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FORCING THE SYSTEM BY A DRIFT

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We establish apriori estimate for the solutions of a degenerate non-divergence nonlinear elliptic equation. For this goal we study forcing the system by a drift.

We consider a nonlinear elliptic equation of non-divergence type

$$\sum_{i,j=1}^{n} a_{ij}(x,u(x),Du(x))D^2u(x)+f(x,u,Du(x))=0. \tag{1}$$

Let $B_{2r}\subset\mathbb{R}^n$ be a ball with radius $2r, r\geq 1$. The solution of equation (1) is searched from $C(\overline{B}_{2r})\cap W^{2,n}_{loc}(B_{2r})$. Here $a_{ij}=a_{ji}$, i.d. correspondingly $A(x,y,p)$ is a symmetric matrix of size $n\times n$ and $\forall y\in R$, $\forall x,p,\xi\in\mathbb{R}^n$ coefficients satisfy

$$\begin{gathered}\Lambda^{-1}\lambda(p)\omega(x)|\xi|^2\leq(\xi,A(x,y,p)\xi)\leq\Lambda\lambda(p)\omega(x)|\xi|^2\\ f(x,y,p)\leq\tfrac{\Lambda}{k}(1+\lambda(p))(1+|p|)\end{gathered} \tag{2}$$

for some $\Lambda\geq 1, k>1$. Let $\lambda\colon\mathbb{R}^n\to\mathbb{R}_+$ be a some continuous mapping for which there exists λ_0 and $M>0$ such that $\lambda(z)\geq\lambda_0$ for $|z|\geq M$, $\omega(x)$ is the Muckenhoupt weight function (see [3]). Let $u\colon\overline{B}_{2r}\to\mathbb{R}$ be a bounded and continuous solution of (1).

The regularity estimates for a solution of divergence form equation were investigated by De Giorgi and Nash (see [7, 12]). Investigations by Serrin [13] and Ladyzhenskaya, Uraltsev [11], De Giorgi and Nash show that these estimates are valid for quasilinear elliptic equations of divergent type [1, 2, 6, 14]). A corresponding result for the non-divergence equations was obtained by Krylov and Safanov. These authors used probability methods (see [9, 10]).

The strategy of the proof results for system (1) relies on a well-known probabilistic interpretation of the nonlinear equations. The proof consists in introducing a diffusion process X, solution to the stochastic differential equation (SDE)

$$dX_t=\sigma(X_t,u(X_t)Du(X_t))dW_t,\quad t\geqslant 0,$$

where W is a Wiener process and σ is a contiuous version of the square root of the matricial mappings 2A. The basic idea follows from the theory of diffusion processes, the generator of a diffusion process enjoys some smoothing property if the path of the corresponding process sufficiently visit the surrounding space with a non trivial probability. The argument may be understood as follows: U is smooth: in such a framework, $(u(X_t))_{t\geqslant 0}$ is a martingale.

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In particular, $U(x)$ may be expressed as the expectation $E[u(X_\tau)]$ for any well-controlled stopping time τ.

In the special probability theory, the point is to bound from below the probability that the diffusion process X hits a Borel subset included in B_2, before leaving it. We specifically show that we can force the stochastic system on the areas of degeneracy by an additional drift to push it towards the desired Borel subset. When $|Du|$ is large by an ellipticity condition, the probability theory can be applied. Theorem 1 says that the probability of hitting a Borel subset V of Q_ρ before leaving the ball Q_ρ is bounded from below by a constant only depending on n and Λ and on the proportion of V in Q_ρ. The connection with system (1) may be understood as follows: when u is strong solution of (1), we choose $a(x)$ in the statement of Theorem 1 as $2A(x,u(x),Du(x))$. We deduce that $|Du(X_t)| \geqslant M$. In other words, the resulting drift $(b_t)_{t\geqslant 0}$ just acts when the gradient is small, i.e. bounded. Later we can use Theorem 1 and establish a priori Hölder estimate as in the result by Krylov and Safanov.

The goal of this paper is to prove a similar result for degenerate quasilinear elliptic equations of non-divergence form. Therefore firstly we study forcing the system by a drift.

Also, we suppose that the following condition is satisfied: let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a Lipschitz continuous mapping such that $\forall x, \xi \in \mathbb{R}^n$

$$\Lambda^{-1}\lambda(x)\omega(x)|\xi|^2 \le (\xi, a(x)\xi) \le \Lambda\bar{\lambda}(x)\omega(x)|\xi|^2, \tag{3}$$

$a(x) = \sigma\sigma^*(x)$, for some $\Lambda \ge 1$ and some mapping $\bar{\lambda}\colon \mathbb{R}^n \to [0,1]$.

Let $(\Omega, F, (F_t)_{t\ge 0}, P)$ be a filtered probability space satisfying the usual conditions endowed with an $(F_t)_{t\ge0}$ Brownian motion $(W_t)_{t\ge0}$, α be a positive real and Q_1 be some hypercube of $\mathbb{R}^n$ of radius 1; Q_ρ is the hypercube of same center as Q_1 but of radius ρ; $Q(z,s)$ is the hypercube at the center z and radius ρ, $Q(0,1) \equiv Q_1$.

Under these conditions on the coefficients, we can build a drift to force the system to hit a prescribed Borel subset of large measure with a non-zero probability.

Out the prof of results for system (1) relies on a probabilistic interpetation of the nonlinear PDE. We need the main Theorem 1. Later we choose $a(x)$ in the statement of Theorem 1 as $2A(x,u(x),Du(x))$.

Theorem 1. *Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a Lipschitz continuous mapping such that $a(x) = \sigma\sigma^*(x)$ satisfies the condition* (3) *and $\alpha \in (0,+\infty)$. Then, there exist positive constants $\mu_0, \varepsilon_0, R_0$ and $(\Gamma_p(\mu))_{1\le p<2}$, only depending on $d, \alpha, \lambda, \omega$, such that, for any $\rho \in (0,1)$, any hypercubes $Q_{\rho/8} \subset Q_1 \subset \mathbb{R}^n$ and any square integrable F_0-measurable random variable X_0 with values in $\mathbb{R}^n$, we can find an integrable n-dimensional progressively-measurable process $(b_t)_{t\ge0}$ and the process X, solution to the SDE*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \omega(X_s)\sigma(X_s)dW_s, t \ge 0,$$

for which it fulfills $\forall t \ge 0$, $\lambda(X_t) \ge \alpha \Rightarrow b_t = 0$, and also $E_0 \int_0^{+\infty} |b_t|^p dt \ge \Gamma_p \rho^{p-2}$ $(\forall p \in [1,2))$. Moreover, for any Borel subset $V \subset Q_\rho$ one has

$$|Q_\rho \backslash V| \le \mu_0 |Q_\rho| \Rightarrow P_0\left\{T_V < (R_0\rho^2) \cap S_{Q_\rho}\right\} \ge \varepsilon_0,$$

a.e. on the event $\{X_0 \in Q_{\rho/8}\}$. Here T_V is the first hitting time of V and S_{Q_ρ} is the first exit time Q_ρ by X.

Proof. Let δ be a small real, for example, at least than $\frac{1}{4}$. By a scaling argument we take $\rho = 1$. We mean some a function X_t of δ that only depends on $d, \alpha, \Lambda, \omega$ and that tends to zero as δ tends to zero. We can assume that $X_0 \in Q_{1/8}$ a.e., if need we can change the values of X_0. Also let $|Q_1 \backslash V| \leq q_0$, where q_0 is a universal constant, only depending on n, such that, for any Borel subset $V \subset Q(0,1)$ one has $|Q(0,1)\backslash V| \leq q_0$. Then we can find a constant $K_0 > 0$, only depending on n and $x_\infty \in Q_{1/8} \cap V$ such that, for any $r \in \left(0, \frac{3}{4}\right)$,

$$|Q(x_\infty, r)\backslash V| \leq K_0 |Q_1 \backslash V|^{\frac{1}{2}} \cdot \mathbb{R}^n \tag{4}$$

Now we construct b and X. We consider the following local dynamics. For a finite stopping time T and two F_t-measurable random variables $N \colon \Omega \to Z$ and $J_0 = \Omega \to \mathbb{R}^n$, we define the drift $b_t^{T,J_0,N} = \delta^{-2N}(x_\infty - J_0)$, $T \leq t \leq T + \delta^{2n}$. For a smooth function $\psi \colon R \to [0,1]$, matching 1 on $(-\infty, \frac{\alpha}{2}]$ and vanishing on $[\alpha, +\infty)$, we solve the SDE

$$J_t^{T,J_0,N} = J_0 + \int_T^t \psi(\lambda(J_s^{T,J_0,N})) b_s^{T,J_0,N} ds + \int_T^t \omega(J_s^{T,J_0,N}) \sigma(J_s^{T,J_0,N}) dW_s, \ T \leq t \leq T + \delta^{2N}.$$

We define $(X_t)_{t \geq 0}$ as follows. Let $T_0 = 0$ be an initial time, $X_0 \in Q(x_\infty, \frac{1}{4})$ as so x_∞ and X_0 are in $Q_{1\backslash 8}$. We take $X_0 \neq x_\infty$. Then there exists a random integer number n_0 such that $X_0 \in Q(x_\infty, \delta^{n_0}) \backslash Q(x_\infty, \delta^{n_0+1})$. Now we set $T_1 = \delta^{2n_0}$ and $X_t = J_t^{0,X_0,n_0}$ for $t \in [0, T_1]$. In another case $X_0 = x_\infty$ then we take $b_t = 0$ for $t \geq 0$ and $(X_t)_{t \geq 0} = S_{X_0}(0, \sigma)$. In this case we set $T_{k+1} = \infty$ and $n_k = +\infty$, $\forall k \geq 0$. The construction is over.

Since it is an initialization, we stop later by one step. Let $n_0 < +\infty$ and $X_{T_1} = x_\infty$. We take $b_t = 0$ for $t \geq T_1$ and define $(X_t)_{t \geq T_1}$ as the solution of

$$X_t = X_{T_1} + \int_{T_1}^t \omega(X_s) \sigma(X_s) dW_s$$

for $t \geq T_1$. In this case we set $T_{k+1} = \infty$, $n_k = \infty$ for $\forall k \geq 1$. The construction is over.

Now we are doing some iteration. Let $n_0 < \infty$, $X_{T_1} \neq x_\infty$. Then there exists a random number n_1 such that $X_{T_1} \in Q(x_\infty, \delta^{n_1}) \backslash Q(x_\infty, \delta^{n_1+1})$. Then for $t \in [T_1, T_2]$ we put $T_2 = T_1 + \delta^{2n_1}$, $X_t = J_t^{T_1, X_{T_1}, n_1}$. Now we apply stop later one step to X_{T_2}. We take, in the case $X_{T_2} = x_\infty$, $b_t = 0$ at $t \geq T_2$ and define $(X_t)_{t \geq T_2}$ as the solution of

$$X_t = X_{T_2} + \int_{T_2}^t \omega(X_s) \sigma(X_s) dW_s$$

and take for any $k \geq 2$ $T_{k+1} = \infty$, $n_k = \infty$. The construction is over.

Now we do some iteration. It is clearly, the random times $(T_k)_{k \geq 0}$ are stopping times, i.d. T_{k+1} is F_{T_k} measurable. Let S be an exit time of X from the hypercube $Q(x_\infty, \frac{3}{4})$ and we also introduce stopping times τ_1, τ_2. These are discrete stopping times for filtration $(F_{T_k})_{k \geq 0}$. With $k \geq \tau_1$, if $t \in [T_k, T_{k+1})$, then $dX_t = \omega(X_t)\sigma(X_t) dW_t$ and with $0 \leq k < \tau_1$, $T_{k+1} = T_k + \delta^{2n_k}$, $dX_t = \delta^{-2n_k} \psi(\lambda(X_t))(x_\infty - X_{T_k}) dt + \omega(X_t)\sigma(X_t) dW_t$. For stopping time the parameter τ_2 permits to evaluate the noise inside the system. Our investigation shows that in time less than S there is enough noise in the system. X hits V before leaving the $Q(x_\infty, \frac{3}{4})$.

The reason why we expect such a behavior may be explained as follows.

Later we study growth of $(n_{k\wedge\tau})_{k>0}$ up to exit time. For this goal we used stochastic comparison. We note that for any $0 \le k < \tau_1$

$$n_{k+1} = \infty \Rightarrow X_{T_{k+1}} = x_\infty, n_{k+1} = l \Rightarrow \delta^{l+1} \le \|X_{T_{k+1}} - x_\infty\| < \delta^l,$$

l is integer. Then for $0 \le k < \tau$ and $T_k < S$ from above we deduce, that for $l \ge 0$, $\{n_{k+1} = n_k - l\} = \{\delta^{n_k - l+1} \le \|X_{T_{k+1}} - x_\infty\| < \delta^{n_k - l}\}$. We compare the conditional distribution of $n_{k+1} - n_k$ knowing F_{T_k} with the distribution of some variable ξ_{k+1} with values into integer $l \le 1$ such that

$$Q\{\xi_{k+1} = -l\} = \delta^{2(1+l)},\ l \ge 0,\ Q\{\xi_{k+1} = 1\} = 1 - \frac{\delta^2}{(1-\delta^2)}.$$

Later we get deviation inequality. We have for $\delta \in (0, \delta_1)$ and any $k \ge 1$

$$P_0\left\{n_k \le \frac{k}{2},\ \tau > k,\ T_k < S\right\} \le \delta^{\frac{k}{2}}.$$

Now we evaluate the exit time S, so that

$$S \le T_{k+1} \Rightarrow \sup_{T_k \le t \le T_{k+1}} \left[\delta^{-n_k} \int\limits_{T_k}^{t} 1_{\{\lambda(X_S > \alpha\backslash 2)\}} ds + \left|\int\limits_{T_k}^{t} \omega(X_s)\sigma(X_s) dW_s\right|\right] \ge \left(\frac{2}{3} - \delta^{n_k}\right).$$

By Doob's maximal inequality we obtain on the event $\{T_{k\wedge\tau} < S\}$ at $k \ge 0$

$$P_{T_{k\wedge\tau}}\left\{S \le T_{(k+1)\wedge\tau}\right\} \le \delta^{2(n_k+2)} \left[\frac{2}{3} - \delta^{n_k}\right]^{-2} 1_{\{k<\tau\}}.$$

So that, we get later some calculations, that

$$P_0\Big(\bigcap_{k\ge 0}\Big\{n_{k\wedge\tau} \ge \frac{(n_{k\wedge\tau})}{4}\Big\}\Big) \ge 1 - o(1).$$

To end, we set $R = 1 + \sum_{k\ge 0} \delta^k$. Later we can choose R_0 in the final statement. Since $P_0\{\exists k \ge 0\colon S < T_{k\wedge\tau}\} \le o(1)$, the exit time is greater than all the times $(T_{k\wedge\tau})_{k\ge 0}$. If τ is infinite, then the process $(X_t)_{t\ge 0}$ converges to $x_\infty \in V$ in time less than R on the event $\{\forall k \ge 0, n_k \ge \frac{k}{2}\}$.

Let τ be finite, there are two cases: $1 : \tau = \tau_1$, then $X_T = x_\infty$, the process hits x_∞ in time less than $R; 2\ \tau = \tau_2$, then the process hits V with a non-zero conditional probability and the hitting time is less than R.

Later we consider cases $\tau = +\infty$; $\tau = \tau_1 < \infty$; $\tau < \infty$, $\tau_2 < \tau_1$; combining above cases together. We use the fact

$$E_{T_\tau}\left[\int\limits_{T_\tau}^{T_{\tau+1}} \lambda(X_s)\omega(X_s) ds\right] \ge \delta^{2(n_\tau+2)},$$

also $P_{T_\tau}\{\exists t \in (T_\tau, T_{\tau+1})\colon X_t \in V\} \ge \varepsilon$.

We choose δ, V and use integrability of the drift. Then we have

$$E_0\left[\int\limits_0^{T_{\tau+1}\wedge S} \|b_t\|^P dt\right] \le 1 + \sum_{k\ge 0} E_0[\delta^{(2-P)n_k} 1_{k<\tau, T_k<S}].$$

□

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