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SPECTRA OF ALGEBRAS OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS OF BOUNDED TYPE

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We investigate algebras of block-symmetric analytic functions on spaces $\ell_p(\mathbb{C}^s)$ which are ℓ_p-sums of $\mathbb{C}^s$. We consider properties of algebraic bases of block-symmetric polynomials, intertwining operations on spectra of the algebras and representations of the spectra as a semigroup of analytic functions of exponential type of several variables. All invertible elements of the semigroup are described for the case $p = 1$.

1. Introduction. Let $\mathcal{S}$ be a representation of a group $\mathfrak{G}$ as a subgroup of linear isometric operators on a complex Banach space X. A function $f\colon X \to \mathbb{C}$ is said to be $\mathcal{S}$-*symmetric* if for every $T \in \mathcal{S}$, $f \circ T = f$. Algebras of $\mathcal{S}$-symmetric analytic functions for various groups of symmetry $\mathcal{S}$ were studied by many authors (see [1, 3–5, 7, 13, 19, 20, 34, 35]). We consider the special case when $\mathfrak{G} = \mathfrak{G}_{\mathbb{N}}$ is the group of all permutations (bijections) on the set of natural numbers $\mathbb{N}$.

If X is a Banach space with a symmetric basis (e_n), then every $x \in X$ can be uniquely represented as

$$x = (x_1, x_2, \ldots) = \sum_{n=1}^{\infty} x_n e_n$$

and the basis $(e_{\sigma(n)})$ is equivalent to (e_n) for any permutation $\sigma \in \mathfrak{G}_{\mathbb{N}}$ (see [27]). Let S_∞ be the representation of $\mathfrak{G}_{\mathbb{N}}$ as the set of perturbation of basis vectors, that is,

$$X \ni x = \sum_{n=1}^{\infty} x_n e_n \mapsto T_\sigma(x) = \sum_{n=1}^{\infty} x_{\sigma(n)} e_n, \quad \sigma \in \mathfrak{G}_{\mathbb{N}}.$$

In the literature, S_∞-symmetric functions often are called *symmetric*. Symmetric functions of infinitely many variables are important objects in the nonlinear functional analysis [13, 20] and are applicable in different areas of the information theory and statistical physics [11, 33, 38].

Let us recall that a function $f\colon X \to \mathbb{C}$ on a complex Banach space X is *analytic* if it is continuous and the restriction of f to any finite dimensional subspace of X is analytic. Every analytic function f can be represented by its Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad x \in X,$$

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where f_n are n-homogeneous polynomials, that is, f_n are analytic and $f(\lambda x) = \lambda^n f(x)$, $\lambda \in \mathbb{C}$. An analytic function f on a Banach space X is said to be a function of *bounded type* if it is bounded on all bounded subsets of X. We denote by $H_b(X)$ the topological algebra of all analytic functions of bounded type and by $H_{bs}(X)$ its closed subalgebra consisting of symmetric functions for the case when X has a symmetric basis. It is well-known that $H_b(X)$ is a Fréchet algebra with respect to the countable family of norms

$$\|f\|_r = \sup_{\|x\|\le r} |f(x)|, \quad r \in \mathbb{Q}_+.$$

Let ϕ be a linear continuous functional on $H_b(X)$. Then ϕ is continuous as a linear functional on a normed space $\big(H_b(X), \|\cdot\|_r\big)$ for some $r > 0$. The infimum of such r is called the *radius function* of ϕ and denoted by $R(\phi)$. In [2] is proved that $R(\phi)$ can be computed by the following formula

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{\frac{1}{n}}, \tag{1}$$

where ϕ_n is the restriction of the functional ϕ to the normed subspace of n-homogeneous polynomials (with respect to the norm $\|\cdot\|_1$). Moreover, it is proved in [16] that formula (1) is still true for any subalgebra $H^0 \subset H_b(X)$, a continuous functional ϕ on H^0, and in this case, ϕ_n is the restriction of ϕ to the subspace of n-homogeneous polynomials in H^0. For more information about polynomials and analytic functions on Banach spaces we refer the reader to [12].

The algebra of all symmetric polynomials on a Banach space X with a symmetric basis is denoted by $\mathcal{P}_s(X)$.

Algebras $H_{bs}(X)$ and $\mathcal{P}_s(X)$ were investigated by many authors ([1, 8, 9, 29]). In particular, it is known that $\mathcal{P}_s(\ell_p)$, $1 \le p < \infty$ admits the following algebraic basis of *power* symmetric n-homogeneous polynomials $n \ge \lceil p \rceil$,

$$F_n(x) = \sum_{i=1}^{n} x_i^n, \quad x = (x_1, \dots, x_i, \dots) \in \ell_p,$$

where $\lceil p \rceil$ is the smallest integer, greater than p. If $X = c_0$ or ℓ_∞, the symmetric polynomials on X are just constants [14, 15]. In the case $X = \ell_1$, there is another important algebraic basis, so-called the basis of *elementary* symmetric polynomials

$$G_n(x) = \sum_{l_1<\dots<l_n}^{\infty} x_{l_1} \cdots x_{l_n}, \quad x \in \ell_1. \tag{2}$$

In this paper, we consider other representations of $\mathfrak{S}_{\mathbb{N}}$ in Banach spaces. If X is a direct sum of infinitely many of "blocks" which consists of linear subspaces that are isomorphic each to other, then $\mathfrak{S}_{\mathbb{N}}$ may act as the group of permutations of the "blocks". For this case we can consider the algebra of block-symmetric analytic functions. Note that such kinds of algebras are much more complicated and in the general case have no algebraic basis (see e.g. [21, 22, 24–26, 37]). Note that if $\dim X < \infty$, then block-symmetric polynomials are investigated in the classical theory of invariants and combinatorics [18, 32, 36].

This research is a continuation of investigations in [8–10] for symmetric analytic functions. Also, some presented results were obtained in [26] for block-symmetric analytic functions for a partial case of two-dimensional blocks. In Section 2, we consider properties of block-symmetric polynomials and algebraic bases of block-symmetric polynomials. In Section 3, we investigated algebras of block-symmetric analytic functions of bounded type on ℓ_p; throughout in this paper we assume that $1 \le p < \infty$. We consider spectra of the algebras of block-symmetric analytic functions (sets of continuous nonzero linear multiplicative functionals) and some algebraic structure on the spectra. In Section 4, for the case $p = 1$, we found a representation of the spectrum in a group of analytic functions of exponential type on $\mathbb{C}^s$.

2. Bases of block-symmetric polynomials. Let us denote by $\ell_p(\mathbb{C}^s)$, $1 \le p < \infty$ the vector space of all sequences

$$x = (x_1, x_2, \ldots, x_j, \ldots), \tag{3}$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{\infty} \sum_{r=1}^{s} |x_j^{(r)}|^p$ is convergent. We say that elements x_j in (3) are *vector coordinates* of x. The space $\ell_p(\mathbb{C}^s)$ endowed with norm

$$\|x\| = \Big(\sum_{j=1}^{\infty} \sum_{r=1}^{s} \Big| x_j^{(r)} \Big|^p \Big)^{1/p}$$

is a Banach space. Since any vector of $\ell_p(\mathbb{C}^s)$ can be represented as $(x^{(1)}, \ldots, x^{(s)})$, where $x^{(i)} \in \ell_p$, $x^{(i)} = \sum_{k=1}^{\infty} x_k^{(i)} e_k$, $i = 1, \ldots, s$, we can write $\ell_p(\mathbb{C}^s) \simeq \underbrace{\ell_p \times \ldots \times \ell_p}_{s} \simeq \mathbb{C}^s \otimes \ell_p$.

A polynomial P on the space $\ell_p(\mathbb{C}^s)$ is called *block-symmetric (or vector-symmetric)* if:

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation $\sigma \in \mathfrak{S}_{\mathbb{N}}$, where $x_j \in \mathbb{C}^s$ for all $j \in \mathbb{N}$. Let us denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

The algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ was considered in [6, 25]. Note that in combinatorics, block-symmetric polynomials on finite-dimensional spaces are called *MacMahon symmetric polynomials* (see e.g. [32]).

For a multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbb{Z}_+^s$ let $|\mathbf{k}| = k_1 + k_2 + \cdots + k_s$ and $\mathbf{k}! = k_1! k_2! \cdots k_s!$. Also, we say that $\mathbf{r} \le \mathbf{k}$ if $r_1 \le k_1$,..., $r_s \le k_s$.

In [25], it was proved that so-called *power* block-symmetric polynomials

$$H^{\mathbf{k}}(x) = H^{k_1, k_2, \ldots, k_s}(x) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |k| \ge \lceil p \rceil}}^{s} (x_j^{(r)})^{k_r} \tag{4}$$

form an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, $1 \le p < \infty$, where $x = (x_1, \ldots, x_m, \ldots) \in \ell_p(\mathbb{C}^s)$, $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$, and $\lceil p \rceil$ is the smallest integer, greater than p. It means that every polynomial in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ can be uniquely represented as a finite algebraic combination of polynomials $H^{\mathbf{k}}$, $|k| \ge \lceil p \rceil$.

In the case of the space $\ell_1(\mathbb{C}^s)$ there is an algebraic basis of *elementary* block-symmetric polynomials:

$$R^{\mathbf{k}}(x) = R^{k_1, k_2, \ldots, k_s}(x) = \sum_{\substack{i_1 < \ldots < i_{k_1} \\ j_1 < \ldots < j_{k_2} \\ \ldots \\ l_1 < \ldots < l_{k_s} \\ i_{k_p} \ne j_{k_q} \ne \ldots \ne l_{k_r}}}^{\infty} x_{i_1}^{(1)} \ldots x_{i_{k_1}}^{(1)} x_{j_1}^{(2)} \ldots x_{j_{k_2}}^{(2)} \ldots x_{l_1}^{(s)} \ldots x_{l_{k_s}}^{(s)}, \tag{5}$$

$\mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbb{Z}_+^s$, $(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(s)}) \in \mathbb{C}^s$. The connection between the basis of power block-symmetric polynomials and the basis of elementary block-symmetric polynomials is given by an analogue of the Newton formula [22, 23].

Lemma 1. *The following equality holds:*

$$\|R^{\mathbf{k}}\| = \frac{1}{\mathbf{k}!} = \frac{1}{k_1!\dots k_s!}. \tag{6}$$

Proof. In [9], it was proved that $\|G_n\| = \frac{1}{n!}$, where $\{G_n\}$ is the basis of elementary symmetric polynomials in $\mathcal{P}_s(\ell_1)$ defined by (2). Thus,

$$\|R^{\mathbf{k}}\| = \sup_{\|x\|_{\ell_1(\mathbb{C}^s)}\le 1} |R^{\mathbf{k}}(x)| = \sup_{\|x\|_{\ell_1(\mathbb{C}^s)}\le 1} \Bigg| \sum_{\substack{i_1<\dots<i_{k_1}\\ j_1<\dots<j_{k_2}\\ \dots\\ l_1<\dots<l_{k_s}\\ i_{k_p}\ne j_{k_q}\ne\dots\ne l_{k_r}}}^{\infty} x^{(1)}_{i_1}\dots x^{(1)}_{i_{k_1}} x^{(2)}_{j_1}\dots x^{(2)}_{j_{k_2}}\dots x^{(s)}_{l_1}\dots x^{(s)}_{l_{k_s}} \Bigg| \le$$

$$\le \sup_{\|x\|_{\ell_1(\mathbb{C}^s)}\le 1} \sum_{\substack{i_1<\dots<i_{k_1}\\ j_1<\dots<j_{k_2}\\ \dots\\ l_1<\dots<l_{k_s}\\ i_{k_p}\ne j_{k_q}\ne\dots\ne l_{k_r}}}^{\infty} |x^{(1)}_{i_1}|\dots|x^{(1)}_{i_{k_1}}||x^{(2)}_{j_1}|\dots|x^{(2)}_{j_{k_2}}|\dots|x^{(s)}_{l_1}|\dots|x^{(s)}_{l_{k_s}}| \le$$

$$\le \prod_{j=1}^{s} \sum_{i_1<\dots<i_{k_j}}^{\infty} |x^{(j)}_{i_1}|\dots|x^{(j)}_{i_{k_j}}| \le \frac{1}{k_1!\dots k_s!}.$$

To get equality (6) it is enough to check that $\lim\limits_{n\to\infty} R^{\mathbf{k}}(v_n) = \frac{1}{\mathbf{k}}$, where $\|v_n\| = 1$, and

$$v_n = \frac{1}{n}\Bigg(\underbrace{\begin{pmatrix}1\\ \dots\\ 1\end{pmatrix},\dots,\begin{pmatrix}1\\ \dots\\ 1\end{pmatrix}}_{n},\begin{pmatrix}0\\ \dots\\ 0\end{pmatrix},\dots\Bigg).$$

This fact is proved below (see formula (12)) for a more general case. □

For every $\sigma \in \mathfrak{S}_{\mathbb{N}}$ we denote by T_σ the linear operator on $\ell_1(\mathbb{C}^s)$ associated with σ by the formula

$$T_\sigma\Big(\sum_{k=1}^{\infty} x^{(1)}_k e_k,\dots,\sum_{k=1}^{\infty} x^{(s)}_k e_k\Big) = \Big(\sum_{k=1}^{\infty} x^{(1)}_{\sigma(k)} e_k,\dots,\sum_{k=1}^{\infty} x^{(s)}_{\sigma(k)} e_k\Big).$$

For any $x, y \in \ell_1(\mathbb{C}^s)$ we denote $x \sim y$ if there exists a permutation σ on $\mathbb{N}$ such that $x = T_\sigma(y)$. If $x \sim y$, then evidently, $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for all $\mathbf{k}$.

A vector $x \in \ell_1(\mathbb{C}^s)$ is said to be *finite* if there is $n_0 \in \mathbb{N}$ such that $x^{(j)}_m = 0$ for every $m > n_0$ and $1 \le j \le s$. Thus, a finite vector has just a finite number of nonzero vector coordinates.

Theorem 1. *Suppose that x and y are such that either x or y is finite in $\ell_p(\mathbb{C}^s)$, $1 \le p < \infty$, or all vector coordinates x_i and y_i of both x and y respectively, are nonzero vectors in $\mathbb{C}^s$. If $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for every $\mathbf{k}$ with $|\mathbf{k}| \ge \lceil p\rceil$, then $x \sim y$.*

Proof. Suppose that $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for all multi-indexes $\mathbf{k}$, $|\mathbf{k}| \ge n$, and $x \not\sim y$. If $x_m = (x^{(1)}_m,\dots,x^{(s)}_m) = y_j = (y^{(1)}_j,\dots,y^{(s)}_j) \ne 0$ for some m and j, we can remove the vector coordinates x_m in x and y_j in y obtaining new elements x' and y' such that $H^{\mathbf{k}}(x') = H^{\mathbf{k}}(y')$ for all $\mathbf{k}$, $|\mathbf{k}| \ge \lceil p\rceil$ and $x' \not\sim y'$. If x or y is finite, then repeating this finitely many times we will reduce the situation to the case when $0 \neq x_m \neq y_j$ for some m and every j. If both x and y are not finite and all their vector coordinate are nonzero, then the multiplicity of any vector coordinate of x (and of y) is finite. If the multiplicity of x_m in x, say, is greater than in y, then removing a finite number of vector coordinates we will get the situation when one vector has a vector coordinate x_m but another has not. If the multiplicity of each vector coordinate of x is equal to the multiplicity of the same vector coordinate of y and vice-versa, then there is a permutation of all vector coordinates of x which maps x to y, that is $x \sim y$. So assuming that $x \not\sim y$ we can suppose, without loss of generality that there is a vector coordinate $x_m \neq 0$ such that $x_m \neq y_j$ for every $j \in \mathbb{N}$.

We claim that there is a vector $t = (t_1, \ldots, t_s) \in \mathbb{C}^s$ such that

$$t_1 x_m^{(1)} + \cdots + t_s x_m^{(s)} \neq t_1 y_j^{(1)} + \cdots + t_s y_j^{(s)} \tag{7}$$

for all $j \in \mathbb{N}$. Indeed, since $x_m \neq y_1$, then there is a vector $t^0 = (t_1^0, \ldots, t_s^0) \in \mathbb{C}^s$ such that

$$t_1^0 x_m^{(1)} + \cdots + t_s^0 x_m^{(s)} \neq t_1^0 y_1^{(1)} + \cdots + t_s^0 y_1^{(s)}.$$

By the continuity of linear forms, this inequality must be true in some neighbourhood U_1 of the point $(t_1^0, \ldots, t_s^0)$. Of course, it is true in a closed ball $V_1 \subset U_1$. If

$$t_1 x_m^{(1)} + \cdots + t_s x_m^{(s)} = t_1 y_2^{(1)} + \cdots + t_s y_2^{(s)}$$

for every $t \in V_1$, then $(x_m^{(1)}, \ldots, x_m^{(s)}) = (y_2^{(1)}, \ldots, y_2^{(s)})$ that contradicts the assumption. Thus, there is a closed ball $V_2 \subset V_1$ such that

$$t_1 x_m^{(1)} + \cdots + t_s x_m^{(s)} \neq t_1 y_2^{(1)} + \cdots + t_s y_2^{(s)}$$

for every $t \in V_2$. If x or y is finite, then there is a nonempty open set U such that (7) is true for every $t \in U$. Otherwise, we will get a chain of closed balls $V_1 \supset V_2 \supset \cdots$ which has a common point t. Let us consider the following linear operator $A_t \colon \ell_p(\mathbb{C}^s) \to \ell_p$,

$$A_t \colon (x_m^{(1)}, \ldots, x_m^{(s)})_{m=1}^{\infty} \mapsto \big(t_1 x_m^{(1)} + \cdots + t_s x_m^{(s)}\big)_{m=1}^{\infty}.$$

The vector t was chosen so that $A_t(x) \neq A_t(y)$. By [1, Theorem 1.3] we obtain that $F_k\big(A_t(x)\big) \neq F_k\big(A_t(y)\big)$ for some $k \in \mathbb{N}$. Clearly, the map $x \mapsto F_k\big(A_t(x)\big)$ is a k-homogeneous polynomial in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$.

Since polynomials $\{H^{\mathbf{k}}\}$, $|\mathbf{k}| \geq \lceil p \rceil$ form an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, it follows that $H^{\mathbf{k}}(x) \neq H^{\mathbf{k}}(y)$ for some multi-index $\mathbf{k}$. A contradiction. □

Corollary 1. *Suppose that x and y are as in Theorem 1. If there is a number $n \in \mathbb{N}$ such that $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for every $\mathbf{k}$ with $|\mathbf{k}| \geq n$, then $x \sim y$.*

Proof. If $x, y \in \ell_p(\mathbb{C}^s)$, then $x, y \in \ell_q(\mathbb{C}^s)$ for every $q \geq p$. Let us take $n \leq q < \infty$. Then, by Theorem 1, $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ whenever $|\mathbf{k}| \geq \lceil q \rceil \geq n$ implies that $x \sim y$ in $\ell_q(\mathbb{C}^s)$. But from here it evidently follows that $x \sim y$ in $\ell_p(\mathbb{C}^s)$. □

Note that the statement of Theorem 1 will be not longer correct if we remove all restrictions to x and y. For example, if $x = (x_1, \ldots, x_m, \ldots) \in \ell_p(\mathbb{C}^s)$ such that all $x_j \neq 0$ and $y = (x_1, 0, x_2, 0, \ldots, x_m, 0, \ldots)$, then $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for every multi-index $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p \rceil$ but $x \not\sim y$.

Corollary 2. *Let x and y be arbitrary vectors in $\ell_1(\mathbb{C}^s)$. If there exists a number n such that $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for every $\mathbf{k}$ with $|\mathbf{k}| \geq n$, then $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for every $\mathbf{k}$ with $|\mathbf{k}| \geq \lceil p \rceil$. Moreover, $P(x) = P(y)$ for every $P \in \mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$.*

Proof. If x is a finite vector, then we set $\widetilde{x} = x$, otherwise, let $\widetilde{x}$ be a vector obtained by removing of zero vector coordinates in x. By the same way we construct $\widetilde{y}$ from y. Then $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(\widetilde{x})$ and $H^{\mathbf{k}}(y) = H^{\mathbf{k}}(\widetilde{y})$ for all $\mathbf{k}$ with $|\mathbf{k}| \geq \lceil p \rceil$. By Corollary 1, $\widetilde{x} \sim \widetilde{y}$. Thus, $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(y)$ for all $\mathbf{k}$ with $|\mathbf{k}| \geq \lceil p \rceil$. Since $\{H^{\mathbf{k}}\colon |\mathbf{k}| \geq \lceil p \rceil\}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, $P(x) = P(y)$ for every block-symmetric polynomial P. □

3. Algebra of block-symmetric analytic functions. Let us denote by $H_{bvs}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric analytic functions of bounded type on $\ell_p(\mathbb{C}^s)$. That is, $H_{bvs}(\ell_p(\mathbb{C}^s))$ is the completion of $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ in $H_b(\ell_p)$. We denote by $M_{bvs}(\ell_p(\mathbb{C}^s))$ the spectrum of $H_{bvs}(\ell_p(\mathbb{C}^s))$, that is, the set of nonzero continuous complex valued homomorphisms of $H_{bvs}(\ell_p(\mathbb{C}^s))$. Clearly that for every $x \in \ell_p(\mathbb{C}^s)$ it is defined the *point evaluation* complex homomorphism δ_x, $\delta_x(f) = f(x)$, $f \in H_{bvs}(\ell_p(\mathbb{C}^s))$. On the other hand, if $x \sim y$, then $\delta_x = \delta_y$. Note that there are complex homomorphisms which are not point evaluation (some examples are below).

Since the set of polynomials $\{H^{\mathbf{k}}\}$, $|\mathbf{k}| \geq \lceil p \rceil$ forms an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, every analytic function $f \in H_{bvs}(\ell_p(\mathbb{C}^s))$ can be represented by

$$f(x) = f(0) + \sum_{n=\lceil p \rceil}^{\infty} \sum_{|\mathbf{k}_1|+\cdots+|\mathbf{k}_m|=n,\, |\mathbf{k}_j| \geq \lceil p \rceil} c_{\mathbf{k}_1} \cdots c_{\mathbf{k}_m} H^{\mathbf{k}_1}(x) \cdots H^{\mathbf{k}_m}(x) \tag{8}$$

and the series converges absolutely for every $x \in \ell_1(\mathbb{C}^s)$ and uniformly on all bounded subsets. Hence, if $\phi \in M_{bvs}(\ell_1(\mathbb{C}^s))$, then by the continuity, linearity, and multiplicativity of ϕ,

$$\phi(f) = f(0) + \sum_{n=\lceil p \rceil}^{\infty} \sum_{|\mathbf{k}_1|+\cdots+|\mathbf{k}_m|=n,\, |\mathbf{k}_j| \geq \lceil p \rceil} c_{\mathbf{k}_1} \cdots c_{\mathbf{k}_m} \phi\big(H^{\mathbf{k}_1}\big) \cdots \phi\big(H^{\mathbf{k}_m}\big).$$

Thus, the homomorphism ϕ is completely defined by its values on polynomials $\{H^{\mathbf{k}}\}$.

To describe the spectrum $M_{bvs}(\ell_p(\mathbb{C}^s))$, we consider an algebraic operation on $\ell_p(\mathbb{C}^s)$ which preserves the relation of equivalence and can be extended to the spectrum. For given $x, y \in \ell_p(\mathbb{C}^s)$, $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$ where $x_i = (x_i^{(1)}, \ldots, x_i^{(s)})$, $y_i = (y_i^{(1)}, \ldots, y_i^{(s)}) \in \mathbb{C}^s$, $i \geq 1$ we set

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$$

and define

$$\mathcal{T}_y(f)(x) := f(x \bullet y). \tag{9}$$

We will say that $x \to x \bullet y$ is the *intertwining* and the operator $\mathcal{T}_y$ is the *intertwining operator*.

Proposition 1. *Let $x, y \in \ell_p(\mathbb{C}^s)$, and $|\mathbf{k}| \geq \lceil p \rceil$. The following elementary properties of intertwining are obvious:*

1. $H^{\mathbf{k}}(x \bullet y) = H^{\mathbf{k}}(x) + H^{\mathbf{k}}(y)$,
2. $\|x \bullet y\|^p = \|x\|^p + \|y\|^p$,
3. $H^{\mathbf{k}}(x^{\bullet m}) = mH^{\mathbf{k}}(x)$, *where* $x^{\bullet m} = \underbrace{x \bullet (\cdots (x \bullet x)}_{m} \cdots)$,
4. *if* $p = 1$, *then* $R^{\mathbf{k}}(x \bullet y) = \sum_{\mathbf{r} \leq \mathbf{k}} R^{\mathbf{r}}(x) R^{\mathbf{k}-\mathbf{r}}(y)$, *where* $\mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{k} - \mathbf{r} = (k_1 - r_1, \ldots, k_n - r_n)$.

Proposition 2. *The operator $\mathcal{T}_y$ is a continuous homomorphism of the algebra $H_{bvs}(\ell_p(\mathbb{C}^s))$ into itself.*

Proof. Let $x, y \in \ell_p(\mathbb{C}^s)$ and $\|x\| \le r, \|y\| \le r$. Then $\|x \bullet y\| = \sqrt[p]{\|x\|^p + \|y\|^p} \le \sqrt[p]{2}r$. Therefore, for every polynomial $P \in \mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$,

$$|\mathcal{T}_y(P(x))| \le \sup_{\|x\bullet y\| \le \sqrt[p]{2}r} P(x \bullet y) = \|P\|_{\sqrt[p]{2}r}.$$

Thus,$\mathcal{T}_y$ is a bounded and so continuous linear operator on the dense subspace $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ of the Fréchet space $H_{bvs}(\ell_p(\mathbb{C}^s))$ into itself. Hence, $\mathcal{T}_y$ can be uniquely extended by the linearity and continuity to the whole space $H_{bvs}(\ell_p(\mathbb{C}^s))$. So $\mathcal{T}_y$ is well-defined and continuous on $H_{bvs}(\ell_p(\mathbb{C}^s))$.

The fact that $\mathcal{T}_y$ is a homomorphism follows from the equalities

$$\mathcal{T}_y(f(x) + g(x)) = f(x \bullet y) + g(x \bullet y) = \mathcal{T}_y(f(x)) + \mathcal{T}_y(g(x)),$$
$$\mathcal{T}_y(\lambda f(x)) = \lambda f(x \bullet y) = \lambda \mathcal{T}_y(f(x)),$$
$$\mathcal{T}_y(f(x)g(x)) = f(x \bullet y)g(x \bullet y) = \mathcal{T}_y(f(x))\mathcal{T}_y(g(x)).$$

□

Following [8], we define the symmetric convolution on the space $H_{bvs}(\ell_1(\mathbb{C}^s))'$ of linear continuous functionals on $H_{bvs}(\ell_p(\mathbb{C}^s))$.

Definition 1. For any $f \in H_{bvs}(\ell_p(\mathbb{C}^s))$ and $\theta \in H_{bvs}(\ell_p(\mathbb{C}^s))'$, its *symmetric convolution* is defined according to

$$(\theta \star f)(x) = \theta[T_x(f)].$$

Definition 2. For any $\phi, \theta \in H_{bvs}(\ell_p(\mathbb{C}^s))'$, its *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(\theta \star f) = \phi(y \mapsto \theta(T_y f)).$$

Theorem 2. *The set $M_{bvs}(\ell_p(\mathbb{C}^s))$ with the operation "$\star$" is a cancellative semigroup. That is, the restriction of the symmetric convolution to $M_{bvs}(\ell_p(\mathbb{C}^s))$ is commutative, associative and $\phi \star \theta = \psi \star \theta$ implies $\phi = \psi$. Moreover, for every multi-index $\mathbf{k}$, $|\mathbf{k}| \ge \lceil p \rceil$,*

$$(\phi \star \theta)(H^{\mathbf{k}}) = \phi(H^{\mathbf{k}}) + \theta(H^{\mathbf{k}}). \tag{10}$$

Proof. Let us prove, first, equality (10). We have

$$(\theta \star H^{\mathbf{k}})(x) = \theta(T_x(H^{\mathbf{k}})) = \theta(H^{\mathbf{k}}(x) + H^{\mathbf{k}}) = H^{\mathbf{k}}(x) + \theta(H^{\mathbf{k}}).$$

Therefore,

$$(\phi \star \theta)(H^{\mathbf{k}}) = \phi(H^{\mathbf{k}}(x) + \theta(H^{\mathbf{k}})) = \phi(H^{\mathbf{k}}) + \theta(H^{\mathbf{k}}).$$

From this equality and formula (8) it follows the associativity and commutativity of $\varphi \star \theta \in M_{bvs}(\ell_p(\mathbb{C}^s))$. Also, if $\phi \star \theta = \psi \star \theta$, then $\phi\big(H^{\mathbf{k}}\big) = \psi\big(H^{\mathbf{k}}\big)$ for every $\mathbf{k}$ and so $\phi = \psi$. □

4. Representation of the spectrum by functions of exponential type.
Let $A_{uvs}\big(rB_{\ell_p(\mathbb{C}^s)}\big)$ be the completion of $H_{bvs}(\ell_p)$ with respect to the norm

$$\|f\|_r = \sup_{\|x\| \le r} |f(x)|.$$

Clearly, $A_{uvs}\big(rB_{\ell_p(\mathbb{C}^s)}\big) \supset H_{bvs}(\ell_p)$ and $A_{uvs}\big(rB_{\ell_p(\mathbb{C}^s)}\big)$ is the Banach algebra of all uniformly continuous block-symmetric analytic functions on the ball $rB_{\ell_p(\mathbb{C}^s)} \subset \ell_p(\mathbb{C}^s)$ of radius r.

$H_{bvs}(\ell_p)$ is the projective limit of algebras $A_{uvs}\big(rB_{\ell_p(\mathbb{C}^s)}\big)$, $r>0$ and $M_{bvs}(\ell_p(\mathbb{C}^s))$ is the union of the spectra of $A_{uvs}\big(rB_{\ell_p(\mathbb{C}^s)}\big)$.

Following [2] and [16], we can define the radius function $R(\phi)$ of a complex homomorphism $\phi\in M_{bvs}(\ell_p(\mathbb{C}^s))$ as the infimum of all r such that ϕ is continuous on $A_{uvs}\big(rB_{\ell_1(\mathbb{C}^s)}\big)$ and calculate it using formula (1), where ϕ_n is the restriction of the functional ϕ to the subspace of n-homogeneous block-symmetric polynomials.

Let $f(z)$ be an entire function of s variables: $f(z)=\sum\limits_{k_i\ge 0} a_{k_1\dots k_s}z_1^{k_1}\dots z_s^{k_s}$ and $\nu=(\nu_1,\dots,\nu_s)$ be a vector in $\mathbb{C}^s$, $\nu_j>0$. Let us recall that f is a function of *exponential type* ν if for every $\varepsilon>0$ there exists a positive number A_ε such that

$$|f(z)|\le A_\varepsilon\exp\sum_{j=1}^{s}(\nu_j+\varepsilon)|z_j|.$$

It is well-known (see e.g. [31, p. 139]) that f has type ν if and only if

$$\overline{\lim_{|\mathbf{k}|\to\infty}}\sqrt[|\mathbf{k}|]{\frac{k_1!\dots k_s!|a_k|}{\nu_1^{k_1}\dots\nu_n^{k_s}}}=1. \tag{11}$$

We will say [30] that $f(z)$, where $z\in\mathbb{C}^s$, has *plane* zeros if the set of zeros is a union of affine subspaces of codimension one.

Let $\mathbb{C}\{t_1,\dots,t_s\}$ be the space of all power series over $\mathbb{C}^s$. We denote by $\mathcal{R}$ and $\mathcal{H}$ the following maps from $M_{bvs}(\ell_1(\mathbb{C}^s))$ into $\mathbb{C}\{t_1,\dots,t_s\}$

$$\mathcal{R}(\varphi)(t)=\sum_{|k|=1}^{\infty}\prod_{i=1}^{s}t_i^{k_i}\varphi(R^{k_i}),\ \text{and}\ \ \mathcal{H}(\varphi)(t)=\sum_{|k|=1}^{\infty}\prod_{i=1}^{s}t_i^{k_i}\varphi(H^{k_i}),$$

where $t=(t_1,\dots,t_s)\in\mathbb{C}^s$, $\varphi\in M_{bvs}(\ell_1(\mathbb{C}^s))$.

Proposition 3. *$\mathcal{R}(\varphi)(t)$ is a function of exponential type for every fixed $\varphi\in M_{bvs}(\ell_1(\mathbb{C}^s))$ and $\mathcal{R}(\varphi)(0)=1$.*

Proof. Note that the $|\mathbf{k}|$-homogeneous polynomial of the power series $\mathcal{R}(\varphi)(t)$ can be written as

$$P_{|\mathbf{k}|}(t)=\sum_{k_1+\dots+k_s=|\mathbf{k}|}t_1^{k_1}\cdots t_s^{k_s}\varphi(R^{k_1}\cdots R^{k_s}).$$

Let $\mathfrak{p}_s(n)$ be the number of partitions of a natural number n into $n=k_1+\dots+k_s$ and $\mathfrak{p}(n)$ be the number of all partitions of n. Clearly, $\mathfrak{p}_s(n)\le\mathfrak{p}(n)$ and according to well-known Hardy-Ramanujan Asymptotic Partition Formula [17] $\mathfrak{p}(n)\sim\dfrac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}$ $(n\to\infty)$. Hence, $\limsup\limits_{n\to\infty}\sqrt[n]{\mathfrak{p}(\mathfrak{n})}=1$. Using Lemma 1, formulas (1) and (11)

$$\limsup_{|\mathbf{k}|\to\infty}\sqrt[|\mathbf{k}|]{k_1!\dots k_s!|\varphi_{|\mathbf{k}|}(P_{|\mathbf{k}|})|}\le\limsup_{|\mathbf{k}|\to\infty}\sqrt[|\mathbf{k}|]{\sum_{k_1+\dots+k_s=|\mathbf{k}|}k_1!\dots k_s!\|\varphi_{|\mathbf{k}|}\|\|R^{\mathbf{k}}\|}=$$

$$=\limsup_{|\mathbf{k}|\to\infty}\sqrt[|\mathbf{k}|]{\mathfrak{p}(|\mathbf{k}|)k_1!\dots k_s!\frac{1}{k_1!\dots k_s!}\|\varphi_{|\mathbf{k}|}\|}=R(\varphi).$$

Thus, $\mathcal{R}(\varphi)(t)$ is entire and of exponential type $(\theta_1,\dots,\theta_s)$ such that each θ_j does not exceed $R(\varphi)$. Also, $\mathcal{R}(\varphi)(0)=\varphi(R^0)=\varphi(1)=1$. □

Theorem 3. *The following identities hold*

1. $\mathcal{H}(\varphi \star \theta) = \mathcal{H}(\varphi) + \mathcal{H}(\theta)$.
2. $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$.

Proof. The first statement follows from Theorem 2. To prove the second statement we observe that

$$R^{\mathbf{k}}(x \bullet y) = \sum_{\mathbf{r} \le \mathbf{k}} R^{\mathbf{r}}(x) R^{\mathbf{k}-\mathbf{r}}(y).$$

Thus,

$$(\theta \star R^{\mathbf{k}})(x) = \theta(T_x(R^{\mathbf{k}})) = \theta\Big(\sum_{\mathbf{r} \le \mathbf{k}} R^{\mathbf{r}}(x) R^{\mathbf{k}-\mathbf{r}}\Big) = \sum_{\mathbf{r} \le \mathbf{k}} R^{\mathbf{r}}(x)\theta(R^{\mathbf{k}-\mathbf{r}}).$$

Therefore,

$$(\varphi \star \theta)\Big(R^{\mathbf{k}}\Big) = \varphi\Big(\sum_{\mathbf{r} \le \mathbf{k}} R^{\mathbf{r}}(x)\theta(R^{\mathbf{k}-\mathbf{r}})\Big) = \sum_{\mathbf{r} \le \mathbf{k}} \varphi(R^{\mathbf{r}})\theta(R^{\mathbf{k}-\mathbf{r}}).$$

On the other hand,

$$\mathcal{R}(\varphi)\mathcal{R}(\theta)(t) = \sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i} \varphi(R^{k_i}) \sum_{|\mathbf{l}|=1}^{\infty} \prod_{i=1}^{s} t_i^{l_i} \theta(R^{l_i}) =$$

$$= \sum_{|\mathbf{n}|=1}^{\infty} \sum_{|\mathbf{k}|+|\mathbf{l}|=|\mathbf{n}|} \prod_{i=1}^{s} t_i^{k_i+l_i} \varphi(R^{k_i})\theta(R^{l_i}) = \sum_{|\mathbf{n}|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i+l_i} \sum_{|\mathbf{k}|+|\mathbf{l}|=|\mathbf{n}|} \varphi(R^{k_i})\theta(R^{l_i}) =$$

$$= \sum_{|\mathbf{n}|=1}^{\infty} \prod_{i=1}^{s} t_i^{n_i} (\varphi \star \theta)(R^{n_i}) = \mathcal{R}(\varphi \star \theta).$$

□

Lemma 2. *If $\varphi = \delta_x$, then for every $x \in \ell_1(\mathbb{C}^s)$,*

$$\mathcal{R}(\delta_x)(t_1, \dots, t_s) = \prod_{i=1}^{\infty}(1 + x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s) = \sum_{n=0}^{\infty} G_n(x^{(1)} t_1 + \dots + x^{(s)} t_s),$$

where $(x_i^{(1)}, \dots, x_i^{(s)}) \in \mathbb{C}^s$, $i \ge 1$, $G_0 = 1$ and

$$G_n(x^{(1)} t_1 + \dots + x^{(s)} t_s) = \sum_{k_1 < k_2 < \dots < k_s}^{\infty} (x_{k_1}^{(1)} t_1 + \dots + x_{k_1}^{(s)} t_s) \cdot \ldots \cdot (x_{k_s}^{(1)} t_1 + \dots + x_{k_s}^{(s)} t_s).$$

Proof. For any $x \in \ell_1(\mathbb{C}^s)$, the product $\prod_{i=1}^{\infty}(1 + x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s)$ is absolutely convergent if the series $\sum_{i=1}^{\infty}(x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s)$ is absolutely convergent. But

$$\sum_{i=1}^{\infty} |x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s| \le \sum_{i=1}^{\infty}(|x_i^{(1)}||t_1| + \dots + |x_i^{(s)}||t_s|) =$$

$$= |t_1| \sum_{i=1}^{\infty} |x_i^{(1)}| + \dots + |t_s| \sum_{i=1}^{\infty} |y_i| \le \max\{|t_1|, \dots, |t_s|\}\Big(\sum_{i=1}^{\infty} |x_i^{(1)}| + \dots + \sum_{i=1}^{\infty} |x_i^{(s)}|\Big) \le$$

$$\le \max\{|t_1|, \dots, |t_s|\} \sum_{i=1}^{\infty} \big(|x_i^{(1)}| + \dots + |x_i^{(s)}|\big) < \infty,$$

and so $\prod_{i=1}^{\infty}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s)$ is absolutely convergent for all $x \in \ell_1(\mathbb{C}^s)$ and $(t_1,\ldots,t_s) \in \mathbb{C}^s$. Since for every $1 \le m < \infty$,

$$\sum_{|k|=0}^{m}\prod_{i=1}^{s} t_i^{k_i}\delta_x(R^k) = \prod_{i=1}^{m}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s),$$

and the series and product are absolutely convergent, we obtain that

$$\mathcal{R}(\delta_x)(t_1,\ldots,t_s) = \prod_{i=1}^{\infty}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s).$$

It is well-known from combinatorics [28] that $\sum\limits_{n=0}^{\infty} t^n G_n(x) = \prod\limits_{i=1}^{\infty}(1+x_i t)$. Thus,

$$\sum_{n=0}^{\infty} G_n(x^{(1)}t_1+\cdots+x^{(s)}t_s) = \prod_{i=1}^{\infty}(1+x_i^{(1)}t_1+\cdots+x_i^{(s)}t_s).$$

□

Let us construct some examples of elements of the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\ell_1(\mathbb{C}^s)$ which are not point evaluation functionals.

Let $(\alpha_1,\ldots,\alpha_s)$ be an nonzero vector in $\mathbb{C}^s$. Consider the following sequence of elements in $\ell_1(\mathbb{C}^s)$

$$\mathfrak{e}_n(\alpha_1,\ldots,\alpha_s) = \Bigg(\underbrace{\begin{pmatrix}0\\ \ldots\\ 0\end{pmatrix},\ldots,\begin{pmatrix}\alpha_1\\ \ldots\\ \alpha_s\end{pmatrix}}_{n},\begin{pmatrix}0\\ \ldots\\ 0\end{pmatrix},\ldots\Bigg)$$

of the space $\ell_1(\mathbb{C}^s)$ and for every $n \in \mathbb{N}$, put

$$v_n(\alpha_1,\ldots,\alpha_s) = \frac{1}{n}(\mathfrak{e}_1(\alpha_1,\ldots,\alpha_s)+\mathfrak{e}_2(\alpha_1,\ldots,\alpha_s)+\cdots+\mathfrak{e}_n(\alpha_1,\ldots,\alpha_s)) \in \ell_1(\mathbb{C}^s).$$

Then $\delta_{v_n(\alpha_1,\ldots,\alpha_s)}\big(H^{0,\ldots,1,\ldots,0}\big) = \alpha_i$ for every $n \in \mathbb{N}$, $i = 1,\ldots,s$, and for $|\mathbf{k}| > 1$,

$$\delta_{v_n(\alpha_1,\ldots,\alpha_s)}(H^{\mathbf{k}}) = \frac{n\alpha_1^{k_1}\cdots\alpha_s^{k_s}}{n^{|\mathbf{k}|}} \to 0 \quad (n\to\infty).$$

Note that for every n, $\|v_n\| = |\alpha_1|+\cdots+|\alpha_s|$, and $R\big(\delta_{v_n(\alpha_1,\ldots,\alpha_s)}\big) \le \|v_n\|$. Thus, $\delta_{v_n(\alpha_1,\ldots,\alpha_s)}$ belongs to the spectrum of $A_{uvs}\big(rB_{\ell_1(\mathbb{C}^s)}\big)$ for some $r \ge |\alpha_1|+\cdots+|\alpha_s|$. Since the spectrum of a Banach algebra is a compact set, the sequence of complex homomorphisms $\delta_{v_n(\alpha_1,\ldots,\alpha_s)}$ must have an accumulation point $\phi_{(\alpha_1,\ldots,\alpha_s)}$ in the spectrum and so in $M_{bvs}(\ell_1(\mathbb{C}^s))$. Hence, $\phi_{(\alpha_1,\ldots,\alpha_s)}(H^{0,\ldots,1,\ldots,0}) = \alpha_i$, $i=1,\ldots,s$, $\phi_{(\alpha_1,\ldots,\alpha_s)}(H^{\mathbf{k}}) = 0$ if $|\mathbf{k}| > 1$.

Clearly, $\mathcal{H}(\phi_{(\alpha_1,\ldots,\alpha_s)}) = \alpha_1+\cdots+\alpha_s$. To find $\mathcal{R}(\phi_{(\alpha_1,\ldots,\alpha_s)})$ note that

$$R^{\mathbf{k}}(v_n(\alpha_1,\ldots,\alpha_s)) = \frac{\alpha_1^{k_1}\cdots\alpha_s^{k_s}}{n^{k_1}\cdots n^{k_s}} C_n^{|\mathbf{k}|} C_{|\mathbf{k}|}^{k_1} C_{|\mathbf{k}|-k_1}^{k_2}\cdots C_{|\mathbf{k}|-k_1-k_2}^{k_3}\cdots C_{|\mathbf{k}|-k_1-\cdots-k_{s-2}}^{k_{s-1}},$$

where $C_n^m = \frac{n!}{m!(n-m)!}$ are the binomial coefficients. Hence,

$$\phi_{(\alpha_1,\ldots,\alpha_s)}(R^{\mathbf{k}}) = \lim_{n\to\infty} R^{\mathbf{k}}(v_n(\alpha_1,\ldots,\alpha_s)) = \lim_{n\to\infty}\frac{\alpha_1^{k_1}\cdots\alpha_s^{k_s}n!}{n^{|\mathbf{k}|}(n-|\mathbf{k}|)!k_1!\cdots k_s!} = \frac{\alpha_1^{k_1}\cdots\alpha_s^{k_s}}{k_1!\cdots k_s!}, \quad (12)$$

and so

$$\mathcal{R}(\phi_{(\alpha_1,\ldots,\alpha_s)})(t_1,\ldots,t_s) = \lim_{n\to\infty}\sum_{|\mathbf{k}|=0}^{n}\prod_{i=1}^{s} t_i^{k_i}\phi(R^{\mathbf{k}}) = \lim_{n\to\infty}\sum_{|\mathbf{k}|=0}^{n}\frac{\prod_{i=1}^{s}(\alpha_i t_i)^{k_i}}{\prod_{i=1}^{s}k_i!} = \exp\Big(\sum_{i=1}^{s}\alpha_i t_i\Big). \tag{13}$$

Proposition 4. *If $\psi = \delta_y \star \phi_{(\alpha_1,\ldots,\alpha_s)}$ for some $y \in \ell_1(\mathbb{C}^s)$ and $0 \neq (\alpha_1,\ldots,\alpha_s) \in \mathbb{C}^s$. Then there is no $x \in \ell_1(\mathbb{C}^s)$ such that $\psi = \delta_x$.*

Proof. If such a point x exists, then

$$\psi\big(H^{0,\ldots,1,\ldots,0}\big) = \alpha_i + H^{0,\ldots,1,\ldots,0}(y) = H^{0,\ldots,1,\ldots,0}(x),$$

where the multi-index $0,\ldots,1,\ldots,0$ means $\underbrace{0,\ldots,1}_{i},\ldots,0$. But, on the other hand, $\psi\big(H^{\mathbf{k}}\big) = H^{\mathbf{k}}(y) = H^{\mathbf{k}}(x)$. for every $\mathbf{k}$ such that $|\mathbf{k}| > 1$. From Corollary 2 it follows that $H^{0,\ldots,1,\ldots,0}(y) = H^{0,\ldots,1,\ldots,0}(x)$, but it contradicts the assumption that $(\alpha_1,\ldots,\alpha_s) \neq 0$. □

Proposition 5. *The set of invertible elements of the semigroup $(M_{bvs}(\ell_1(\mathbb{C}^s)),\star)$ coincides with the set of all complex homomorphisms of the form $\phi_{(\alpha_1,\ldots,\alpha_s)}$, $(\alpha_1,\ldots,\alpha_s) \in \mathbb{C}^s$, and*

$$\mathcal{R}(\phi_{(\alpha_1,\ldots,\alpha_s)})(t_1,\ldots,t_s) = \exp\Big(\sum_{i=1}^{s}\alpha_i t_i\Big).$$

Proof. Since by Theorem 3, $\mathcal{R}(\varphi\star\theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$, it follows that $\phi_{(-\alpha_1,\ldots,-\alpha_s)}$ is inverse to $\phi_{(\alpha_1,\ldots,\alpha_s)}$. So $\phi_{(\alpha_1,\ldots,\alpha_s)}$ is invertible for every $(\alpha_1,\ldots,\alpha_s) \in \mathbb{C}^s$. On the other hand, if φ is invertible and $\psi = \varphi^{-1}$, then

$$\mathcal{R}(\psi)(t_1,\ldots,t_s) = \frac{1}{\mathcal{R}(\varphi)(t_1,\ldots,t_s)}$$

is an entire function of exponential type and has no zeros. Thus, we have that

$$\mathcal{R}(\varphi)(t_1,\ldots,t_s) = \exp\Big(\sum_{i=1}^{s}\alpha_i t_i\Big)$$

for some complex numbers $\alpha_1,\ldots,\alpha_s$. By formula (13), $\varphi = \phi_{(\alpha_1,\ldots,\alpha_s)}$. □

Corollary 3. *Let Φ be a homomorphism on the subspace of block-symmetric polynomials in $H_{bvs}(\ell_1(\mathbb{C}^s))$ to itself such that $\Phi(H^{\mathbf{k}}) = -H^{\mathbf{k}}$ for every $\mathbf{k} = (k_1,\ldots,k_s)$. Then Φ is discontinuous.*

Proof. If Φ is continuous, it may be extended to a continuous homomorphism $\tilde{\Phi}$ of $H_{bvs}(\ell_1(\mathbb{C}^s))$. Then for $x \in \ell_1(\mathbb{C}^s)$,

$$H^{\mathbf{k}}(x) + \Phi(H^{\mathbf{k}})(x) = 0 \tag{14}$$

for all $\mathbf{k}$. This equality is true, in particular, for $x_0 = (\mathbf{1},0,\ldots,0,\ldots)$, where $\mathbf{1} = \underbrace{(1,1,\ldots,1)}_{s}$.

Let us denote $\psi = \delta_{x_0}\circ\tilde{\Phi}$. From the continuity of $\tilde{\Phi}$ we have that $\psi \in M_{bvs}(\ell_1(\mathbb{C}^s))$. From equality (14) it follows that $\delta_{x_0}\star\psi = \delta_0$, that is, δ_{x_0} is invertible and $\psi = \delta_{x_0}^{-1}$. But, according to the Proposition 5, δ_{x_0} is not invertible. □

Theorem 4. *Let $\varphi = \phi_{(\alpha_1,\dots,\alpha_s)} \star \delta_x$ for some $x \neq 0$ and $(\alpha_1,\dots,\alpha_s) \in \mathbb{C}^s$. Then $\mathcal{R}(\varphi)(t)$ is an entire function with plane zeros, that is* $\ker \mathcal{R}(\varphi)$ *consists of hyperplanes.*

Proof. From Theorem 3, Lemma 2 and formula (13) we have that

$$\mathcal{R}(\varphi) = \exp\Big(\sum_{i=1}^{s} \alpha_i t_i\Big) \prod_{i=1}^{\infty} (1 + x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s).$$

The set of zeros of this function is the union of sets $\{t \in \mathbb{C}^s \colon 1 + x_i^{(1)} t_1 + \dots + x_i^{(s)} t_s = 0\}$. But each of these sets is a hyperplane, providing $(x_i^{(1)},\dots,x_i^{(s)}) \neq 0$. □

We do not know: *Do there exist elements in $M_{bvs}(\ell_1(\mathbb{C}^s))$ different from φ as in Theorem 4?* This question is open even in the case $s = 1$ (see [10]).

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