xi(a)\mathscr{A}\xi(b)=0$ $(\forall\ a,b\in\mathscr{A})$. So, if $\mathscr{A}$ is a prime ring, then the only additive map which is both a derivation and a homoderivation is the zero map. An additive map ψ from $\mathscr{A}$ into itself is called a generalized homoderivation with associated homoderivation ξ if $\psi(ab)=\psi(a)\psi(b)+\psi(a)b+a\xi(b)$ $(\forall\ a,b\in\mathscr{A})$, denoted by (ψ,ξ). It is easy to see that every homoderivation is a generalized homoderivation, but the inverse, in general, is not true, for example, take $\psi(a)=a$ and $\xi(a)=-a$. It is clear that a generalized homoderivation ψ is also a generalized derivation if $\psi(a)\psi(b)=0$ $(\forall\ a,b\in\mathscr{A})$. So, if $\mathscr{A}$ is a prime ring, then the only additive map which is both a generalized derivation and a generalized homoderivation is a left centralizer (a left multiplier). Note that if z is a nonzero element in the center of $\mathscr{A}$ and (ψ,ξ) is a generalized homoderivation of $\mathscr{A}$, then $\psi(z)$ (also $\xi(z)$) is not necessary an

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element in the center of $\mathscr{A}$. Also, if ξ is a homoderivation of $\mathscr{A}$, then $-\xi$ is not necessary a homoderivation, (also for a generalized homoderivation). For example, let $\xi(a) = -a$ be a homoderivation of $\mathscr{A}$, but $-\xi(a)$ is not a homoderivation, since char$(\mathscr{A}) \neq 2$. If $\mathscr{S} \subseteq \mathscr{A}$, then a map $f\colon \mathscr{A} \to \mathscr{A}$ preserves on $\mathscr{S}$ if $f(\mathscr{S}) \subseteq \mathscr{S}$. If $f\colon \mathscr{A} \to \mathscr{A}$ preserves on $\mathscr{S}$ and there exists a positive integer $n(a) > 1$ such that $f^{n(a)} = 0$, the map f is said to be zero-power valued on $\mathscr{S}$, El Sofy [4].

El Sofy [4], Melaibari et al. [6], Alharfie et al. [1–3], and Rehman et al. [11] had shown that if $\mathscr{A}$ is a prime ring and $\xi\colon \mathscr{A} \to \mathscr{A}$ is a homoderivation of $\mathscr{A}$, then $\mathscr{A}$ is a commutative ring if $\mathscr{A}$ satisfies certain algebraic identities. A growing body of research has been done on homoderivation mappings in rings under different conditions; for example, see [8–10].

Motivated by these results, we prove some results regarding generalized homoderivations. To achieve our aim, we will use the following lemmas.

2. Preliminary results. The following basic commutator identities are useful in the sequel

$$(\forall a,b,c \in \mathscr{A})\colon \quad [ab,c] = a[b,c] + [a,c]b, \quad [a,bc] = b[a,c] + [a,b]c.$$

We begin our discussion with the following basic lemmas that will be frequently used in our results.

Lemma 1 ([5], Lemma 4)**.** *Let $\mathscr{A}$ be a prime ring. If z and az are in the center of $\mathscr{A}$. Then a is in the center of $\mathscr{A}$ or $z = 0$.*

Lemma 2 ([7], Lemma 2.5)**.** *Let $\mathscr{A}$ be a prime ring $\mathscr{A}$. If $[a,b] \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a,b \in \mathscr{A})$, then $\mathscr{A}$ is commutative.*

Lemma 3. *Let $\mathscr{A}$ be a prime ring. If $ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a,b \in \mathscr{A})$, then $\mathscr{A}$ is commutative.*

Proof. Suppose that $ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a,b \in \mathscr{A})$. Interchanging a and b, we get $ba \in \mathscr{Z}(\mathscr{A})$. Comparing two last relations, we obtain $[a,b] \in \mathscr{Z}(\mathscr{A})$. Therefore, applying Lemma 2, we find that $\mathscr{A}$ is commutative. □

3. The main results.

Theorem 1. *Let $\mathscr{A}$ be a prime ring. If (ψ,ξ) is a generalized homoderivation of $\mathscr{A}$ satisfying any of the conditions:*
(i) $(\forall a,b \in \mathscr{A})\colon\ \psi(a)\psi(b) - ab \in \mathscr{Z}(\mathscr{A})$, *(ii)* $(\forall a,b \in \mathscr{A})\colon\ \psi(a)\psi(b) + ab \in \mathscr{Z}(\mathscr{A})$.

Then $\mathscr{A}$ is commutative or any generalized homoderivation (ψ,ξ) is of the form $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall a \in \mathscr{A})$.

Proof. *(i)* If $\psi = 0$, then from the condition, we arrive at $ab \in \mathscr{Z}(\mathscr{A})$ $(\forall a,b \in \mathscr{A})$, and by Lemma 3, we obtain that $\mathscr{A}$ is commutative. So, from now on we will assume that $\psi \neq 0$.

Suppose that

$$(\forall a,b \in \mathscr{A})\colon \quad \psi(a)\psi(b) - ab \in \mathscr{Z}(\mathscr{A}). \tag{1}$$

Case (1). If $\mathscr{Z}(\mathscr{A}) = \{0\}$. Then condition (1) gives

$$(\forall a,b \in \mathscr{A})\colon \quad \psi(a)\psi(b) - ab = 0. \tag{2}$$

Substituting bt instead of b in (2), we obtain $\psi(a)\psi(bt) - abt = 0$ $(\forall\ a,b,t \in \mathscr{A})$, that is

$$(\forall a,b,t \in \mathscr{A})\colon \quad \psi(a)\psi(b)\psi(t) + \psi(a)\psi(b)t + \psi(a)b\xi(t) - abt = 0. \tag{3}$$

Taking b by t and a by b in (2), we infer that $\psi(b)\psi(t) = bt$. By using the previous expression in (3), we find that $\psi(a)bt + \psi(a)\psi(b)t + \psi(a)b\xi(t) - abt = 0$. This implies that $\psi(a)bt + \psi(a)b\xi(t) + (\psi(a)\psi(b) - ab)t = 0$. Using (2) in the above relation, we have $\psi(a)bt + \psi(a)b\xi(t) = 0$, that is $\psi(a)b(t + \xi(t)) = 0$. Hence $\psi(a)\mathscr{A}(t + \xi(t)) = \{0\}$ and by primeness of $\mathscr{A}$ we obtain so $\psi(a) = 0$ or $\xi(t) + t = 0$. In the case $\psi(a) = 0$, using the last expression in (2), we get $ab = 0$ ($\forall\ a, b \in \mathscr{A}$, and by applying Lemma 3, to get $\mathscr{A}$ is commutative. In the case $\xi(t)+t = 0$, i.e. $\xi(t) = -t$ ($\forall\ t \in \mathscr{A}$). Putting a by at in (2), we get $\psi(at)\psi(b) - atb = 0$ ($\forall\ a, b, t \in \mathscr{A}$), that is $\psi(a)\psi(t)\psi(b) + \psi(a)t\psi(b) + a\xi(t)\psi(a) - atb = 0$. By using the equality $\xi(t) = -t$ in the above expression, we have $\psi(a)\psi(t)\psi(b)+\psi(a)t\psi(b)-at\psi(b) - atb = 0$. This further implies $(\psi(a)\psi(t) - at)\psi(b) + \psi(a)t\psi(b) - atb = 0$. Take t in place of b in (2) and then using it in the above relation, we obtain

$$(\forall a, b, t \in \mathscr{A}): \quad \psi(a)t\psi(b) - atb = 0. \tag{4}$$

Writing $t\psi(b)$ instead of t in (4), we see that

$$(\forall a, b, t \in \mathscr{A}): \quad \psi(a)t\psi(b)^2 - at\psi(b)b = 0. \tag{5}$$

Right multiplying (4) by $\psi(b)$, we find that

$$(\forall a, b, t \in \mathscr{A}): \quad \psi(a)t\psi(b)^2 - atb\psi(b) = 0. \tag{6}$$

Comparing (5) and (6), we deduce $at[\psi(b), b] = 0$. Putting a by $[\psi(b), b]$ in the previous expression, to get $[\psi(b), b]t[\psi(b), b] = 0$, hence $[\psi(b), b]\mathscr{A}[\psi(b), b] = \{0\}$, and by primeness of $\mathscr{A}$, we obtain $[\psi(b), b] = 0$. By linearizing the last expression, we conclude that

$$(\forall a, b \in \mathscr{A}): \quad [\psi(a), b] + [\psi(b), a] = 0. \tag{7}$$

Substituting bt instead of b in (7), we get $[\psi(a), bt] + [\psi(bt), a] = 0\ \forall\ a, b, t \in \mathscr{A}$), that is

$$[\psi(a), b]t + b[\psi(a), t] + [\psi(b)\psi(t) + \psi(b)t + b\xi(t), a] = 0.$$

Application of the equality $\xi(t) = -t$ in the last relation yields

$$[\psi(a), b]t + b[\psi(a), t] + [\psi(b)\psi(t) + \psi(b)t - bt, a] = 0,$$

that is $[\psi(a), b]t+b[\psi(a), t]+[(\psi(b)\psi(t)-bt)+\psi(b)t, a] = 0$. By using (2) in the last relation, to get $[\psi(a), b]t + b[\psi(a), t] + [\psi(b)t, a] = 0$, we deduce that

$$[\psi(a), b]t + b[\psi(a), t] + \psi(b)[t, a] + [\psi(b), a]t = 0,$$

hence $([\psi(a), b] + [\psi(b), a])t + b[\psi(a), t] + \psi(b)[t, a] = 0$. Using (7) in the above relation, we obtain $b[\psi(a), t] + \psi(b)[t, a] = 0$ ($\forall a, b, t \in \mathscr{A}$). Having first written in this inequality rb instead of b and also left multiplying it by r, we get $rb[\psi(a), t] + \psi(rb)[t, a] = 0$ and $rb[\psi(a), t]+r\psi(b)[t, a] = 0$, respectively. Comparing this two equalities, we infer that $(\psi(rb)-r\psi(b))[t, a] = 0$. Putting t by st in the last expression and using it, we obtain that $(\psi(rb) - r\psi(b))s[t, a] = 0$, hence $(\psi(rb) - r\psi(b))\mathscr{A}[t, a] = \{0\}$, and so $\psi(rb) - r\psi(b) = 0$ or $[t, a] = 0$. If $[t, a] = 0$ ($\forall\ a, t \in \mathscr{A}$). Therefore, $\mathscr{A}$ is commutative. If $\psi(rb) - r\psi(b) = 0$, then

$$(\forall\ b, r \in \mathscr{A}): \quad \psi(rb) = r\psi(b). \tag{8}$$

From the definition of ψ, we conclude that $\psi(rb) = \psi(r)\psi(b)+\psi(r)b+r\xi(b)$, and by using (2) and the equality $\xi(t) = -t$ in the last relation, we have $\psi(rb) = rb + \psi(r)b - rb$, that is

$$(\forall\ b, r \in \mathscr{A}): \quad \psi(rb) = \psi(r)b. \tag{9}$$

Comparing (9) and (8), we see that $\psi(r)b-r\psi(b) = 0$ $(\forall\ b, r \in \mathscr{A})$. Replacing a by r in (2), to get $\psi(r)\psi(b)-rb = 0$. Adding two last expressions, we get $\psi(r)\psi(b)-rb+\psi(r)b-r\psi(b) = 0$, that is $\psi(r)\psi(b)+\psi(r)b-r\psi(b)-rb = 0$, and hence $\psi(r)(\psi(b)+b)-r(\psi(b)+b) = 0$, which further implies $(\psi(r)-r)(\psi(b)+b) = 0$ $(\forall\ b, r \in \mathscr{A})$. Take rt in place of r in the last relation, we infer that $(\psi(rt)-rt)(\psi(b)+b) = 0$ $(\forall\ b, r, t \in \mathscr{A})$. By using (9) in the previous expression, we find that $(\psi(r)t - rt)(\psi(b) + b) = 0$ $(\forall\ b, r, t \in \mathscr{A})$, that is $(\psi(r) - r)t(\psi(b) + b) = 0$ $(\forall\ b, r, t \in \mathscr{A})$. Putting r by b in the previous relation, we get $(\psi(b) - b)t(\psi(b) + b) = 0$ $(\forall\ b, t \in \mathscr{A})$, hence $(\psi(b)-b)\mathscr{A}(\psi(b)+b) = 0$ $(\forall\ b \in \mathscr{A})$. Thus $\psi(b)-b = 0$ or $\psi(b)+b = 0$, that is $\psi(b) = \pm b$ $(\forall\ b \in \mathscr{A})$, as desired. On other hand, the equality $\xi(t) = -t$ gives $\xi(b) = -b$ $(\forall b \in \mathscr{A})$, as desired.

Case (2). Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Let $0 \neq z \in \mathscr{Z}(\mathscr{A})$. Replacing b by z in (1), we have

$$(\forall a \in \mathscr{A}): \quad \psi(a)\psi(z) - az \in \mathscr{Z}(\mathscr{A}). \tag{10}$$

Again, replacing a by z and b by a in (1), we get $\psi(z)\psi(a) - za \in \mathscr{Z}(\mathscr{A})$. Comparing the above expression with (10), we obtain

$$(\forall a \in \mathscr{A}): \quad [\psi(a), \psi(z)] \in \mathscr{Z}(\mathscr{A}). \tag{11}$$

Writing az instead of a in (11), we see that $[\psi(az), \psi(z)] \in \mathscr{Z}(\mathscr{A})$, which further gives

$$[\psi(a)\psi(z) + \psi(a)z + a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A}),$$

that is $[\psi(a), \psi(z)]\psi(z) + [\psi(a), \psi(z)]z + [a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A})$ $(\forall a \in \mathscr{A})$. Applying of (11) in the above relation, we find that

$$(\forall a \in \mathscr{A}): \quad [\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A}). \tag{12}$$

Substituting az for a in (12), we conclude that $[\psi(az), \psi(z)]\psi(z) + [az\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a \in \mathscr{A})$. It follows that $[\psi(a)\psi(z) + \psi(a)z + a\xi(z), \psi(z)]\psi(z) + z[a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a \in \mathscr{A})$, that is

$$[\psi(a)\psi(z), \psi(z)]\psi(z) + [\psi(a)z, \psi(z)]\psi(z) + [a\xi(z), \psi(z)]\psi(z) + z[a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A})$$

$(\forall\ a \in \mathscr{A})$. Hence,

$$[\psi(a), \psi(z)]\psi(z)^2 + [\psi(a), \psi(z)]\psi(z)z + [a\xi(z), \psi(z)]\psi(z) + z[a\xi(z), \psi(z)] \in \mathscr{Z}(\mathscr{A})$$

$(\forall\ a \in \mathscr{A})$, it follows that

$$([\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)])\psi(z) + ([\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)])z \in \mathscr{Z}(\mathscr{A})$$

$(\forall\ a \in \mathscr{A})$. By using (12) in the above expression, we arrive at

$$(\forall a \in \mathscr{A}): \quad ([\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)])\psi(z) \in \mathscr{Z}(\mathscr{A}). \tag{13}$$

Again, by using (12) and Lemma 1 in (13), we have $[\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)] = 0$ or $\psi(z) \in \mathscr{Z}(\mathscr{A})$.

Subcase (1). Suppose that $\psi(z) \in \mathscr{Z}(\mathscr{A})$. Putting $b = z$ in (1), we get

$$\psi(a)\psi(z) - az \in \mathscr{Z}(\mathscr{A}) \tag{14}$$

$(\forall\ a \in \mathscr{A})$, it implies that $[\psi(a)\psi(z) - az, a] = 0$, that is $[\psi(a)\psi(z), a] = 0$. Using the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$ in the last relation, we obtain $[\psi(a), a]\psi(z) = 0$. Again, by using the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$ in the last expression, we infer that $[\psi(a), a]\mathscr{A}\psi(z) = \{0\}$, hence $[\psi(a), a] = 0$ or $\psi(z) = 0$. Let $\psi(z) = 0$. By using the last relation in (14), we find that $-az \in \mathscr{Z}(\mathscr{A})$, and so $a \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a \in \mathscr{A})$, thus $\mathscr{A}$ is commutative. Let $[\psi(a), a] = 0$. By linearizing the previous expression, we conclude that

$$(\forall\ a, b \in \mathscr{A}): \quad [\psi(a), b] + [\psi(b), a] = 0. \tag{15}$$

Taking b by bz in (15), we arrive at $[\psi(a), bz] + [\psi(bz), a] = 0$, this further implies $([\psi(a), b] + [\psi(b), a])z + [\psi(b)\psi(z) + b\xi(z), a] = 0$. Application of (15) in the above relation gives $[\psi(b)\psi(z) + b\xi(z), a] = 0$, that is

$$(\forall\ a, b \in \mathscr{A}): \quad [\psi(b)\psi(z), a] + [b\xi(z), a] = 0. \tag{16}$$

In particular, for $b = z$ in (16) and using the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$, we have $[z\xi(z), a] = 0$, and hence $z[\xi(z), a] = 0$, it implies that $[\xi(z), a] = 0$, that is

$$\xi(z) \in \mathscr{Z}(\mathscr{A}). \tag{17}$$

Using (17) in (16), we get $[\psi(b)\psi(z), a] + [b, a]\xi(z) = 0$. Adding $[\pm bz, a]$ to the last expression, we obtain $[(\psi(b)\psi(z) - bz) + bz, a] + [b, a]\xi(z) = 0$, that is $[\psi(b)\psi(z) - bz, a] + [bz, a] + [b, a]\xi(z) = 0$. Putting a by b in (14) and then using it in the last relation, we see that $[bz, a] + [b, a]\xi(z) = 0$, hence $[b, a]z + [b, a]\xi(z) = 0$, which further implies $[b, a](z + \xi(z)) = 0$. Using (17) in the previous expression, we find that $[b, a]\mathscr{A}(z + \xi(z)) = \{0\}$, thus $[b, a] = 0$ or $z + \xi(z) = 0$.

If $[b, a] = 0$ $(\forall\ a, b \in \mathscr{A})$, then $\mathscr{A}$ is commutative.

If $z + \xi(z) = 0$, then $\xi(z) = -z$. Substituting bz instead of b in (1), we infer that $\psi(a)\psi(bz) - abz \in \mathscr{Z}(\mathscr{A})$, that is $\psi(a)\psi(b)\psi(z) + \psi(a)\psi(b)z + \psi(a)b\xi(z) - abz \in \mathscr{Z}(\mathscr{A})$, and hence $\psi(a)\psi(b)\psi(z) + \psi(a)b\xi(z) + (\psi(a)\psi(b) - ab)z \in \mathscr{Z}(\mathscr{A})$. By using (1) in the above relation, we get $\psi(a)\psi(b)\psi(z) + \psi(a)b\xi(z) \in \mathscr{Z}(\mathscr{A})$. Application of the equality $\xi(z) = -z$ in the above expression yields $\psi(a)\psi(b)\psi(z) - \psi(a)bz \in \mathscr{Z}(\mathscr{A})$, that is

$$\psi(a)(\psi(b)\psi(z) - bz) \in \mathscr{Z}(\mathscr{A}).$$

Using (14) in the above relation and by applying Lemma 1, we obtain $\psi(a) \in \mathscr{Z}(\mathscr{A})$ or $\psi(b)\psi(z) - bz = 0$. Let $\psi(a) \in \mathscr{Z}(\mathscr{A})$. Using the last expression in (1), we see that $-ab \in \mathscr{Z}(\mathscr{A})$, and so $ab \in \mathscr{Z}(\mathscr{A})$, and by Lemma 3, to get $\mathscr{A}$ is commutative. Suppose that

$$(\forall\ b \in \mathscr{A}): \quad \psi(b)\psi(z) - bz = 0. \tag{18}$$

Putting $b = z$ in (18), we find that $\psi(z)^2 - z^2 = 0$, and so $(\psi(z) - z)(\psi(z) + z) = 0$. By using the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$ in the last relation, we conclude that $(\psi(z) - z)\mathscr{A}(\psi(z) + z) = \{0\}$, hence $\psi(z) - z = 0$ or $\psi(z) + z = 0$, that is $\psi(z) = \pm z$. Using the previous expression in

(18), we have $\pm\psi(b)z - bz = 0$. It follows that $\pm\psi(b) - b = 0$, and so $\psi(b) = \pm b$ $(\forall\ b \in \mathscr{A})$, as desired. From the definition of ψ and using the last relation, we infer that $\psi(ab) = \psi(a)\psi(b) + \psi(a)b + a\xi(b)$, and since $\psi(b) = \pm b$ $(\forall\ b \in \mathscr{A})$, we get $\pm ab = ab + (\pm ab) + a\xi(b)$, that is $0 = ab + a\xi(b)$, and hence $a(b + \xi(b)) = 0$. It implies that $(b + \xi(b))a(b + \xi(b)) = 0$, and so $b + \xi(b) = 0$, thus $\xi(b) = -b$ $(\forall\ b \in \mathscr{A})$, as desired.

Subcase (2). Suppose that $[\psi(a), \psi(z)]\psi(z) + [a\xi(z), \psi(z)] = 0$, that is

$$[\psi(a), \psi(z)]\psi(z) + a[\xi(z), \psi(z)] + [a, \psi(z)]\xi(z) = 0,$$

which further gives

$$(\forall\ a \in \mathscr{A}): \quad [\psi(a)\psi(z), \psi(z)] + a[\xi(z), \psi(z)] + [a, \psi(z)]\xi(z) = 0. \tag{19}$$

In particular, for $a = z$ in (19), we have $z[\xi(z), \psi(z)] = 0$, and hence $[\xi(z), \psi(z)] = 0$. Application of the last expression in (19) gives us

$$[\psi(a), \psi(z)]\psi(z) + [a, \psi(z)]\xi(z) = 0 \tag{20}$$

$(\forall\ a \in \mathscr{A})$. Taking $b = z$ in (1), we obtain $\psi(a)\psi(z) - az \in \mathscr{Z}(\mathscr{A})$, it follows that $[\psi(a)\psi(z) - az, \psi(z)] = 0$, hence $[\psi(a)\psi(z), \psi(z)] = [az, \psi(z)]$. By using the last relation in (20), we find that $[az, \psi(z)] + [a, \psi(z)]\xi(z) = 0$, that is $[a, \psi(z)]z + [a, \psi(z)]\xi(z) = 0$, and hence $[a, \psi(z)](z+\xi(z)) = 0$. Replacing a by ra in the previous expression and using it, we conclude that $[r, \psi(z)]a(z + \xi(z)) = 0$, thus $[r, \psi(z)]\mathscr{A}(z + \xi(z)) = \{0\}$, and so $[r, \psi(z)] = 0$ or $z+\xi(z) = 0$. If $[r, \psi(z)] = 0$ $(\forall\ r \in \mathscr{A})$, then $\psi(z) \in \mathscr{Z}(\mathscr{A})$. Now, we argue as in Subcase (1).

If $z + \xi(z) = 0$, then $\xi(z) = -z$. Putting $b = a = z$ in (1), we get $\psi(z)^2 - z^2 \in \mathscr{Z}(\mathscr{A})$, and since $z^2 \in \mathscr{Z}(\mathscr{A})$, we obtain

$$\psi(z)^2 \in \mathscr{Z}(\mathscr{A}). \tag{21}$$

Equation (13) gives $[\psi(a), \psi(z)]\psi(z)^2 + [a\xi(z), \psi(z)]\psi(z) \in \mathscr{Z}(\mathscr{A})$. Using (11) and (21) in the above relation, we see that $[a\xi(z), \psi(z)]\psi(z) \in \mathscr{Z}(\mathscr{A})$. By using the equality $\psi(z) = -z$ in the last expression, we find that $-z[a, \psi(z)]\psi(z) \in \mathscr{Z}(\mathscr{A})$, that is

$$(\forall\ a \in \mathscr{A}): \quad [a, \psi(z)]\psi(z) \in \mathscr{Z}(\mathscr{A}). \tag{22}$$

Take $\psi(z)a$ in place of a in (22), we have $\psi(z)[a, \psi(z)]\psi(z) \in \mathscr{Z}(\mathscr{A})$. Apply (22) in the last relation and Lemma 1, to get $\psi(z) \in \mathscr{Z}(\mathscr{A})$ or $[a, \psi(z)]\psi(z) = 0$. If $\psi(z) \in \mathscr{Z}(\mathscr{A})$, the arguments are the same as in Subcase (1). Suppose that $[a, \psi(z)]\psi(z) = 0$. Replacing a by ra in the previous expression and using it, we infer that $[r, \psi(z)]a\psi(z) = 0$. Again, replacing a by ar in the last relation, then right multiplying it by r and finally subtracting them, we get $[r, \psi(z)]a[r, \psi(z)] = 0$, that is $[r, \psi(z)]\mathscr{A}[r, \psi(z)] = \{0\}$. Hence $[r, \psi(z)] = 0$ $(\forall\ r \in \mathscr{A})$, and so $\psi(z) \in \mathscr{Z}(\mathscr{A})$, now, the same as in Subcase (1).

(ii) If $\mathrm{char}(\mathscr{A}) = 2$, then our hypothesis, $\psi(a)\psi(b)+ab \in \mathscr{Z}(\mathscr{A})$ becomes $\psi(a)\psi(b)-ab \in \mathscr{Z}(\mathscr{A})$ as in *(i)*, and so, to get $\mathscr{A}$ is commutative or any generalized homoderivation (ψ, ξ) is of the form $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$.

So, from now on we will assume that $\mathrm{char}(\mathscr{A}) \neq 2$. If $\psi = 0$, then from our hypothesis, we find that $ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, and by applying Lemma 3, we obtain $\mathscr{A}$ is commutative.

So, from now on we will assume that $\psi \neq 0$.

We have

$$(\forall a, b \in \mathscr{A}): \quad \psi(a)\psi(b) + ab \in \mathscr{Z}(\mathscr{A}). \tag{23}$$

Firstly. Suppose that $\mathscr{Z}(\mathscr{A}) = \{0\}$. Using last condition in (23), we infer that

$$(\forall a, b \in \mathscr{A}): \quad \psi(a)\psi(b) + ab = 0. \tag{24}$$

In (24) replace b by bt, where $t \in \mathscr{A}$, we have $\psi(a)\psi(bt) + abt = 0$ $(\forall\ a, b, t \in \mathscr{A})$, that is $\psi(a)(\psi(b)\psi(t) + \psi(b)t + b\xi(t)) + abt = 0$ $(\forall\ a, b, t \in \mathscr{A})$. This implies that $\psi(a)\psi(b)\psi(t) + \psi(a)\psi(b)t + \psi(a)b\xi(t) + abt = 0$ $(\forall\ a, b, t \in \mathscr{A})$, that is $\psi(a)\psi(b)\psi(t) + \psi(a)b\xi(t) + (\psi(a)\psi(b) + ab)t = 0$ $(\forall\ a, b, t \in \mathscr{A})$. By using (24) in the last expression, we get $\psi(a)\psi(b)\psi(t) + \psi(a)b\xi(t) = 0$. Application of (24) in the last relation yields $-ab\psi(t) + \psi(a)b\xi(t) = 0$. Putting b by $\psi(b)$ in the previous expression, we see that $-a\psi(b)\psi(t) + \psi(a)\psi(b)\xi(t) = 0$. By using (24) in the last relation, we find that $-a\psi(b)\psi(t) - ab\xi(t) = 0$. Taking b by t and a by b in (24) and then using it in the last expression, we conclude that $abt - ab\xi(t) = 0$. It follows that $ab(t - \xi(t)) = 0$. Put $t - \xi(t)$ in place of a in the last relation, we arrive at $(t - \xi(t))b(t - \xi(t)) = 0\ \forall\ b, t \in \mathscr{A}$, that is $(t - \xi(t))\mathscr{A}(t - \xi(t)) = \{0\}$. Thus $t - \xi(t) = 0$, that is $\xi(t) = t$ $(\forall\ t \in \mathscr{A})$. From the definition of ξ, we get $\xi(ab) = \xi(a)\xi(b) + \xi(a)b + a\xi(b)$ $(\forall\ a, b \in \mathscr{A})$. Using the equality $\xi(t) = t$ in the previous expression, we obtain $ab = 3ab$ $(\forall\ a, b \in \mathscr{A})$, thus $2ab = 0$ $(\forall\ a, b \in \mathscr{A})$. Since $\operatorname{char}(\mathscr{A}) \neq 2$, we have $ab = 0$ $(\forall\ a, b \in \mathscr{A})$, and by Lemma 3, we get $\mathscr{A}$ is commutative.

Secondly. Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Let $0 \neq z \in \mathscr{Z}(\mathscr{A})$. Now, the same as in Theorem 1 in Case (2), except, in Subcase (1) of Case (2), we will get two relations as the equality $\xi(z) = -z$ and (18) become as $\xi(z) = z$ and

$$(\forall\ b \in \mathscr{A}): \quad \psi(b)\psi(z) + bz = 0. \tag{25}$$

In particular, for $b = z$ in (25), we have $\psi(z)^2 + z^2 = 0$, that is

$$\psi(z)^2 = -z^2. \tag{26}$$

Right multiplying (25) by $\psi(z)$, we obtain $\psi(b)\psi(z)^2 + bz\psi(z) = 0$. By using (26) in the last relation, we have $-\psi(b)z^2 + bz\psi(z) = 0$, that is $\psi(b)z - b\psi(z) = 0$, and hence

$$(\forall\ b \in \mathscr{A}): \quad \psi(b)z = b\psi(z). \tag{27}$$

From the definition of ψ, we get $\psi(bz) = \psi(b)\psi(z) + \psi(b)z + b\xi(z)$. Applying $\xi(z) = z$ and (25) in the above expression, we obtain $\psi(bz) = -bz + \psi(b)z + bz$, that is $\psi(bz) = \psi(b)z$. By using (27) in the last relation, we infer that

$$(\forall\ b \in \mathscr{A}): \quad \psi(bz) = b\psi(z). \tag{28}$$

Again, from the definition of ψ, we find that $\psi(zb) = \psi(z)\psi(b) + \psi(z)b + z\xi(b)$. Using (25) in the above expression and since $\psi(z) \in \mathscr{Z}(\mathscr{A})$ (see the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$ in Subcase (1) in Theorem 1), we get $\psi(zb) = -bz + \psi(z)b + z\xi(b)$. Comparing the last relation and (28), we obtain $b\psi(z) = -bz + \psi(z)b + z\xi(b)$, and since $\psi(z) \in \mathscr{Z}(\mathscr{A})$, we find that $0 = -bz + z\xi(b)$, that is $-b + \xi(b) = 0$, and so $\xi(b) = b$ $(\forall\ b \in \mathscr{A})$. Now, from the definition of ξ, we get

$$\xi(ab) = \xi(a)\xi(b) + \xi(a)b + a\xi(b),$$

and since $\xi(b) = b$ $(\forall\ b \in \mathscr{A})$, we obtain $ab = ab + ab + ab$, and so $2ab = 0$, that is $2ab2a = 0$. Hence $2a\mathscr{A}2a = \{0\}$, thus $2a = 0$ $(\forall\ a \in \mathscr{A})$, and therefore $\operatorname{char}(\mathscr{A}) = 2$, a contradiction with $\operatorname{char}(\mathscr{A}) \neq 2$. □

Now, suppose we add the condition ψ is zero-power valued on $\mathscr{A}$ to the assumptions of Theorem 1. In that case, we can remove part of the result that any generalized homoderivation (ψ, ξ) is of the form $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$, and state only that $\mathscr{A}$ is commutative, as in the following result.

Corollary 1. *Let $\mathscr{A}$ be a prime ring. If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$, ψ is zero-power valued on $\mathscr{A}$, and satisfies*
(i) $(\forall a, b \in \mathscr{A})\colon\ \psi(a)\psi(b) - ab \in \mathscr{Z}(\mathscr{A})$, *(ii)* $(\forall a, b \in \mathscr{A})\colon\ \psi(a)\psi(b) + ab \in \mathscr{Z}(\mathscr{A})$.
Then $\mathscr{A}$ is commutative.

Proof. (i) From Theorem 1 (i), we have $\mathscr{A}$ is commutative or any generalized homoderivation (ψ, ξ) is of the form $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$. So, we may assume that $\psi(a) = \pm a$ $(\forall\ a \in \mathscr{A})$. From our assumption, we have ψ is a zero-power valued on $\mathscr{A}$, and so there exists a positive integer $n(a) > 1$ such that $\psi^{n(a)}(a) = 0$ $(\forall\ a \in \mathscr{A})$, and $\psi^{n(a)-1}(a) \neq 0$ $(\forall\ a \in \mathscr{A})$. From the equalities $\psi(a) = \pm a$ we obtain $\psi^{n(a)-1}(\psi(a)) = \psi^{n(a)-1}(\pm a)$, that is $\psi^{n(a)}(a) = \psi^{n(a)-1}(\pm a)$. It implies that $\psi^{n(a)}(a) = \pm\psi^{n(a)-1}(a)$. By using the equality $\psi^{n(a)}(a) = 0$ in the last expression, we see that $0 = \pm\psi^{n(a)-1}(a)$, that is $0 = \psi^{n(a)-1}(a)$, a contradiction with $\psi^{n(a)-1}(a) \neq 0$.

(ii) Same as in (i). □

Corollary 2. *Let $\mathscr{A}$ be a prime ring with* $\mathrm{char}(\mathscr{A}) \neq 2$. *If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$ satisfying $\psi(a)\psi(b) + ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, then $\mathscr{A}$ is commutative.*

Proof. From Theorem 1(ii), we have $\mathscr{A}$ is commutative or any generalized homoderivation (ψ, ξ) is of the form $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$. Suppose that $\psi(a) = \pm a$. By using the last expression in our hypothesis, that is $(\pm a)(\pm b) + ab \in \mathscr{Z}(\mathscr{A})$, we deduce that $ab + ab \in \mathscr{Z}(\mathscr{A})$, hence $2ab \in \mathscr{Z}(\mathscr{A})$. Since $\mathrm{char}(\mathscr{A}) \neq 2$, we get $ab \in \mathscr{Z}(\mathscr{A})$, and by applying Lemma 3, we obtain $\mathscr{A}$ is commutative. □

Theorem 2. *Let $\mathscr{A}$ be a prime ring. If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$ satisfying any one of the following conditions:*
(i) $(\forall a, b \in \mathscr{A})\colon\ \psi(a)\psi(b) - ba \in \mathscr{Z}(\mathscr{A})$, *(ii)* $(\forall a, b \in \mathscr{A})\colon\ \psi(a)\psi(b) + ba \in \mathscr{Z}(\mathscr{A})$.
Then $\mathscr{A}$ is commutative.

Proof. (i) If $\psi = 0$, then from our hypothesis, we arrive at $ba \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, and by Lemma 3, we obtain $\mathscr{A}$ is commutative.

So, from now on we will assume that $\psi \neq 0$.

Assume that

$$(\forall\ a, b \in \mathscr{A})\colon \quad \psi(a)\psi(b) - ba \in \mathscr{Z}(\mathscr{A}). \tag{29}$$

Firstly. Suppose that $\mathscr{Z}(\mathscr{A}) = \{0\}$. Applying this condition in (29), we see that

$$(\forall\ a, b \in \mathscr{A})\colon \quad \psi(a)\psi(b) - ba = 0. \tag{30}$$

Substitution bt for b in (30), where $t \in \mathscr{A}$, yields $\psi(a)\psi(bt) - bta = 0$, that is $\psi(a)(\psi(b)\psi(t) + \psi(b)t + b\xi(t)) - bta = 0$. This implies that $\psi(a)\psi(b)\psi(t) + \psi(a)\psi(b)t + \psi(a)b\xi(t) - bta = 0$. By using (30) in the above relation, we obtain $ba\psi(t) + bat + \psi(a)b\xi(t) - bta = 0$, that is

$$(\forall\ a, b, t \in \mathscr{A})\colon \quad b(a\psi(t) + [a, t]) + \psi(a)b\xi(t) = 0. \tag{31}$$

Replacing b by rb in (31), we get

$$(\forall\ a, b, t, r \in \mathscr{A})\colon \quad rb(a\psi(t) + [a, t]) + \psi(a)rb\xi(t) = 0. \tag{32}$$

Left multiplying (31) by r, we obtain $rb(a\psi(t) + [a, t]) + r\psi(a)b\xi(t) = 0$ $(\forall\ a, b, t, r \in \mathscr{A})$. Comparing above and (32), we infer that $[\psi(a), r]b\xi(t) = 0$, that is $[\psi(a), r]\mathscr{A}\xi(t) = \{0\}$, and so $[\psi(a), r] = 0$ or $\xi(t) = 0$. If $[\psi(a), r] = 0$, then $\psi(a) \in \mathscr{Z}(\mathscr{A})$, but $\mathscr{Z}(\mathscr{A}) = \{0\}$, and so $\psi(a) = 0$, a contradiction with $\psi \neq 0$. Now, let $\xi(t) = 0$. Using the previous expression in (31), we find that $b(a\psi(t) + [a, t]) = 0$. This implies that $(a\psi(t) + [a, t])b(a\psi(t) + [a, t]) = 0$, and hence $(a\psi(t) + [a, t])\mathscr{A}(a\psi(t) + [a, t]) = \{0\}$, and so

$$(\forall\ a, t \in \mathscr{A}): \quad a\psi(t) + [a, t] = 0. \tag{33}$$

Writing ra instead of a in (33), we have $ra\psi(t) + [ra, t] = 0$, that is

$$(\forall\ a, t, r \in \mathscr{A}): \quad a\psi(t) + r[a, t] + [r, t]a = 0 \tag{34}$$

Left multiplying (33) by r, we obtain $ra\psi(t) + r[a, t] = 0$ $(\forall\ a, t, r \in \mathscr{A})$. Comparing last condition with (34), we get $[r, t]a = 0$, which further implies $[r, t]a[r, t] = 0$. Hence $[r, t] = 0$ $(\forall\ t, r \in \mathscr{A})$, and so $\mathscr{A}$ is commutative.

Secondly. Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Now, the same as in Theorem 1 in Case (2), we get $\mathscr{A}$ is commutative or $\psi(a) = \pm a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$. It is sufficient to suppose that $\psi(a) = \pm a$. By using the last relation in (29), we obtain $(\pm a)(\pm b) - ba \in \mathscr{Z}(\mathscr{A})$, that is $ab - ba \in \mathscr{Z}(\mathscr{A})$. It implies that $[a, b] \in \mathscr{Z}(\mathscr{A})$, and by applying Lemma 2, $\mathscr{A}$ is commutative.

(*ii*) If $\text{char}(\mathscr{A}) = 2$, then our hypothesis, $\psi(a)\psi(b) + ba \in \mathscr{Z}(\mathscr{A})$ becomes as in our hypothesis in part (*i*), that is $\psi(a)\psi(b) - ba \in \mathscr{Z}(\mathscr{A})$, and so we get $\mathscr{A}$ is commutative.

So, from now on we will assume that $\text{char}(\mathscr{A}) \neq 2$.

If $\psi = 0$, then from our hypothesis, to get $ba \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, and by Lemma 3, we obtain $\mathscr{A}$ is commutative. So, from now on we will assume that $\psi \neq 0$.

Suppose that

$$(\forall\ a, b \in \mathscr{A}): \quad \psi(a)\psi(b) + ba \in \mathscr{Z}(\mathscr{A}). \tag{35}$$

Firstly. Let $\mathscr{Z}(\mathscr{A}) = \{0\}$. By application of this condition in (35) we have

$$(\forall\ a, b \in \mathscr{A}): \quad \psi(a)\psi(b) + ba = 0. \tag{36}$$

Substituting bt for b in (36), where $t \in \mathscr{A}$, we get $\psi(a)\psi(bt) + bta = 0$, that is

$$\psi(a)\psi(b)\psi(t) + \psi(a)\psi(b)t + \psi(a)b\xi(t) + bta = 0.$$

By using (36) in the above expression, we obtain $-ba\psi(t) - bat + \psi(a)b\xi(t) + bta = 0$. It follows that $-b(a\psi(t) + [a, t]) + \psi(a)b\xi(t) = 0$, and hence $b(a\psi(t) + [a, t]) - \psi(a)b\xi(t) = 0$. Now, the same as in (*i*) in (31), we get $\mathscr{A}$ is commutative.

Secondly. Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Now, the same arguments are as in Theorem 1 (*ii*). □

Theorem 3. *Let $\mathscr{A}$ be a prime ring. If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$ satisfuing $\psi(ab) - ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, then $\mathscr{A}$ is commutative or $\psi(a) = -\xi(a)$ $(\forall\ a \in \mathscr{A})$.*

Proof. If $\psi = 0$, then from our hypothesis, we find that $ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, and by applying Lemma 3, we obtain $\mathscr{A}$ is commutative, as desired. So, from now on we will assume that $\psi \neq 0$.

Suppose that

$$(\forall\ a, b \in \mathscr{A}): \quad \psi(ab) - ab \in \mathscr{Z}(\mathscr{A}). \tag{37}$$

Case (1). Let $\mathscr{Z}(\mathscr{A}) = \{0\}$. Then condition (37) gives

$$(\forall\ a, b \in \mathscr{A}):\quad \psi(ab) - ab = 0. \tag{38}$$

Replacing b by bt in (38), we obtain $\psi(abt) - abt = 0$, this implies that

$$\psi(ab)\psi(t) + ab\xi(t) + (\psi(ab) - ab)t = 0.$$

Using (38) in the previous expression, we get $\psi(ab)\psi(t) + ab\xi(t) = 0$ $(\forall\ a, b, t \in \mathscr{A})$. By using (38) in above, we see that $ab\psi(t) + ab\xi(t) = 0$, and so $ab(\xi(t) + \psi(t)) = 0$. Putting a by $\xi(t) + \psi(t)$ in the last relation, we find that $(\xi(t) + \psi(t))b(\xi(t) + \psi(t)) = 0$, and so $(\xi(t) + \psi(t))\mathscr{A}(\xi(t) + \psi(t)) = \{0\}$, hence $\psi(t) = -\xi(t)$ $(\forall\ t \in \mathscr{A})$, as desired.

Case (2). Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Let $0 \neq z \in \mathscr{Z}(\mathscr{A})$. Replacing b by bz in (37), we have $\psi(abz) - abz \in \mathscr{Z}(\mathscr{A})$, it implies that $\psi(ab)\psi(z) + \psi(ab)z + ab\xi(z) - abz \in \mathscr{Z}(\mathscr{A})$, that is $\psi(ab)\psi(z) + ab\xi(z) + (\psi(ab) - ab)z \in \mathscr{Z}(\mathscr{A})$. Application of (37) in the above expression, gives $\psi(ab)\psi(z) + ab\xi(z) \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$. Adding $\pm ab\psi(z)$ to $\psi(ab)\psi(z) + ab\xi(z)$, we infer that

$$(\forall\ a, b \in \mathscr{A}):\quad (\psi(ab) - ab)\psi(z) + ab(\psi(z) + \xi(z)) \in \mathscr{Z}(\mathscr{A}). \tag{39}$$

We have (37) commutes with any element in $\mathscr{A}$, in particular, with $\psi(z)$. Thus condition (39) gives $[ab(\psi(z) + \xi(z)), \psi(z)] = 0$, that is $[ab, \psi(z)](\psi(z) + \xi(z)) + ab[\psi(z) + \xi(z), \psi(z)] = 0$, which further implies

$$(\forall\ a, b \in \mathscr{A}):\quad [ab, \psi(z)](\psi(z) + \xi(z)) + ab[\xi(z), \psi(z)] = 0. \tag{40}$$

In particular, for $a = b = z$ in (40), we obtain $z^2[\xi(z), \psi(z)] = 0$, and so $[\xi(z), \psi(z)] = 0$, and by using the last expression in (40), we see that $[ab, \psi(z)](\psi(z) + \xi(z)) = 0$. Taking $b = z$ in the previous relation, we find that $[a, \psi(z)](\psi(z) + \xi(z)) = 0$. Substituting ra for a in the previous expression and using it, where $r \in \mathscr{A}$, we conclude that $[r, \psi(z)]a(\psi(z) + \xi(z)) = 0$, that is $[r, \psi(z)]\mathscr{A}(\psi(z) + \xi(z)) = \{0\}$, and so $[r, \psi(z)] = 0$ or $\psi(z) + \xi(z) = 0$.

Subcase (1). If $\psi(z) + \xi(z) = 0$, then $\psi(z) = -\xi(z)$. By using this equality in the relation $\psi(ab)\psi(z) + ab\xi(z) \in \mathscr{Z}(\mathscr{A})$, we have

$$(\psi(ab) - ab)\xi(z) \in \mathscr{Z}(\mathscr{A}) \tag{41}$$

$(\forall\ a, b \in \mathscr{A})$. Using Lemma 1 and (37) in (41), we obtain $\psi(ab) - ab = 0$ or $\xi(z) \in \mathscr{Z}(\mathscr{A})$. In the case $\psi(ab) - ab = 0$, the same as in (38), to get $\psi(a) = -\xi(a)$ $(\forall\ a \in \mathscr{A})$, as desired.

Now, consider the case

$$\xi(z) \in \mathscr{Z}(\mathscr{A}). \tag{42}$$

We know that $\psi(az) = \psi(za)$, and by using the definition of ψ, in the last relation, we find that

$$\psi(a)\psi(z) + \psi(a)z + a\xi(z) = \psi(z)\psi(a) + \psi(z)a + z\xi(a).$$

By using the equality $\psi(z) = -\xi(z)$ in the above expression, we get

$$-\psi(a)\xi(z) + \psi(a)z + a\xi(z) = -\xi(z)\psi(a) - \xi(z)a + z\xi(a).$$

Applying (42) in the above relation, we obtain

$$-\xi(z)\psi(a) + \psi(a)z + \xi(z)a = -\xi(z)\psi(a) - \xi(z)a + z\xi(a),$$

that is $\psi(a)z + \xi(z)a = -\xi(z)a + z\xi(a)$. It implies that

$$z(\xi(a) - \psi(a)) = 2\xi(z)a \tag{43}$$

($\forall\ a \in \mathscr{A}$). Replacing a by ab in (43), we find that $z(\xi(ab) - \psi(ab)) = 2\xi(z)ab$, which further gives $z(\xi(a)\xi(b) + \xi(a)b + a\xi(b) - \psi(a)\psi(b) - \psi(a)b - a\xi(b)) = 2\xi(z)ab$, that is

$$(\forall\ a, b \in \mathscr{A}): \quad z(\xi(a)\xi(b) + \xi(a)b - \psi(a)\psi(b) - \psi(a)b) = 2\xi(z)ab \tag{44}$$

Right multiplying (43) by b, we conclude that $z(\xi(a)b - \psi(a)b) = 2\xi(z)ab$ ($\forall\ a, b \in \mathscr{A}$). Comparing the last equality and (44), we have $z(\xi(a)\xi(b) - \psi(a)\psi(b)) = 0$, that is $\xi(a)\xi(b) - \psi(a)\psi(b) = 0$. Putting $b = z$ in the last expression and using the equality $\psi(z) = -\xi(z)$ and (42), we get $(\xi(a) + \psi(a))\mathscr{A}\xi(z) = \{0\}$, and so $\xi(a) + \psi(a) = 0$ or $\xi(z) = 0$. If $\xi(a) + \psi(a) = 0$, then

$$\psi(a) = -\xi(a) \tag{45}$$

($\forall\ a \in \mathscr{A}$).

Now, assume $\xi(z) = 0$. By using this equality in (43), we get $z(\xi(a) - \psi(a)) = 0$, and so $\xi(a) - \psi(a) = 0$, that is

$$\psi(a) = \xi(a) \tag{46}$$

($\forall\ a \in \mathscr{A}$). If $\mathrm{char}(\mathscr{A}) = 2$, then (46) becomes $\psi(a) = -\xi(a)$ ($\forall\ a \in \mathscr{A}$), as desired.

We then suppose that $\mathrm{char}(\mathscr{A}) \neq 2$. Putting $a = z$ in (46) and then using the equality $\xi(z) = 0$, we infer that $\psi(z) = 0$. Replacing b by bt in (37), we have $\psi(abt) - abt \in \mathscr{Z}(\mathscr{A})$. It implies that

$$\psi(ab)\psi(t) + \psi(ab)t + ab\xi(t) - abt \in \mathscr{Z}(\mathscr{A}),$$

that is $\psi(ab)\psi(t) + ab\xi(t) + (\psi(ab) - ab)t \in \mathscr{Z}(\mathscr{A})$. Using (46) in the above expression, we get $(\psi(ab) + ab)\psi(t) + (\psi(ab) - ab)t \in \mathscr{Z}(\mathscr{A})$. By using (37) in the above relation, we obtain $[(\psi(ab) + ab)\psi(t), t] = 0$. In particular, for $b = z$ in the previous expression, we have $[(\psi(az) + az)\psi(t), t] = 0$, which yields that $[(\psi(a)\psi(z) + \psi(a)z + a\xi(z) + az)\psi(t), t] = 0$.

Applying the equalities $\xi(z) = 0$ and $\psi(z) = 0$ in the above relation, we find that $[(\psi(a)z + az)\psi(t), t] = 0$, that is $z[(\psi(a) + a)\psi(t), t] = 0$. This implies that $[(\psi(a) + a)\psi(t), t] = 0$, and hence

$$(\psi(a) + a)[\psi(t), t] + [(\psi(a) + a), t]\psi(t) = 0 \tag{47}$$

($\forall\ a, t \in \mathscr{A}$). Putting $a = z$ in (47) and using the equality $\psi(z) = 0$ in the above expression, we have $z[\psi(t), t] = 0$, that is $[\psi(t), t] = 0$. By using the last relation in (47), we get $[(\psi(a) + a), t]\psi(t) = 0$. Replacing a by ab in the last expression, we see that $[(\psi(ab) + ab), t]\psi(t) = 0$. Adding $\pm[ab, t]\psi(t)$ to the last relation, we infer that $[(\psi(ab) - ab) + 2ab, t]\psi(t) = 0$. Using (37) in the previous expression, we find that $2[ab, t]\psi(t) = 0$. In particular, for $b = z$ in the last relation, we have $2z[a, t]\psi(t) = 0$, that is $2[a, t]\psi(t) = 0$. By using the condition $\mathrm{char}(\mathscr{A}) \neq 2$ in the last expression, we get $[a, t]\psi(t) = 0$. Replacing a by ra in the last relation and using it, we obtain $[r, t]a\psi(t) = 0$, that is $[r, t]\mathscr{A}\psi(t) = \{0\}$, hence $[r, t] = 0$ or $\psi(t) = 0$. If $\psi(t) = 0$ ($\forall\ t \in \mathscr{A}$), we have a contradiction with $\psi \neq 0$. If $[r, t] = 0$ ($\forall\ r, t \in \mathscr{A}$), then $\mathscr{A}$ is commutative.

Subcase (2). If $[a, \psi(z)] = 0$, then $\psi(z) \in \mathscr{Z}(\mathscr{A})$. Adding $\pm ab\psi(z)$ to $\psi(ab)\psi(z) + ab\xi(z)$ and using (37) and the condition $\psi(z) \in \mathscr{Z}(\mathscr{A})$, we get

$$(\forall\ a, b \in \mathscr{A}): \quad ab(\psi(z) + \xi(z)) \in \mathscr{Z}(\mathscr{A}). \tag{48}$$

Taking $b = a = z$ in (48), we obtain $\psi(z) + \xi(z) \in \mathscr{Z}(\mathscr{A})$, and by using the previous expression in (48) and by applying Lemma 1, we infer that $ab \in \mathscr{Z}(\mathscr{A})$ or $\psi(z) + \xi(z) = 0$. In the case $\psi(z) + \xi(z) = 0$, the same as in Subcase (1). If $ab \in \mathscr{Z}(\mathscr{A})$, by Lemma 3 $\mathscr{A}$ is commutative. $\square$

Corollary 3. *Let $\mathscr{A}$ be a prime ring with* $\mathrm{char}(\mathscr{A}) \neq 2$. *If* (ψ, ξ) *is a generalized homoderivation of* $\mathscr{A}$ *satisfying* $\psi(ab) - ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$. *Then* $\mathscr{A}$ *is commutative or any generalized homoderivation* (ψ, ξ) *is of the form* $\psi(a) = a$ *and* $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$.

Proof. From Theorem 3, we have $\mathscr{A}$ is commutative or $\psi(a) = -\xi(a)$ $(\forall\ a \in \mathscr{A})$. If $\mathscr{A}$ is commutative, as desired. Now, suppose that $\psi(a) = -\xi(a)$ $(\forall\ a \in \mathscr{A})$.

Case (1). Let $\mathscr{Z}(\mathscr{A}) = \{0\}$. Then from the definition of ψ, we have $\psi(ab) = \psi(a)\psi(b) + \psi(a)b + a\xi(b)$, and by using the equality $\psi(a) = -\xi(a)$ in the last expression, we get

$$(\forall\ a, b \in \mathscr{A}): \quad -\xi(ab) = \xi(a)\xi(b) - \xi(a)b + a\xi(b) \tag{49}$$

From the definition of ξ, we obtain $\xi(ab) = \xi(a)\xi(b) + \xi(a)b + a\xi(b)$, and comparing (49) and the previous expression, we infer that $2\xi(ab) = 2\xi(a)b$. Since $\mathrm{char}(\mathscr{A}) \neq 2$, we find that

$$(\forall\ a, b \in \mathscr{A}): \quad \xi(ab) = \xi(a)b. \tag{50}$$

Applying of the equality $\psi(a) = -\xi(a)$ and (50) in (38), we see that $(\xi(a) + a)b = 0$, that is $(\xi(a) + a)b(\xi(a) + a) = 0$, and so $\xi(a) + a = 0$, and hence $\xi(a) = -a$ by using the last expression in the equality $\psi(a) = -\xi(a)$, we conclude that $\psi(a) = a$ $(\forall\ a \in \mathscr{A})$, as desired.

Case (2). Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Let $0 \neq z \in \mathscr{Z}(\mathscr{A})$. Now, the same as in Case (2) in Theorem 3, we get $\mathscr{A}$ is commutative or we obtain the equality $\psi(a) = -\xi(a)$. Now, the same as above, we get (50), that is $\xi(ab) = \xi(a)b$, and by using the previous expression and $\psi(a) = -\xi(a)$ in (37), we obtain $(\xi(a) + a)b \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$. In particular, for $b = z$ in $(\xi(a) + a)b$, we infer that $\xi(a) + a \in \mathscr{Z}(\mathscr{A})$, and by using the last condition, from the condition $(\xi(a)+a)b \in \mathscr{Z}(\mathscr{A})$ by applying Lemma 1, we find that $\xi(a)+a = 0$ or $b \in \mathscr{Z}(\mathscr{A})$. If $b \in \mathscr{Z}(\mathscr{A})$ $(\forall\ b \in \mathscr{A})$, then $\mathscr{A}$ is commutative. If $\xi(a) + a = 0$, then $\xi(a) = -a$, and by using the previous expression in $\psi(a) = -\xi(a)$, we conclude that $\psi(a) = a$ $(\forall\ a \in \mathscr{A})$, as desired. □

Corollary 4. *Let $\mathscr{A}$ be a prime ring. If* (ψ, ξ) *is a generalized homoderivation of* $\mathscr{A}$, *zero-power valued on* $\mathscr{A}$, *satisfies* $\psi(ab) - ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$. *Then* $\mathscr{A}$ *is commutative.*

Proof. **Case (1).** Suppose that $\mathrm{char}(\mathscr{A}) \neq 2$, the same as in Corollary 3, we have $\mathscr{A}$ is commutative or any generalized homoderivation (ψ, ξ) is of the form $\psi(a) = a$ and $\xi(a) = -a$ $(\forall\ a \in \mathscr{A})$. Now, we may assume $\psi(a) = a$ $(\forall\ a \in \mathscr{A})$. Since ψ is a zero-power valued on $\mathscr{A}$, we get $\psi^{n(a)}(a) = 0$ $(\forall\ a \in \mathscr{A})$ and for some a positive integer $n(a) > 1$, and $\psi^{n(a)-1}(a) \neq 0$ $(\forall\ a \in \mathscr{A})$. Now, from the equality $\psi(a) = a$, we obtain $\psi^{n(a)}(a) = \psi^{n(a)-1}(a)$. By using $\psi^{n(a)}(a) = 0$ in the last expression, we see that $0 = \psi^{n(a)}(a) = \psi^{n(a)-1}(a)$, that is $\psi^{n(a)-1}(a) = 0$, a contradiction with $\psi^{n(a)-1}(a) \neq 0$.

Case (2). Let $\mathrm{char}(\mathscr{A}) = 2$.

Then from Theorem 3, we have $\psi(a) = -\xi(a)$ becomes $\psi(a) = \xi(a)$, since $\mathrm{char}(\mathscr{A}) = 2$. Also, our hypothesis $\psi(ab) - ab \in \mathscr{Z}(\mathscr{A})$ becomes $\psi(ab) + ab \in \mathscr{Z}(\mathscr{A})$, because $\mathrm{char}(\mathscr{A}) = 2$. Now, since $\psi(a) = \xi(a)$, we get $\xi(ab) + ab \in \mathscr{Z}(\mathscr{A})$, that is

$$(\xi(a) + a)(\xi(b) + b) \in \mathscr{Z}(\mathscr{A}) \tag{51}$$

$(\forall\ a, b \in \mathscr{A})$. Since ψ is a zero-power valued on $\mathscr{A}$, and $\psi(a) = \xi(a)$, we obtain ξ is a zero-power valued on $\mathscr{A}$, and so there exists a positive integer $n(a) > 1$ such that $\xi^{n(a)}(a) = 0$ $(\forall\ a \in \mathscr{A})$. Now, putting a by $A = a - \xi(a) + \xi^2(a) - ... + (-1)^{n(a)-1}\xi^{n(a)-1}(a)$ and b

by $B = b - \xi(b) + \xi^2(b) - ... + (-1)^{n(b)-1}\xi^{n(b)-1}(b)$ in (51), that is, $(\xi(A) + A)(\xi(B) + B) \in \mathscr{Z}(\mathscr{A})$ $\forall\, a, b \in \mathscr{A}$. This implies that $\big(\xi(a) - \xi^2(a) + \xi^3(a) - ... + (-1)^{n(a)-1}\xi^{n(a)}(a) + +a - \xi(a) + \xi^2(a) - ... + (-1)^{n(a)-1}\xi^{n(a)-1}(a)\big)$ $\big(\xi(b) - \xi^2(b) + \xi^3(b) - ... + (-1)^{n(b)-1}\xi^{n(b)}(b) + +b - \xi(b) + \xi^2(b) - ... + (-1)^{n(b)-1}\xi^{n(b)}(b)\big) \in \mathscr{Z}(\mathscr{A})$ $(\forall a, b \in \mathscr{A})$. But $\xi^{n(a)}(a) = 0 = \xi^{n(b)}(b)$ and hence, $\big(\xi(a) - \xi^2(a) + \xi^3(a) - ... + (-1)^{n(a)-2}\xi^{n(a)-1}(a) + a - \xi(a) + \xi^2(a) - ... + (-1)^{n(a)-1}\xi^{n(a)-1}(a)\big)\big(\xi(b) - \xi^2(b) + \xi^3(b) - ... + (-1)^{n(b)-2}\xi^{n(b)-1}(b) + b - \xi(b) + \xi^2(b) - ... + (-1)^{n(b)-1}\xi^{n(b)-1}(b)\big) \in \mathscr{Z}(\mathscr{A})$ $(\forall a, b \in \mathscr{A})$. It follows that $ab \in \mathscr{Z}(\mathscr{A})$ and by applying Lemma 3, we obtain $\mathscr{A}$ is commutative. □

Theorem 4. *Let $\mathscr{A}$ be a prime ring. If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$ satisfying $\psi(ab) + ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\, a, b \in \mathscr{A})$, then $\mathscr{A}$ is commutative or any generalized homoderivation ψ is homoderivation $\xi(a)$ that is $\psi(a) = \xi(a)$ $(\forall\, a \in \mathscr{A})$.*

Proof. The proof is the same as in Theorem 3, except in Subcase (1) of Case (2) when we get the relations the equality $\psi(z) = -\xi(z)$ and (42), become $\psi(z) = \xi(z)$ and $\xi(z) \in \mathscr{Z}(\mathscr{A})$. We know that $\psi(az) = \psi(za)$, and by using the definition of ψ, in the last relation, we find that

$$\psi(a)\psi(z) + \psi(a)z + a\xi(z) = \psi(z)\psi(a) + \psi(z)a + z\xi(a).$$

Using the equality $\psi(z) = \xi(z)$ in the previous expression, to get

$$\psi(a)\xi(z) + \psi(a)z + a\xi(z) = \xi(z)\psi(a) + \xi(z)a + z\xi(a).$$

By using $\xi(z) \in \mathscr{Z}(\mathscr{A})$ in the above relation, we obtain

$$\xi(z)\psi(a) + \psi(a)z + \xi(z)a = \xi(z)\psi(a) + \xi(z)a + z\xi(a),$$

that is $\psi(a)z = z\xi(a)$. Hence $z(\psi(a) - \xi(a)) = 0$, and so $\psi(a) - \xi(a) = 0$, thus $\psi(a) = \xi(a)$ $(\forall\, a \in \mathscr{A})$, as desired. □

Corollary 5. *Let $\mathscr{A}$ be a prime ring. If (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$, a zero-power valued on $\mathscr{A}$, which satisfies $\psi(ab) + ab \in \mathscr{Z}(\mathscr{A})$ $(\forall\, a, b \in \mathscr{A})$. Then $\mathscr{A}$ is commutative.*

Proof. From Theorem 4, we have $\mathscr{A}$ is commutative or any generalized homoderivation ψ is homoderivation $\xi(a)$, that is $\psi(a) = \xi(a)$ $(\forall\, a \in \mathscr{A})$. Suppose that $\psi(a) = \xi(a)$ $(\forall\, a \in \mathscr{A})$.
Case (1). If $\mathscr{Z}(\mathscr{A}) = \{0\}$. Then from our hypothesis, we have $\psi(ab) + ab = 0$. Application of the equality $\psi(a) = \xi(a)$ in the previous expression yields $\xi(ab) + ab = 0$. It implies that $\xi(a)\xi(b) + \xi(a)b + a\xi(b) + ab = 0$, that is

$$\forall\, a, b \in \mathscr{A}: \quad (\xi(a) + a)(\xi(b) + b) = 0. \tag{52}$$

Since ψ is zero-power valued on $\mathscr{A}$, from the equality $\psi(a) = \xi(a)$ we obtain ξ is zero-power valued on $\mathscr{A}$, and so there exists a positive integer $n(a) > 1$ such that $\xi^{n(a)}(a) = 0$ $(\forall\, a \in \mathscr{A})$. Now, taking a by $a - \xi(a) + \xi^2(a) - ... + (-1)^{n(a)-1}\xi^{n(a)-1}(a)$ and b by $b - \xi(b) + \xi^2(b) - ... + (-1)^{n(b)-1}\xi^{n(b)-1}(b)$ in (52), and the same as in (51), we infer that $ab = 0$, and by Lemma 3, $\mathscr{A}$ is commutative.
Case (2). Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. By using the equality $\psi(a) = \xi(a)$ in our hypothesis, we arrive at $\xi(ab) + ab \in \mathscr{Z}(\mathscr{A})$, which further implies $\xi(a)\xi(b) + \xi(a)b + a\xi(b) + ab \in \mathscr{Z}(\mathscr{A})$, and hence

$$(\forall\, a, b \in \mathscr{A}): \quad (\xi(a) + a)(\xi(b) + b) \in \mathscr{Z}(\mathscr{A}). \tag{53}$$

Since ψ is zero-power valued on $\mathscr{A}$, and from the equality $\psi(a) = \xi(a)$, we obtain ξ is zero-power valued on $\mathscr{A}$, and so there exists a positive integer $n(a) > 1$ such that $\xi^{n(a)}(a) = 0$

$(\forall\ a \in \mathscr{A})$. Now, taking a by $a - \xi(a) + \xi^2(a) - ... + (-1)^{n(a)-1}\xi^{n(a)-1}(a)$ and b by $b - \xi(b) + \xi^2(b) - ... + (-1)^{n(b)-1}\xi^{n(b)-1}(b)$ in (53), we obtain $ab \in \mathscr{Z}(\mathscr{A})$, and by applying Lemma 3 in the last relation, $\mathscr{A}$ is commutative, as desired. □

Theorem 5. *Let $\mathscr{A}$ be a prime ring $\mathscr{A}$. Suppose (ψ, ξ) is a generalized homoderivation of $\mathscr{A}$ which satisfies any one of the following condiitions:*
(i) $(\forall a, b \in \mathscr{A})$: $\psi(ab) - ba \in \mathscr{Z}(\mathscr{A})$, (ii) $(\forall a, b \in \mathscr{A})$: $\psi(ab) + ba \in \mathscr{Z}(\mathscr{A})$.
Then $\mathscr{A}$ is commutative.

Proof. (*i*) If $\psi = 0$, then from our hypothesis, we find that $ba \in \mathscr{Z}(\mathscr{A})$ $(\forall\ a, b \in \mathscr{A})$, and by Lemma 3, we obtain $\mathscr{A}$ is commutative, as desired.

So, from now on we will suppose that $\psi \neq 0$. Assume that

$$(\forall a, b \in \mathscr{A}): \quad \psi(ab) - ba \in \mathscr{Z}(\mathscr{A}). \tag{54}$$

Firstly. Suppose that $\mathscr{Z}(\mathscr{A}) = \{0\}$. Using this condition in (54), we see that

$$(\forall\ a, b \in \mathscr{A}): \quad \psi(ab) - ba = 0. \tag{55}$$

Replacing b by bt in (55), we get $\psi(abt) - bta = 0$, which implies that

$$\psi(ab)\psi(t) + \psi(ab)t + ab\xi(t) - bta = 0.$$

By using (55) in the above expression, we obtain $ba\psi(t) + bat + ab\xi(t) - bta = 0$, that is

$$(\forall\ a, b, t \in \mathscr{A}): \quad b(a\psi(t) + [a, t]) + ab\xi(t) = 0 \tag{56}$$

Replacing b by rb in (56) and then left multiplying it by r and then subtracting them, where $r \in \mathscr{A}$, we find that $[a, r]b\xi(t) = 0$, that is $[a, r]\mathscr{A}\xi(t) = \{0\}$. Hence $[a, r] = 0$ $(\forall\ a, r \in \mathscr{A})$ or $\xi(t) = 0$ $(\forall\ t \in \mathscr{A})$. If $[a, r] = 0$ $(\forall\ a, r \in \mathscr{A})$, then $\mathscr{A}$ is commutative. Now, suppose that $\xi(t) = 0$ $(\forall\ t \in \mathscr{A})$. Application of the last relation in (56) yields $b(a\psi(t) + [a, t]) = 0$, that is $(a\psi(t) + [a, t])b(a\psi(t) + [a, t]) = 0$, and hence $a\psi(t) + [a, t] = 0$ $(\forall\ a, t \in \mathscr{A})$. Replacing a by ra in the previous equality, then left multiplying it by r and finally subtracting them, where $r \in \mathscr{A}$, we obtain $[r, t]a = 0$, which further gives $[r, t]a[r, t] = 0$. Hence $[r, t] = 0$ $(\forall\ r, t \in \mathscr{A})$, and so $\mathscr{A}$ is commutative.

Secondly. Suppose that $\mathscr{Z}(\mathscr{A}) \neq \{0\}$. Let $0 \neq z \in \mathscr{Z}(\mathscr{A})$. Substituting bz for b in (54), we have $\psi(abz) - bza \in \mathscr{Z}(\mathscr{A})$, this further implies $\psi(ab)\psi(z) + \psi(ab)z + ab\xi(z) - bza \in \mathscr{Z}(\mathscr{A})$, that is $\psi(ab)\psi(z) + ab\xi(z) + (\psi(ab) - ba)z \in \mathscr{Z}(\mathscr{A})$. By using (54) in the above expression, we get

$$(\forall a, b \in \mathscr{A}): \quad \psi(ab)\psi(z) + ab\xi(z) \in \mathscr{Z}(\mathscr{A}). \tag{57}$$

Adding $-ba\psi(z)$ and $ba\psi(z)$ to $\psi(ab)\psi(z) + ab\xi(z)$, we get

$$(\forall a, b \in \mathscr{A}): \quad (\psi(ab) - ba)\psi(z) + ba\psi(z) + ab\xi(z) \in \mathscr{Z}(\mathscr{A}). \tag{58}$$

Using (54) in (58), we obtain $[ba\psi(z) + ab\xi(z), \psi(z)] = 0$. In particular, for $b = z$ in the above expression, we infer that

$$(\forall a \in \mathscr{A}): \quad [a(\psi(z) + \xi(z)), \psi(z)] = 0. \tag{59}$$

Again, putting $a = z$ in (59), we find that $[\psi(z) + \xi(z), \psi(z)] = 0$, and by using the last expression in (59), we conclude that $[a, \psi(z)](\psi(z) + \xi(z)) = 0$. Writing ab instead of a in the previous expression and using it, we have $[a,\psi(z)]b(\psi(z)+\xi(z))=0$, and so $[a,\psi(z)]=0$ or $\psi(z)+\xi(z)=0$.

Case (1). If $\psi(z)+\xi(z)=0$, then $\psi(z)=-\xi(z)$. By using this equality in (57), we get $(-\psi(ab)+ab)\xi(z)\in\mathscr{Z}(\mathscr{A})$, which implies that

$$(\forall a,b\in\mathscr{A}):\quad (\psi(ab)-ab)\xi(z)\in\mathscr{Z}(\mathscr{A}). \tag{60}$$

Putting $b=z$ in (54), we obtain $\psi(az)-za\in\mathscr{Z}(\mathscr{A})$, it follows that

$$\psi(a)\psi(z)+\psi(a)z+a\xi(z)-za\in\mathscr{Z}(\mathscr{A}).$$

Applying of the equality $\psi(z)=-\xi(z)$ in the above relation, we see that $-\psi(a)\xi(z)+\psi(a)z+a\xi(z)-za\in\mathscr{Z}(\mathscr{A})$, that is $\psi(a)\xi(z)-\psi(a)z-a\xi(z)+za\in\mathscr{Z}(\mathscr{A})$, and hence $(\psi(a)-a)\xi(z)-(\psi(a)-a)z\in\mathscr{Z}(\mathscr{A})$. Taking ab in place of a in the above expression, we find that $(\psi(ab)-ab)\xi(z)-(\psi(ab)-ab)z\in\mathscr{Z}(\mathscr{A})$. By using (60) in the above relation, we conclude that $-(\psi(ab)-ab)z\in\mathscr{Z}(\mathscr{A})$, that is $\psi(ab)-ab\in\mathscr{Z}(\mathscr{A})$. Comparing the previous expression and (54), we have $[a,b]\in\mathscr{Z}(\mathscr{A})$, and by Lemma 2, we obtain $\mathscr{A}$ is commutative.

Case (2). If $[a,\psi(z)]=0$, then $\psi(z)\in\mathscr{Z}(\mathscr{A})$. Using this condition in (58), we get $ba\psi(z)+ab\xi(z)\in\mathscr{Z}(\mathscr{A})$. In particular, for $b=z$ in the above relation, we obtain

$$(\forall a\in\mathscr{A}):\quad a(\psi(z)+\xi(z))\in\mathscr{Z}(\mathscr{A}). \tag{61}$$

Again, taking $a=z$ in (61), we conclude that $\psi(z)+\xi(z)\in\mathscr{Z}(\mathscr{A})$. By using the condition $\psi(z)\in\mathscr{Z}(\mathscr{A})$ in the last expression, we infer that $\xi(z)\in\mathscr{Z}(\mathscr{A})$. Application of the last relation and the condition $\psi(z)\in\mathscr{Z}(\mathscr{A})$ in (61) and by applying Lemma 1, we find that $a\in\mathscr{Z}(\mathscr{A})$ or $\psi(z)+\xi(z)=0$. If $a\in\mathscr{Z}(\mathscr{A})$ ($\forall\ a\in\mathscr{A}$), then $\mathscr{A}$ is commutative. If $\psi(z)+\xi(z)=0$, then $\psi(z)=-\xi(z)$, now, the same as in Case (1), we obtain that $\mathscr{A}$ is commutative.

(ii) The same as in (i). □

We provide an example to show that Corollaries 1, 4, and 5 do not hold if ψ is not zero-power valued on $\mathscr{A}$.

Example 1. Let $\mathscr{A}=M_2(\mathbb{Z}_2)$, $\psi(a)=\pm a$, $\xi(a)=-a$. Moreover, all the conditions, except "ψ is zero-power valued on $\mathscr{A}$," of our corollaries are satisfied, but $\mathscr{A}$ is non-commutative. That is the condition "ψ is zero-power valued on $\mathscr{A}$," in our corollaries is essential.

Remark 1. Let $\mathscr{A}$, ψ, and ξ as in Example 1. Moreover, all the conditions, except "char$(\mathscr{A})\neq 2$" of Corollary 2 are satisfied, but $\mathscr{A}$ is non-commutative. That is the condition "char$(\mathscr{A})\neq 2$" in Corollary 2 is essential.

We provide an example to show that all theorems and all corollaries do not hold if a ring is not prime.

Example 2. Let $\mathscr{A}=\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix} : a,b,c\in\mathbb{Z}\right\}$. Define $\psi\colon\mathscr{A}\to\mathscr{A}$ by

$$\psi\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&c\\0&0&0\\0&0&0\end{pmatrix},$$

and $\xi\colon \mathscr{A} \to \mathscr{A}$ by $\xi = 0$. Clearly ψ is a generalized homoderivation with associated homoderivation ξ. The ring $\mathscr{A}$ is not prime and its elements of the form $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b \in \mathbb{Z}$, are in its centre. Moreover, all the conditions, except primeness of all theorems and all corollaries are satisfied but $\mathscr{A}$ is non-commutative. That is the condition of primeness in all theorems and all corollaries is essential.

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