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ON COMPACT CLASSES OF SOLUTIONS OF DIRICHLET PROBLEM IN SIMPLY CONNECTED DOMAINS

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The article is devoted to compactness of solutions of the Dirichlet problem for the Beltrami equation in some simply connected domain. In terms of prime ends, we have proved corresponding results for the case when the maximal dilatations of these solutions satisfy certain integral constraints. The first section is devoted to a presentation of well-known definitions that are necessary for the formulation of the main results. In particular, here we have given a definition of a prime end corresponding to Näkki's concept. The research tool that was used to establish the main results is the method of moduli for families of paths. In this regard, in the second section we study mappings that satisfy upper bounds for the distortion of the modulus, and in the third section, similar lower bounds. The main results of these two sections include the equicontinuity of the families of mappings indicated above, which is obtained under integral restrictions on those characteristics. The proof of the main theorem is done in the fourth section and is based on the well-known Stoilow factorization theorem. According to this, an open discrete solution of the Dirichlet problem for the Beltrami equation is a composition of some homeomorphism and an analytic function. In turn, the family of these homeomorphisms is equicontinuous (Section 2). At the same time, the equicontinuity of the family of corresponding analytic functions in composition with some (auxiliary) homeomorphisms reduces to using the Schwartz formula, as well as the equicontinuity of the family of corresponding inverse homeomorphisms (Section 3).

1. Introduction. In our recent joint publication [22], we proved the compactness theorem of the classes of solutions of the Dirichlet problem for the Beltrami equation in a simply connected Jordan domain whose characteristics satisfy the constraints of the integral type. In this article we are talking about solutions defined in an arbitrary simply connected domain. Since such domains do not have to be Jordanian, this is a slight relaxation of conditions compared to [22]. Note that, such solutions of the Dirichlet problem exist ai this case (see, e.g., [15]). These solutions, generally speaking, do not have a homeomorphic extension to the boundary of the domain in the usual sense, since simply connected domains do not have to be locally connected on their boundary. However, such an extension exists in terms of the so-called prime ends.

Let D be a domain in $\mathbb{C}$. In what follows, a mapping $f : D \to \mathbb{C}$ is assumed to be *sense-preserving,* moreover, we assume that f has partial derivatives almost everywhere.

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Put

$$f_{\overline{z}} = \frac{(f_x + if_y)}{2}, \qquad f_z = \frac{(f_x - if_y)}{2}.$$

The *complex dilatation* of f at $z \in D$ is defined as follows

$$\mu(z) = \mu_f(z) := \begin{cases} \frac{f_{\overline{z}}}{f_z}, & \text{for } f_z \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

The *maximal dilatation* of f at z is the following function

$$K_{\mu}(z) = K_{\mu_f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{1}$$

Note that the Jacobian of f at $z \in D$ is calculated by the formula

$$J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2.$$

Since we assume that the map f is sense preserving, the Jacobian of this map is nonnegative at all points of its differentiability.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mu : D \to \mathbb{D}$ be a Lebesgue measurable function. Without reference to some mapping f, we define the *maximal dilatation* corresponding to its complex dilatation μ by (1).

It is easy to see that

$$K_{\mu_f}(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}$$

whenever partial derivatives of f exist at $z \in D$ and, in addition, $J(z, f) \neq 0$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$ and ω be an open set in $\mathbb{R}^k$, $k \in \{1, \ldots, n-1\}$. Recall some definitions (see, for example, [6], [8]).

A continuous mapping $\sigma \colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* in $\mathbb{R}^n$. A *surface* is an arbitrary $(n-1)$-dimensional surface σ in $\mathbb{R}^n$. A surface σ is called *a Jordan surface*, if $\sigma(x) \neq \sigma(y)$ for $x \neq y$. Below we will use σ instead of $\sigma(\omega) \subset \mathbb{R}^n$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$ and $\partial\sigma$ instead of $\overline{\sigma(\omega)} \setminus \sigma(\omega)$.

A Jordan surface $\sigma \colon \omega \to D$ is called a *cut* of D, if σ separates D, that is $D \setminus \sigma$ has more than one component, $\partial\sigma \cap D = \varnothing$ and $\partial\sigma \cap \partial D \neq \varnothing$.

A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in D is called *a chain*, if:

(i) the set σ_{m+1} is contained in exactly one component d_m of the set $D \setminus \sigma_m$, wherein $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$;

(ii) $\bigcap\limits_{m=1}^{\infty} d_m = \varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent*, if for each $m \in \{1, 2, \ldots\}$ the domain d_m contains all the domains d'_k, except for a finite number, and for each $k \in \{1, 2, \ldots\}$ the domain d'_k also contains all domains d_m, except for a finite number.

The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{R}^n$, then the set $I(K) = \bigcap\limits_{m=1}^{\infty} \overline{d_m}$ is called *the impression of the end* K. Let us to show that

$$I(K) = \bigcap_{m=1}^{\infty} \overline{d_m} \subset \partial D. \tag{2}$$

Let $\xi_0 \in I(K)$. Observe that $\xi_0 \in \partial d_m$ for any $m \geqslant m_0$ and some $m_0 \in \mathbb{N}$.

Indeed, in the contrary case, $\xi_0 \in d_{m_k}$ for some increasing sequence m_k, $k \in \{1, 2, \ldots\}$. Now $\xi_0 \in \bigcap\limits_{k=1}^{\infty} d_{m_k}$ and, since $d_m \supset d_n$ for $m < n$, we have $\xi_0 \in \bigcap\limits_{m=1}^{\infty} d_m$, as well. The latter contradicts with the definition (ii) $\bigcap\limits_{m=1}^{\infty} d_m = \varnothing$. Thus, $\xi_0 \in \partial d_m$ for any $m \geqslant m_0$. By the same reasons, $\xi_0 \in \partial d_m$ for any $m \in \mathbb{N}$, not only for $m \geqslant m_0$. On the other hand, by the definition of domains d_m, we have that $\partial d_m \cap D \subset \sigma_m$ for any $m \in \mathbb{N}$. Since $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$, we have that $\sigma_m \cap \sigma_n = \varnothing$ for $m \neq n$. It follows from this, that the situation $\xi_0 \in \partial d_{l_0} \cap D$ is possible at most for one $l_0 \in \mathbb{N}$. Thus, $\xi_0 \in \partial d_m \cap \partial D$ for any $m \in \{1, 2, \ldots\}$. Therefore, the relation (2) holds, as required.

Throughout the paper, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma\colon [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for every $t \in [a, b]$.

In what follows, M denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$, see [26]. Following [13], we say that the end K is *a prime end*, if K contains a chain of cuts $\{\sigma_m\}$ such that $\lim\limits_{m\to\infty} M(\Gamma(C, \sigma_m, D)) = 0$ for some continuum C in D.

In the following, the following notation is used: the set of prime ends corresponding to the domain D, is denoted by E_D, and the completion of the domain D by its prime ends is denoted $\overline{D}_P$.

Consider the following definition, which goes back to Näkki [13]. We say that the boundary of the domain D in $\mathbb{R}^n$ is *locally quasiconformal*, if each point $x_0 \in \partial D$ has a neighborhood U in $\mathbb{R}^n$, which can be mapped by a quasiconformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ so that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ with the coordinate hyperplane.

For the sets $A, B \subset \mathbb{R}^n$ we set, as usual,

$$\operatorname{diam} A = \sup_{x,y\in A} |x - y|\,, \quad \operatorname{dist}(A, B) = \inf_{x\in A, y\in B} |x - y|.$$

Sometimes we also write $d(A)$ instead of $\operatorname{diam} A$ and $d(A, B)$ instead of $\operatorname{dist}(A, B)$, if no misunderstanding is possible.

The sequence of cuts σ_m, $m \in \{1, 2, \ldots\}$, is called *regular,* if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \varnothing$ for $m \in \mathbb{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end K contains at least one regular chain, then K will be called *regular*.

We say that a bounded domain D in $\mathbb{R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Note that space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g : D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|\,, \tag{3}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0. The specified boundary point is unique and well-defined, see e.g. [6, Theorem 2.1, Remark 2.1], cf. [13, Theorem 4.1]. It is easy to verify that ρ in (3) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map g with the indicated property.

We say that a sequence $x_m \in D$, $m \in \{1, 2, \ldots\}$, converges to a prime end of $P \in E_D$ as $m \to \infty$, write $x_m \to P$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements x_m belong to d_k except for a finite number. Here d_k denotes a sequence of nested domains corresponding to the definition of the prime end P. Note that for a homeomorphism of a domain D onto D', the end of the domain D uniquely corresponds to some sequence of nested domains in the image under the mapping.

Consider the following Cauchy problem:

$$f_{\overline{z}} = \mu(z)\cdot f_z\,, \tag{4}$$

$$\forall\ P\in E_D\colon\quad \lim_{\zeta\to P}{\rm Re}\,f(\zeta)=\varphi(P), \tag{5}$$

where $\varphi : E_D \to \mathbb{R}$ is a predefined continuous function. In what follows, we assume that D is some simply connected domain in $\mathbb{C}$.

The solution of the problem (4)–(5) is called *regular,* if one of two conditions is fulfilled: or $f(z) = const$ in D, or f is an open discrete $W_{\rm loc}^{1,1}(D)$-mapping such that $J(z, f) \neq 0$ for almost any $z \in D$.

Given $z_0 \in D$, a function $\varphi : E_D \to \mathbb{R}$, a function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ and a function $\mathcal{M}(\Omega)$ of open sets $\Omega \subset D$, we denote by $\mathfrak{F}^{\mathcal{M}}_{\varphi,\Phi,z_0}(D)$ the class of all regular solutions $f : D \to \mathbb{C}$ of the Cauchy problem (4)–(5) that satisfy the condition ${\rm Im}\,f(z_0) = 0$ and, in addition,

$$\int\limits_{\Omega}\Phi(K_{\mu}(z))\cdot\frac{dm(z)}{(1+|z|^2)^2}\leqslant \mathcal{M}(\Omega) \tag{6}$$

for any open set $\Omega \subset D$. The following statement generalizes [4, Theorem 2] to the case of arbitrary simply connected domains.

Theorem 1. *Let D be some simply connected domain in $\mathbb{C}$, and let $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a continuous increasing convex function which satisfies the condition*

$$\int\limits_{\delta}^{\infty}\frac{d\tau}{\tau\Phi^{-1}(\tau)}=\infty$$

for some $\delta > \Phi(0)$. Assume that the function $\mathcal{M}$ is bounded, and the function φ in (5) is continuous. Then the family $\mathfrak{F}^{\mathcal{M}}_{\varphi,\Phi,z_0}(D)$ is compact in D.

2. Convergence theorems for mappings with upper estimates for modulus distortion. The proof of the main result is based on the theorems on the global behavior of mappings satisfying the weight Poletsky inequality. Results of a similar type in some other situations have been obtained earlier, see, for example, [23]. The case considered below concerns regular domains and mappings with one normalization condition. This case is considered for the first time in this degree of generality.

Given $p \geqslant 1$, M_p denotes the p-modulus of a family of paths, and the element $dm(x)$ corresponds to a Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$, see [26]. In what follows, we usually write $M(\Gamma)$ instead of $M_n(\Gamma)$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0,r)=\{x\,\in\,\mathbb{R}^n: |x-x_0|=r\}\,, S_i=S(x_0,r_i)\,,\quad i\in\{1,2\},$$
$$A=A(x_0,r_1,r_2)=\{x\,\in\,\mathbb{R}^n: r_1<|x-x_0|<r_2\}. \tag{7}$$

Everywhere below, unless otherwise stated, the closure $\overline{A}$ and the boundary ∂A of the set A are understood in the topology of the space $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$. Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function satisfying the condition $Q(x)\equiv 0$ for $x\in{\Bbb R}^n\setminus D$, and let $p\geqslant 1$. Given sets E and F and a given domain D in $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[0,1]\rightarrow \overline{{\Bbb R}^n}$ joining E and F in D, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0,1)$. According to [11, Chap. 7.6], a mapping $f:D\rightarrow \overline{{\Bbb R}^n}$ is called a *ring Q-mapping at the point $x_0\in \overline{D}\setminus\{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D)))\leqslant \int\limits_{A\cap D} Q(x)\cdot \eta^p(|x-x_0|)\,dm(x) \tag{8}$$

holds for all $0<r_1<r_2< d_0:=\sup\limits_{x\in D}|x-x_0|$ and all Lebesgue measurable functions $\eta:(r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1. \tag{9}$$

A mapping $f:D\rightarrow \overline{{\Bbb R}^n}$ is called a *ring Q-mapping in $\overline{D}\setminus\{\infty\}$ with respect to p-modulus* if (8) holds for any $x_0\in \overline{D}\setminus\{\infty\}$. This definition can also be applied to the point $x_0=\infty$ by inversion: $\varphi(x)=\frac{x}{|x|^2}$, $\infty\mapsto 0$. In what follows, h denotes the so-called chordal metric defined by the equalities

$$h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \quad x\ne\infty\ne y\,, \quad h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}. \tag{10}$$

For a given set $E\subset\overline{{\Bbb R}^n}$, we set

$$h(E):=\sup\limits_{x,y\in E}h(x,y)\,, \tag{11}$$

The quantity $h(E)$ in (11) is called the *chordal diameter* of the set E.

For given sets $A, B\subset \overline{{\Bbb R}^n}$, we put

$$h(A,B)=\inf\limits_{x\in A, y\in B}h(x,y),$$

where h is a chordal metric defined in (10).

Let I be a fixed set of indices and let D_i, $i\in I$, be some sequence of domains. Following [14, Sect. 2.4], we say that a family of domains $\{D_i\}_{i\in I}$ is *equi-uniform with respect to p-modulus* if for any $r>0$ there exists a number $\delta>0$ such that the inequality

$$M_p(\Gamma(F^*, F, D_i))\geqslant \delta \tag{12}$$

holds for any $i\in I$ and any continua $F, F^*\subset D$ such that $h(F)\geqslant r$ and $h(F^*)\geqslant r$.

Given a Lebesgue measurable function $Q:{\Bbb R}^n\rightarrow [0,\infty]$ and a point $x_0\in {\Bbb R}^n$ we set

$$q_{x_0}(t)=\frac{1}{\omega_{n-1}t^{n-1}}\int\limits_{S(x_0, t)}Q(x)\,d\mathcal{H}^{n-1}\,, \tag{13}$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. The following lemma was proved in [21, Lemma 2.1].

Lemma 1. *Let $1 \leqslant p \leqslant n$, and let $\Phi : [0, \infty] \to [0, \infty]$ be a strictly increasing convex function such that the relation*

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{p-1}}} = \infty \tag{14}$$

holds for some $\delta_0 > \tau_0 := \Phi(0)$. Let $\mathfrak{Q}$ be a family of functions $Q : \mathbb{R}^n \to [0, \infty]$ such that

$$\int\limits_{D} \Phi(Q(x)) \frac{dm(x)}{(1+|x|^2)^n} \leqslant M_0 < \infty \tag{15}$$

for some $0 < M_0 < \infty$. Now, for any $0 < r_0 < 1$ and for every $\sigma > 0$ there exists $0 < r_* = r_*(\sigma, r_0, \Phi) < r_0$ such that

$$\int\limits_{\varepsilon}^{r_0} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)} \geqslant \sigma\,, \qquad \varepsilon \in (0, r_*)\,,$$

for any $Q \in \mathfrak{Q}$.

Consider some another auxiliary family of mappings.

For $p \geqslant 1$, a given number $0 < M_0 < \infty$, a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, and a strictly increasing convex function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ denote by $\mathfrak{A}_{\Phi,p,M_0}(D)$ the family all open discrete mappings $f : D \to \mathbb{R}^n$ satisfying relations (8)–(9) with some $Q = Q_f$ in D with respect to p-modulus. The following statement is true (see [21, Theorem 1.2]).

Lemma 2. *Let $p \in (n-1, n)$, and let $\delta_0 > \tau_0 := \Phi(0)$ be such that the condition (14) holds. Now the family $\mathfrak{A}_{\Phi,p,M_0}(D)$ is equicontinuous in D.*

Here the equicontinuity of the family of mappings $\mathfrak{A}_{\Phi,p,M_0}(D)$ should be understood with respect to the spaces (D, d) and $(\mathbb{R}^n, d)$, where d is the Euclidean metric.

Given $p \geqslant 1$, numbers $\delta > 0$, $0 < M_0 < \infty$, a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, a point $a \in D$ and a strictly increasing convex function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ denote by $\mathfrak{F}_{\Phi,a,p,\delta,M_0}(D)$ the family of all homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ satisfying (8)–(9) in $\overline{D}$ for some $Q = Q_f$ such that

$$h(f(a), \partial f(D)) \geqslant \delta, \quad h(\overline{\mathbb{R}^n} \setminus f(D)) \geqslant \delta$$

and, in addition, (15) holds.

Theorem 2. *Let $p \in (n-1, n]$, let D be regular, and let $D_f^{\prime} = f(D)$ be bounded domains with a locally quasiconformal boundary which are equi-uniform with respect to p-modulus over all $f \in \mathfrak{F}_{\Phi,a,p,\delta,M_0}(D)$. If there is $\delta_0 > \tau_0 := \Phi(0)$ such that (14) holds, then any $f \in \mathfrak{F}_{\Phi,a,p,\delta,M_0}(D)$ has a continuous extension $\overline{f} : \overline{D}_P \to \overline{\mathbb{R}^n}$ and, in addition, the family $\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$ of all extended mappings $\overline{f} : \overline{D}_P \to \overline{\mathbb{R}^n}$ is equicontinuous in $\overline{D}_P$.*

Remark 1. In Theorem 2, the equicontinuity should be understood in the sense of mappings acting between the spaces (X, d) and $(X^{\prime}, d^{\prime})$, where $X = \overline{D}_P$ is the replenishment of the domain D by its prime ends, and d is one of the possible metrics that correspond to the topological space $\overline{D}_P$ in (3). In addition, $X^{\prime} = \overline{\mathbb{R}^n}$ and $d^{\prime} = h$ is a chordal (spherical) metric.

An example of a family of plane mappings $f_n(z) = z^n$, $n \in \{1, 2, \ldots\}$, $z \in \mathbb{D}$, indicates the inaccuracy of Theorem 2 for mappings with branching, in particular, this theorem is not true under the normalization condition $f_n(0) = 0$, $n \in \{1, 2, \ldots\}$.

Proof of Theorem 2. Put $f\in\mathfrak{F}_{\Phi,A,p,\delta,M_0}(D)$ and $Q=Q_f(x)$. Given $x\in\mathbb{R}^n$ we set

$$Q'(x)=\begin{cases}Q(x), & x\in D, Q(x)\geqslant 1;\\ 1, & x\in D, Q(x)<1;\\ 1, & x\notin D.\end{cases}$$

Observe that the function $Q'(x)$ satisfies the relation (15) up to some constant. Indeed,

$$\int\limits_D\Phi(Q'(x))\frac{dm(x)}{(1+|x|^2)^n}=\int\limits_{\{x\in D:Q(x)<1\}}\Phi(Q'(x))\frac{dm(x)}{(1+|x|^2)^n}+$$

$$+\int\limits_{\{x\in D:Q(x)\geqslant 1\}}\Phi(Q'(x))\frac{dm(x)}{(1+|x|^2)^n}\leqslant M_0+\Phi(1)\int\limits_{\mathbb{R}^n}\frac{dm(x)}{(1+|x|^2)^n}=M_0'<\infty.$$

In this case, by Lemma 1

$$\int\limits_{\varepsilon}^{r_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\prime\frac{1}{p-1}}(t)}\rightarrow\infty \tag{16}$$

as $\varepsilon\rightarrow 0$ for any $0<r_0<1$ and $\varepsilon\rightarrow 0$, where $q_{x_0}'(t)=\frac{1}{\omega_{n-1}t^{n-1}}\int\limits_{S(x_0,t)}Q'(x)\,d\mathcal{H}^{n-1}$. Besides that,

$$\int\limits_{\varepsilon}^{r_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\prime\frac{1}{p-1}}(t)}<\infty$$

for any $\varepsilon\in(0,r_0)$, because $q_{x_0}'(t)\geqslant 1$ for almost any $t\in(0,r_0)$. Observe that, equi-uniform domains with respect to p-modulus have strongly accessible boundaries with respect to p-modulus, as well (see [23, Remark 1]). In this case, the condition (16) directly implies that $f\in\mathfrak{F}_{\Phi,A,p,\delta,M_0}(D)$ has a continuous extension to $\overline{D}_P$ (see [20, Theorem 3]).

Observe that $f\in\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$ does not equal to infinity for $p\neq n$ (see, e.g., [5, Lemmas 2.6 and 3.1]). Now, the equicontinuity of $\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$ inside D follows by Theorem 4.1 in [17] for $p=n$ and Lemma 2 for $p<n$.

We prove the equicontinuity of the family $\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$ in $E_D:=\overline{D}_P\setminus D$. Let us assume the opposite, namely that there are $\varepsilon_*>0$, $P_0\in E_D$, a sequence $x_m\in\overline{D}_P$, $x_m\rightarrow P_0$ as $m\rightarrow\infty$, and a mapping $f_m\in\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$ such that

$$h(f_m(x_m),f_m(P_0))\geqslant\varepsilon_*\,,\quad m\in\{1,2,\ldots\}. \tag{17}$$

Since f_m as a continuous extension at P_0, we may assume that $x_m\in D$ and, in addition, there is a sequence $x_m'\in\overline{D}_P$, $x_m'\rightarrow P_0$ as $m\rightarrow\infty$, such that

$$h(f_m(x_m),f_m(x_m'))\geqslant\varepsilon_*/2\,,\quad m\in\{1,2,\ldots\}. \tag{18}$$

Let d_m, $m\in\{1,2,\ldots\}$, be a sequence of cuts σ_m corresponding to P_0. By [8, Lemma 2] the sequence σ_m may be chosen such that $\sigma_m\subset S(x_0,r_m)$, where $x_0\in\partial D$ and $r_m\rightarrow 0$ as $m\rightarrow\infty$.

By the definition of the convergence of the sequence x_m to P_0 as $m\rightarrow\infty$, there exists $m_1\in\mathbb{N}$ such that $x_{m_1}\in d_1$. Similarly, there exists $m_2>m_1$ such that $x_{m_2}\in d_2$. Etc. Given

$k \in N$ we may find $m_k > m_{k-1}$ such that $x_{m_k} \in d_k$. Etc. Thus, $x_{m_k} \in d_k$, $k \in \mathbb{N}$. Relabeling the sequence x_{m_k} (if necessary), we may consider that $x_m \in d_m$ for any $m \in \mathbb{N}$. Similarly, me may assume that $x'_m \in d_m$, $m \in \{1, 2, \ldots\}$.

Since the domain D is regular, the space $\overline{D}_P$ contains at least two prime ends P_1 and $P_2 \in E_D$. Let $P_1 \subset E_D$ be a prime end that does not coincide with P_0. Suppose that G_m, $m \in \{1, 2, \ldots\}$, is a sequence of domains that corresponds to a prime end P_1. Since the mapping f_m has a continuous extension on $\overline{D}_P$ for any $m \in \{1, 2, \ldots\}$, we may choose a sequence $\zeta_m \in G_m$, $\zeta_m \to P_1$ as $m \to \infty$, such that $h(f_m(\zeta_m), f_m(P_1)) \to 0$ as $m \to \infty$. Note that

$$h(f_m(a), f_m(\zeta_m)) \geqslant h(f_m(a), f_m(P_1)) - h(f_m(\zeta_m), f_m(P_1)) \geqslant \delta/2\,, \tag{19}$$

for any $m \geqslant m_0$ and some $m_0 \in \mathbb{N}$. We construct a sequence of continua K_m, $m \in \{1, 2, \ldots\}$ as follows. We join the points ζ_1 and a by an arbitrary path in D, which we denote by K_1. Next, we join the points ζ_2 and ζ_1 by a path K'_1, in G_1. Combining the paths K_1 and K'_1, we obtain a path K_2, joining the points a and ζ_2. And so on. Suppose that at some step we have a path K_m, that join the points ζ_m and a. Join the points ζ_{m+1} and ζ_m with a path K'_m, which lies in G_m. Combining the paths K_m and K'_m, we obtain a path K_{m+1}. We show that there is a number $m_1 \in \mathbb{N}$ such that

$$\forall\ m \geqslant m_1\colon \qquad d_m \cap K_m = \varnothing. \tag{20}$$

We prove this from the opposite, namely, suppose that (20) does not hold. Then there is an increasing sequence of numbers $m_k \to \infty$, $k \to \infty$, and points $\xi_k \in K_{m_k} \cap d_{m_k}$, $m \in \{1, 2, \ldots\}$. Then $\xi_k \to P_0$ as $k \to \infty$.

Note that two cases are possible: either all elements ξ_k belong to $D \setminus G_1$ for $k \in \{1, 2, \ldots\}$, or there is a number k_1 such that $\xi_{k_1} \in G_1$. In the second case, consider the sequence ξ_k, $k > k_1$. Note that two cases are possible: or ξ_k for $k > k_1$ belong to $D \setminus G_2$, or there is $k_2 > k_1$ such that $\xi_{k_2} \in G_2$. In the second case, consider the sequence ξ_k, $k > k_2$, and so on. Assume that the element $\xi_{k_{l-1}} \in G_{l-1}$ is already constructed. Note that two cases are possible: either ξ_k belong to $D \setminus G_l$ for $k > k_{l-1}$, or there is a number $k_l > k_{l-1}$ such that $\xi_{k_l} \in G_l$, and etc. This procedure can be both finite or infinite, depending on which we have two possible situations:

1) or there are numbers $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{N}$ such that that $\xi_k \in D \setminus G_{n_0}$ for all $k > l_0$;

2) or for each there is an element ξ_{k_l} such that $\xi_{k_l} \in G_l$, and the sequence k_l is increasing by $l \in \mathbb{N}$.

Consider each of these cases separately and show that in both of them we come to a contradiction. Let situation 1) holds. Observe that all elements of the sequence ξ_k belong to K_{n_0}, hence there exists a subsequence ξ_{k_r}, $r \in \{1, 2, \ldots\}$, convergent as $r \to \infty$ to some point $\xi_0 \in D$. However, $\xi_k \in d_{m_k}$, i.e., $\xi_0 \in \bigcap\limits_{m=1}^{\infty} \overline{d_m}$. Due to (2), $\xi_0 \in \partial D$. The obtained contradiction indicates the impossibility of the case 1). Suppose that case 2) holds, then simultaneously $\xi_k \to P_0$ and $\xi_k \to P_1$ as $k \to \infty$. Since space $\overline{D}_P$ is metric with a metric ρ in (3), by the triangle inequality it follows that $P_1 = P_0$, which contradicts the choice of P_1. The obtained contradiction indicates the validity of the relation (20).

By the relation (20) and by the definition of cuts $\sigma_m \subset S(x_0, r_m)$, we obtain that

$$\Gamma\left(|\gamma_m|, K_m, D\right) > \Gamma(S(x_0, r_m), S(x_0, \widetilde{\varepsilon_0}), D)\,, \qquad m \geqslant 2\,,$$

where $\widetilde{\varepsilon_0}$ is some positive number which may be chosen as $\widetilde{\varepsilon_0} := r_1$. Thus

$$f_m(\Gamma\left(|\gamma_m|, K_m, D\right)) > f_m(\Gamma(S(x_0, r_m), S(x_0, \widetilde{\varepsilon_0}), D))\,,$$

whence, by the definition of the class $\mathfrak{F}_{\Phi,a,p,\delta,M_0}(\overline{D})$

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) \leqslant$$
$$\leqslant M_p(f_m(\Gamma(S(x_0, r_m), S(x_0, \widetilde{\varepsilon_0}), D)) \leqslant \int\limits_{A\cap D} Q_m(x)\cdot\eta^p(|x-x_0|)\,dm(x)\,, \tag{21}$$

where η is any Lebesgue measurable function satisfying (9) for $r_1 \mapsto r_m$ and $r_2 \mapsto \widetilde{\varepsilon_0}$, in addition, $Q_m := Q_{f_m}$ corresponds to the function Q in (8). Let us to prove the inequality

$$M_p(f_m(\Gamma(S(x_0, r_m), S(x_0, \widetilde{\varepsilon_0}), D)) \leqslant \frac{\omega_{n-1}}{I_m^{p-1}}\,, \tag{22}$$

where $I_m = \int\limits_{r_m}^{\widetilde{\varepsilon_0}} \frac{dr}{r^{\frac{n-1}{p-1}} q_{mx_0}^{\frac{1}{p-1}}(r)}$, $q_{mx_0}(t) = \frac{1}{\omega_{n-1}t^{n-1}} \int\limits_{S(x_0,t)} Q_m(x)\,d\mathcal{H}^{n-1}$ and $Q_m := Q_{f_m}$ (we set $Q_m(x) \equiv 1$ for $x \notin D$). To do this, we will reason similarly to the proof of Lemma 1 in [18]. We may assume that $I \neq 0$, since (22) is obviously in this case. We may also assume that $I \neq \infty$, because otherwise we may consider $Q(x)+\delta$ instead of $Q(x)$ in (22), and then go to the limit as $\delta \to 0$. Let $0 \neq I \neq \infty$. Then $q_{x_0}(r) \neq 0$ for $r \in (r_m, \widetilde{\varepsilon_0})$. Put

$$\psi(t) = \begin{cases} 1/[t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)], & t \in (r_m, \widetilde{\varepsilon_0});\\ 0, & t \notin (r_m, \widetilde{\varepsilon_0}).\end{cases}$$

By the Fubini theorem

$$\int\limits_{A} Q_m(x)\cdot\psi^p(|x-x_0|)\,dm(x) = \omega_{n-1} I_m\,, \tag{23}$$

where $A = A(r_m, \widetilde{\varepsilon_0}, x_0)$ is defined in (7). Observe that the function $\eta_1(t) = \psi(t)/I$, $t \in (r_m, \widetilde{\varepsilon_0})$, satisfies (9). Now, by (8) and (23) we obtain relation (22), as required.

Finally, from (21), (22) and from Lemma 1 it follows that

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_m}^{\widetilde{\varepsilon_0}} \frac{dr}{r^{\frac{n-1}{p-1}} q_{mx_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}} \to 0\,, \quad m \to \infty\,, \tag{24}$$

where $q_{mx_0}(t) = \frac{1}{\omega_{n-1}t^{n-1}} \int\limits_{S(x_0,t)} Q_m(x)\,d\mathcal{H}^{n-1}$ and $Q_m := Q_{f_m}$ corresponds to the function Q in (13). The relation (21) contradicts the equi-uniformity of the sequence of domains $D_m^{\prime} := f_m(D)$. Indeed,

$$h(f_m(K_m)) \geqslant \delta/2$$

according to (19), and

$$h(f_m(|\gamma_m|)) > \varepsilon_*/2$$

by the relation (18). Hence, since the sequence of domains $D'_m := f_m(D)$ is equi-uniform, we obtain that

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) = M_p(\Gamma(f_m(|\gamma_m|), f_m(K_m), f_m(D))) \geqslant \delta_* > 0$$

for some $\delta_* > 0$ and any $m \in \{1, 2, \ldots\}$, which contradicts the relation (24). The obtained contradiction indicates the incorrectness of the assumption in (17). □

3. Equicontinuity of families of mappings with inverse Poletsky inequality. Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$ and let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma : [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $\gamma(t) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that f *satisfies the inverse Poletsky inequality* at $y_0 \in f(D)$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^n(|y - y_0|)\, dm(y) \tag{25}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \tag{26}$$

Given domains $D, D' \subset \mathbb{R}^n$, points $a \in D$, $b \in D'$ and a number $M_0 > 0$ denote by $\mathfrak{S}_{a,b,M_0}(D, D')$ the family of open discrete and closed mappings f of D onto D' satisfying the relation (25) for some $Q = Q_f$, $\|Q\|_{L^1(D')} \leqslant M_0$ for any $y_0 \in f(D)$ such that $f(a) = b$.

Theorem 3. *Assume that D has a weakly flat boundary, none of the components of which degenerates into a point, and D' is regular. Then any $f \in \mathfrak{S}_{a,b,M_0}(D, D')$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_P$, while $\overline{f}(\overline{D}) = \overline{D'}_P$ and, in addition, the family $\mathfrak{S}_{a,b,M_0}(\overline{D}, \overline{D'})$ of all extended mappings $\overline{f} : \overline{D} \to \overline{D'}_P$ is equicontinuous in $\overline{D}$.*

Theorem 3 was proved in [24, Theorem 7.1] in the case of a fixed function Q (cf. Theorem 4.2 in [22]). The proof of it may be found in [22, Theorem 4.2].

4. Compactness of families of solutions of the Dirichlet problem.

Proof of Theorem 1. In general, we will use the scheme of proving Theorem 1.2 in [22].

I. Let $f_m \in \mathfrak{F}^{\mathcal{M}}_{\varphi,\Phi,z_0}(D)$, $m \in \{1, 2, \ldots\}$. By Stoilow's factorization theorem (see, e.g., [25, 5(III).V]) a mapping f_m has a representation

$$f_m = \varphi_m \circ g_m\,, \tag{27}$$

where g_m is some homeomorphism, and φ_m is some analytic function. By Lemma 1 in [19], the mapping g_m belongs to the Sobolev class $W^{1,1}_{\rm loc}(D)$ and has a finite distortion. Moreover, by [1, (1).C, Ch. I]

$$f_{mz} = \varphi_{mz}(g_m(z))g_{mz}, \qquad f_{m\overline{z}} = \varphi_{mz}(g_m(z))g_{m\overline{z}} \tag{28}$$

for almost all $z \in D$. Therefore, by the relation (28), $J(z, g_m) \neq 0$ for almost all $z \in D$, in addition, $K_{\mu_{f_m}}(z) = K_{\mu_{g_m}}(z)$.

II. We prove that $\partial g_m(D)$ contains at least two points. Suppose the contrary. Then either $g_m(D) = \mathbb{C}$, or $g_m(D) = \mathbb{C} \setminus \{a\}$, where $a \in \mathbb{C}$. Consider first the case $g_m(D) = \mathbb{C}$. By Picard's theorem $\varphi_m(g_m(D))$ is the whole plane, except perhaps one point $\omega_0 \in \mathbb{C}$. On the other hand, for every $m \in \{1, 2, \ldots\}$ the function $u_m(z) := \mathrm{Re}\, f_m(z) = \mathrm{Re}\,(\varphi_m(g_m(z)))$ is continuous on the compact set $\overline{D}$ under the condition (5) by the continuity of φ. Therefore, there exists $C_m > 0$ such that $|\mathrm{Re}\, f_m(z)| \leqslant C_m$ for any $z \in D$, but this contradicts the fact that $\varphi_m(g_m(D))$ contains all points of the complex plane except, perhaps, one. The situation $g_m(D) = \mathbb{C} \setminus \{a\}$, $a \in \mathbb{C}$, is also impossible, since the domain $g_m(D)$ must be simply connected in $\mathbb{C}$ as a homeomorphic image of the simply connected domain D.

Therefore, the boundary of the domain $g_m(D)$ contains at least two points. Then, according to Riemann's mapping theorem, we may transform the domain $g_m(D)$ onto the unit disk $\mathbb{D}$ using the conformal mapping ψ_m. Let $z_0 \in D$ be a point from the condition of the theorem. By using an auxiliary conformal mapping

$$\widetilde{\psi_m}(z) = \frac{z - (\psi_m \circ g_m)(z_0)}{1 - z\overline{(\psi_m \circ g_m)(z_0)}}$$

of the unit disk onto itself we may consider that $(\psi_m \circ g_m)(z_0) = 0$. Now, by (27) we obtain that

$$f_m = \varphi_m \circ g_m = \varphi_m \circ \psi_m^{-1} \circ \psi_m \circ g_m = F_m \circ G_m\,, \quad m \in \{1, 2, \ldots\}\,,$$

where $F_m := \varphi_m \circ \psi_m^{-1}$, $F_m : \mathbb{D} \to \mathbb{C}$, and $G_m = \psi_m \circ g_m$. Obviously, a function F_m is analytic, and G_m is a regular Sobolev homeomorphism in D. In particular, $\mathrm{Im}\, F_m(0) = 0$ for any $m \in \mathbb{N}$.

III. We prove that the L^1-norms of the functions $K_{\mu_{G_m}}(z)$ are bounded from above by some universal positive constant $C > 0$ over all $m \in \{1, 2, \ldots\}$. Indeed, by the convexity of the function Φ in (6) and by [3, Proposition 5, I.4.3], the slope $[\Phi(t) - \Phi(0)]/t$ is a non-decreasing function. Hence there exist constants $t_0 > 0$ and $C_1 > 0$ such that

$$\Phi(t) \geqslant C_1 \cdot t \qquad \forall\, t \in [t_0, \infty). \tag{29}$$

Fix $m \in \mathbb{N}$. By (6) and (29), we obtain that

$$\int\limits_D K_{\mu_{G_m}}(z)\, dm(z) = \int\limits_{\{z \in D: K_{\mu_{G_m}}(z) < t_0\}} K_{\mu_{G_m}}(z)\, dm(z) + \int\limits_{\{z \in D: K_{\mu_{G_m}}(z) \geqslant t_0\}} K_{\mu_{G_m}}(z)\, dm(z) \leqslant$$

$$\leqslant t_0 \cdot m(D) + \frac{1}{C_1} \int\limits_D \Phi(K_{\mu_{G_m}}(z))\, dm(z) \leqslant$$

$$\leqslant t_0 \cdot m(D) + \frac{\sup\limits_{z \in D}(1 + |z|^2)^2}{C_1} \int\limits_D \Phi(K_{\mu_{G_m}}(z)) \cdot \frac{1}{(1 + |z|^2)^2}\, dm(z) \leqslant$$

$$\leqslant t_0 \cdot m(D) + \frac{\sup\limits_{z \in D}(1 + |z|^2)^2}{C_1} \mathcal{M}(D) < \infty\,,$$

because $\mathcal{M}(D)<\infty$ by the assumption of the theorem.

IV. We prove that each map G_m, $m\in\{1,2,\ldots\}$, has a continuous extension to E_D, in addition, the family of extended maps $\overline{G}_m$, $m\in\{1,2,\ldots\}$, is equicontinuous in $\overline{D}_P$. Indeed, as proved in item **III**, $K_{\mu_{G_m}}\in L^1(D)$. By [7, Theorem 3] (see also [9, Theorem 3.1]) each G_m, $m\in\{1,2,\ldots\}$, is a ring Q-homeomorphism in $\overline{D}$ for $Q=K_{\mu_{G_m}}(z)$, where μ is defined in (4), and K_μ my be calculated by the formula (1). Note that the unit disk $\mathbb{D}$ is a uniform domain as a finitely connected flat domain at its boundary with a finite number of boundary components (see, for example, [12, Theorem 6.2 and Corollary 6.8]). Then the desired conclusion follows by Theorem 2.

V. Let us prove that the inverse homeomorphisms G_m^{-1}, $m\in\{1,2,\ldots\}$, have a continuous extension $\overline{G}_m^{-1}$ to $\partial\mathbb{D}$ in terms of prime ends in D, and $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ is equicontinuous in $\overline{\mathbb{D}}$ as a family of mappings from $\overline{\mathbb{D}}$ to $\overline{D}_P$. Since by the item **IV** mappings G_m, $m\in\{1,2,\ldots\}$, are ring $K_{\mu_{G_m}}(z)$-homeomorphisms in D, the corresponding inverse mappings G_m^{-1} satisfy (25) (in this case, D corresponds the unit disk $\mathbb{D}$ in (26), $f\mapsto G_m$, $Q\mapsto K_{\mu_{G_m}}(z)$, and $f(D)\mapsto D$). Since $G_m^{-1}(0)=z_0$ for any $m\in\{1,2,\ldots\}$, the possibility of a continuous extension of G_m^{-1} to $\partial\mathbb{D}$, and the equicontinuity of $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ as mappings $G_m^{-1}:\overline{\mathbb{D}}\to\overline{D}_P$ follow by Theorem 3.

VI. Since, as proved above the family $\{G_m\}_{m=1}^{\infty}$ is equicontinuous in D, by Arzela-Ascoli criterion there exists an increasing subsequence of numbers m_k, $k\in\{1,2,\ldots\}$, such that G_{m_k} converges locally uniformly in D to some continuous mapping $G:D\to\overline{\mathbb{C}}$ as $k\to\infty$ (see, e.g., [26, Theorem 20.4]). By [22, Lemma 2.1], either G is a homeomorphism with values in $\mathbb{R}^n$, or a constant in $\overline{\mathbb{R}^n}$. Let us prove that the second case is impossible. Let us apply the approach used in proof of the second part of Theorem 21.9 in [26]. Suppose the contrary: let $G_{m_k}(x)\to c=const$ as $k\to\infty$. Since $G_{m_k}(z_0)=0$ for all $k\in\{1,2,\ldots\}$, we have that $c=0$. By item **V**, the family of mappings G_m^{-1}, $m\in\{1,2,\ldots\}$, is equicontinuous in $\mathbb{D}$. Then

$$h(z,G_{m_k}^{-1}(0))=h(G_{m_k}^{-1}(G_{m_k}(z)),G_{m_k}^{-1}(0))\to 0$$

as $k\to\infty$, which is impossible because z is an arbitrary point of D. The obtained contradiction refutes the assumption made above. Thus, $G:D\to\mathbb{C}$ is a homeomorphism.

VII. According to **V**, the family of mappings $\{\overline{G}_m^{-1}\}_{m=1}^{\infty}$ is equicontinuous in $\overline{\mathbb{D}}$. By the Arzela-Ascoli criterion (see, e.g., [26, Theorem 20.4]) we may consider that $\overline{G}_{m_k}^{-1}(y)$, $k\in\{1,2,\ldots\}$, converges to some mapping $\widetilde{F}:\overline{\mathbb{D}}\to\overline{D}$ as $k\to\infty$ uniformly in $\overline{\mathbb{D}}$. Let us to prove that $\widetilde{F}=\overline{G}^{-1}$. For this purpose, we show that $G(D)=\mathbb{D}$. Fix $y\in\mathbb{D}$. Since $G_{m_k}(D)=\mathbb{D}$ for every $k\in\{1,2,\ldots\}$, we obtain that $G_{m_k}(x_k)=y$ for some $x_k\in D$. Since D is regular, the metric space $(\overline{D}_P,\rho)$ is compact. Thus, we may assume that $\rho(x_k,x_0)\to 0$ as $k\to\infty$, where $x_0\in\overline{D}_P$. By the triangle inequality and the equicontinuity of $\{\overline{G}_m\}_{m=1}^{\infty}$ in $\overline{D}_P$ (see **IV**), we obtain that

$$|\overline{G}(x_0)-y|=|\overline{G}(x_0)-\overline{G}_{m_k}(x_k)|\leqslant|\overline{G}(x_0)-\overline{G}_{m_k}(x_0)|+|\overline{G}_{m_k}(x_0)-\overline{G}_{m_k}(x_k)|\to 0$$

as $k\to\infty$. Thus, $\overline{G}(x_0)=y$. Observe that $x_0\in D$, because G is a homeomorphism. Since $y\in\mathbb{D}$ is arbitrary, the equality $G(D)=\mathbb{D}$ is proved. In this case, $G_{m_k}^{-1}\to G^{-1}$ locally uniformly in $\mathbb{D}$ as $k\to\infty$ (see, e.g., [16, Lemma 3.1]). Thus, $\widetilde{F}(y)=G^{-1}(y)$ for every $y\in\mathbb{D}$.

Finally, since $\widetilde{F}(y)=G^{-1}(y)$ for any $y\in\mathbb{D}$ and, in addition, $\widetilde{F}$ has a continuous extension on $\partial\mathbb{D}$, due to the uniqueness of the limit at the boundary points we obtain that

$\widetilde{F}(y)=\overline{G}^{-1}(y)$ for $y\in\overline{\mathbb{D}}$. Therefore, we have proved that $\overline{G}_{m_k}^{-1}\to\overline{G}^{-1}$ uniformly in $\overline{\mathbb{D}}$ with as $k\to\infty$ with respect to the metrics ρ in $\overline{D}_P$.

VIII. By **VII,** for $y=e^{i\theta}\in\partial\mathbb{D}$

$$\operatorname{Re}F_{m_k}(e^{i\theta})=\varphi\left(\overline{G}_{m_k}^{-1}(e^{i\theta})\right)\to\varphi\left(\overline{G}^{-1}(e^{i\theta})\right) \tag{30}$$

as $k\to\infty$ uniformly on $\theta\in[0,2\pi)$. Here we have used that φ is continuous in $\overline{D}_P$. Since by the construction $\operatorname{Im}F_{m_k}(0)=0$ for any $k\in\{1,2,\ldots\}$, by the Schwartz formula (see, e.g., [2, relation (66), sect. 6.3, ch. IV]) the analytic function F_{m_k} is uniquely restored by its real part, namely,

$$F_{m_k}(y)=\frac{1}{2\pi i}\int\limits_{S(0,1)}\varphi\left(\overline{G}_{m_k}^{-1}(t)\right)\frac{t+y}{t-y}\cdot\frac{dt}{t}. \tag{31}$$

Set

$$F(y):=\frac{1}{2\pi i}\int\limits_{S(0,1)}\varphi\left(\overline{G}^{-1}(t)\right)\frac{t+y}{t-y}\cdot\frac{dt}{t}. \tag{32}$$

Let $K\subset\mathbb{D}$ be an arbitrary compact set, and let $y\in K$. By (31) and (32) we obtain that

$$|F_{m_k}(y)-F(y)|\leqslant\frac{1}{2\pi}\int\limits_{S(0,1)}|\varphi(\overline{G}_{m_k}^{-1}(t))-\varphi(\overline{G}^{-1}(t))|\left|\frac{t+y}{t-y}\right||dt|. \tag{33}$$

Since K is compact, there is $0<R_0=R_0(K)<\infty$ such that $K\subset B(0,R_0)$. By the triangle inequality $|t+y|\leqslant 1+R_0$ and $|t-y|\geqslant|t|-|y|\geqslant 1-R_0$ for $y\in K$ and any $t\in\mathbb{S}^1$. Thus

$$\left|\frac{t+y}{t-y}\right|\leqslant\frac{1+R_0}{1-R_0}:=M=M(K).$$

Put $\varepsilon>0$. By (30), for a number $\varepsilon':=\frac{\varepsilon}{M}$ there is $N=N(\varepsilon,K)\in\mathbb{N}$ such that

$$\left|\varphi\left(\overline{G}_{m_k}^{-1}(t)\right)-\varphi\left(\overline{G}^{-1}(t)\right)\right|<\varepsilon'$$

for any $k\geqslant N(\varepsilon)$ and $t\in\mathbb{S}^1$. Now, by (33)

$$|F_{m_k}(y)-F(y)|<\varepsilon\quad\forall\ k\geqslant N. \tag{34}$$

It follows from (34) that the sequence F_{m_k} converges to F as $k\to\infty$ in the unit disk locally uniformly. In particular, we obtain that $\operatorname{Im}F(0)=0$. Note that F is analytic function in $\mathbb{D}$ (see comments before the relation (66) in [2, sect. 6.3, ch. IV]), and

$$\operatorname{Re}F(re^{i\psi})=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\varphi\left(\overline{G}^{-1}(e^{i\theta})\right)\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,d\theta$$

for $z=re^{i\psi}$. By [2, relation (66), sect. 6.3, ch. IV]

$$\lim_{\zeta\to z}\operatorname{Re}F(\zeta)=\varphi(\overline{G}^{-1}(z))\quad\forall\ z\in\partial\mathbb{D}. \tag{35}$$

Observe that F either is a constant or open and discrete (see, e.g., [25, Ch. V, I.6 and II.5]). Thus, $f_{m_k} = F_{m_k} \circ G_{m_k}$ converges to $f = F \circ G$ locally uniformly as $k \to \infty$, where $f = F \circ G$ either is a constant or open and discrete. Moreover, by (35)

$$\lim_{\zeta \to P} \operatorname{Re} f(\zeta) = \lim_{\zeta \to P} \operatorname{Re} F(G(\zeta)) = \varphi(G^{-1}(G(P))) = \varphi(P).$$

IX. Since by **VI** G is a homeomorphism, by [10, Theorem 1] G is a regular solution of the equation (4) for some function $\mu : \mathbb{C} \to \mathbb{D}$. Since the set of points of the function F, where its Jacobian is zero, consist only of isolated points (see [25, Ch. V, 5.II and 6.II]), f is regular solution of the Dirichlet problem (4)–(5) whenever $F \not\equiv const$. Note that the relation (6) holds for the corresponding function $K_\mu = K_{\mu_f}$ (see e.g. [10, Lemma 1]).

Therefore, $f \in \mathfrak{F}^{\mathcal{M}}_{\varphi,\Phi,z_0}(D)$. □

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