THE SHARP BOUND OF THE THIRD HANKEL DETERMINANTS FOR INVERSE OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

B. Rath, K. S. Kumar, D. V. Krishna, G. K. S. Viswanadh

We study the sharp bound for the third Hankel determinant for the inverse function $f$, when it belongs to the class of starlike functions with respect to symmetric points. Let $S_*$ be the class of starlike functions with respect to symmetric points. In the article proves the following statements (Theorem): If $f \in S_*$ then

$$|H_{3,1}(f^{-1})| \leq 1,$$

and the result is sharp for $f(z) = z/(1 - z^2)$.

1. Introduction. Let $\mathcal{A}$ be the family of all analytic normalized mappings $f$ of the form

$$f(z) = \sum_{n=1}^{+\infty} a_n z^n, \quad a_1 = 1,$$

in the open unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and $S$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. Pommereneke [9] characterized the $r^{th}$ Hankel determinant of order $n$, for $f$ with $r, n \in \mathbb{N} = \{1, 2, 3, \ldots \}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \quad (1)$$

In recent years, research on the estimation of an upper bound of the second and third order Hankel determinant is investigated by many authors. Particularly, the problem of estimating $H_{3,1}(f)$ is technically much more difficult [2, 4, 6, 7, 11, 14, 16], and only few sharp bounds have been obtained.

The class of starlike functions with respect to symmetric points is introduced by Sakaguchi [12] and is denoted as $S_*$. These functions satisfy the analytic condition

$$\text{Re} \left( \frac{2zf'(z)}{f(z) - f(-z)} \right) > 0 \quad z \in D. \quad (2)$$

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Recently, when \( f \in \mathcal{S}_s^\ast \), Virendra et al. [15] estimated bounds for the third Hankel determinant, namely \( H_{3,1}(f) \) obtained for \( r = 3, n = 1 \) in (1).

For \( f \in \mathcal{S} \) its inverse function \( f^{-1} \) is given by

\[
f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, \quad |w| < r_0(f) \quad \left( r_0(f) \geq \frac{1}{4} \right).
\]

Ali [1] determined sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő coefficient functional of the inverse functions which belong to the class of strongly starlike functions denoted by \( \mathcal{SS}^\ast(\alpha) \) defined as \( |\arg (zf'(z)/f(z))| < \pi \alpha \), \( 0 < \alpha \leq 1 \). Recently, Sim et al. [14] obtained sharp bound of \( |H_{2,2}(f^{-1})| \) for the class of strongly Ozaki functions denoted by \( \mathcal{F}_\alpha(\lambda) \) is defined as

\[
\Re \{1 + (zf''(z)/f'(z))\} < (1 - 2\lambda)/2 \quad (1/2 \leq \lambda \leq 1).
\]

Motivated by the results obtained by the authors mentioned above, in this paper we are making an attempt to estimate sharp bound for the third Hankel determinant namely \( |H_{3,1}(f^{-1})| \), when \( f \) belongs to the class of \( \mathcal{S}_s^\ast \).

Let \( \mathcal{P} \) be a class of all functions \( p \) having a positive real part in \( \mathbb{D} \):

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.
\]

Every such a function is called the Carathéodory function. In view of (2) and (3), the coefficients of functions in \( \mathcal{S}_s^\ast \) have suitable representation expressed by coefficients of functions in \( \mathcal{P} \). Hence, to estimate the upper bound of \( |H_{3,1}(f^{-1})| \), we build our computation on the well known formulas on coefficients \( c_2 \) (see [9, p. 166]) , \( c_3 \) (see [8]) and \( c_4 \) can be found in [11].

The foundation for proof of our main result are the following lemmas and we adopt some ideas from Libera and Zlotkiewicz [8].

**Lemma 1** ([5]). If \( p \in \mathcal{P} \), then \( |c_i - \mu c_j c_{i-j}| \leq 2 \), satisfies for the values \( i, j \in \mathbb{N} \), with \( i > j \) and \( \mu \in [0, 1] \), which is same as \( |c_{n+k} - \mu c_n c_k| \leq 2 \), for \( n, k \in \mathbb{N} \), with \( \mu \in [0, 1] \).

**Lemma 2** ([9]). For \( p \in \mathcal{P} \), then \( |c_i| \leq 2 \), for \( i \in \mathbb{N} \), equality occurs for the function

\[
p_0 = \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}.
\]

**Lemma 3.** If \( p \in \mathcal{P} \), then

\[
2c_2 = c_1^2 + t\zeta, \quad 4c_3 = c_1^3 + 2c_1t\zeta - c_1t\zeta^2 + 2t(1 - |\zeta|^2)\eta,
\]

and

\[
8c_4 = c_1^4 + 3c_1^2t\zeta + (4 - 3c_1^2)t\zeta^2 + c_1^2t\zeta^3 + 4t(1 - |\zeta|^2)\left(1 - |\eta|^2\right)\xi + 4t(1 - |\zeta|^2)\left(c_1\eta - c\zeta\eta - \bar{\zeta}\eta^2\right),
\]

where \( t := 4 - c_1^2 \), for some \( \zeta, \eta \) and \( \xi \) such that \( |\zeta| \leq 1, |\eta| \leq 1 \) and \( |\xi| \leq 1 \).

**2. Bound for inverse of \( \mathcal{S}_s^\ast \).** We now prove the main theorem of this paper.

**Theorem.** If \( f \in \mathcal{S}_s^\ast \) then

\[
|H_{3,1}(f^{-1})| \leq 1,
\]

and the result is sharp for \( f(z) = z/(1 - z^2) \).
Proof. For \( f \in S^*_s \), there exists an analytic function \( p \in P \) such that
\[
\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \iff 2zf'(z) = p(z) \{ f(z) - f(-z) \}.
\]
Using the series representation for \( f \) and \( p \) in (4), a simple calculation gives
\[
a_2 = \frac{c_1}{2}, \quad a_3 = \frac{c_2}{2}, \quad a_4 = \frac{c_1c_2 + 2c_3}{8} \quad \text{and} \quad a_5 = \frac{c_2^2 + 2c_4}{8}.
\]
Now from the definition (1), we have
\[
w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n (f^{-1}(w))^n = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \left( w + \sum_{n=2}^{\infty} t_n w^n \right)^n.
\]
Upon simplification, we obtain
\[
(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4
+ (t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \ldots = 0.
\]
Equating the coefficients in powers of \( w \) from (6), after simplifying, we get
\[
t_2 = -a_2; \quad t_3 = -a_3 + 2a_2^2; \quad t_4 = -a_4 + 5a_2a_3 - 5a_2^3;
\]
\[
t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4.
\]
From (5) in (7), upon simplification, we obtain
\[
t_2 = -\frac{c_1}{2}; \quad t_3 = \frac{1}{2} \left( c_1^2 - c_2 \right); \quad t_4 = \frac{1}{8} \left( -5c_1^3 + 9c_1c_2 - 2c_3 \right); \quad t_5 = \frac{1}{8} \left( 7c_1^4 - 18c_1^2c_2 + 5c_2^2 + 6c_1c_3 - 2c_4 \right).
\]
Now, in view of (1) with \( r = 3 \) and \( n = 1 \), we have
\[
H_{3,1}(f^{-1}) = \begin{vmatrix} 1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix},
\]
Using the values of \( t_j, (j \in \{2, 3, 4, 5\} \) from (8) in (9), we obtain
\[
H_{3,1}(f^{-1}) = \frac{1}{64} \left( c_1^6 - 6c_1^4c_2 + 13c_1^2c_2^2 - 12c_2^3 + 4c_1c_2c_3 - 4c_3^2 - 4c_1^2c_4 + 8c_2c_4 \right).
\]
Substituting the values of \( c_2, c_3 \) and \( c_4 \) from Lemma 1.3 and taking into account that \( t = (4 - c_1^2) \) in (10), after simplifying, we get
\[
H_{3,1}(f^{-1}) = \frac{(4 - c_1^2)^2}{64} \left( \frac{1}{4} c_1^2s^2 + \frac{1}{4} c_1^2s^4 + \frac{1}{4} (1 - |s|^2) \left( 4c_1^2 - 4c_1^2c_1 \right) \eta + \right.
+ (1 - |s|^2) \left( -1 - |s|^2 \right) \eta^2 - \left( 4 - \frac{c_1^2}{2} \right) \zeta^3 + 2(1 - |s|^2) \left( 1 - |\eta|^2 \right) \zeta \xi \right).
\]
Taking modulus on either side of the above expression, since $|\xi| \leq 1$, using $|\zeta| := x \in [0,1]$, $|\eta| := y \in [0,1]$ and $c_1 := c \in [0,2]$ in (11), we obtain
\[ |H_{3,1}(f^{-1})| \leq \frac{F(c, x, y)}{64}, \]
where $F : \mathbb{R}^3 \to \mathbb{R}$ is defined as
\[ F(c, x, y) = \left(4 - c^2\right)^2 \left(\frac{c^2 x^2}{4} + \left(4 - \frac{c^2}{2}\right) x^3 + \frac{c^2 x^4}{4} + \frac{1}{4} \left(1 - x^2\right) \left(4cx + 4cx^2\right) y + \left(1 + x^2\right) \left(1 - x^2\right) y^2 + 2x \left(1 - x^2\right) \left(1 - y^2\right) \right) \]
Now we will maximize the function $F(c, x, y)$ in the region $\Omega := [0, 2] \times [0, 1] \times [0, 1]$. 

A. On the vertices of $\Omega$, from (13), we have
- $F(0, 0, 0) = 0$, $F(0, 1, 0) = F(0, 1, 1) = 64$, $F(0, 0, 1) = 16$,
- $F(2, 0, 0) = F(2, 0, 1) = F(2, 1, 0) = F(2, 1, 1) = 0$.

B. On the edges of $\Omega$ from (13), we have
- (a) $F(0, 0, y) = 16y^2 \leq 16$ for $y \in (0, 1)$.
- (b) $F(0, 1, y) = 64$ for $y \in (0, 1)$.
- (c) $F(0, x, 0) = 32x + 32x^3 \leq 64$ for $x \in (0, 1)$.
- (d) $F(0, x, 1) = 16 + 64x^3 - 16x^4$ for $x \in (0, 1)$, is an increasing function of $x$. Therefore, $F(0, x, 1) \leq F(0, 1, 1) = 64$.
- (e) $F(c, 0, 1) = (4 - c^2)^2 \leq 16$, for $c \in (0, 2)$.
- (f) $x = 1$ and $y = 1 \lor x = 1$ and $y = 0$, then $F(c, 1, y) = 4(4 - c^2)^2 \leq 64$.
- (g) $F(2, x, 0) = F(2, x, 1) = F(2, 0, y) = F(2, 1, y) = F(c, 0, 0) = 0$ for $c \in (0, 2)$, $x \in (0, 1)$ and $y \in (0, 1)$.

C. Considering the edges of $\Omega$, from (13), we get
- (a) $F(2, x, y) = 0$ for $x \in (0, 1)$, $y \in (0, 1)$.
- (b) If $x \in (0, 1)$, $y \in (0, 1)$ then
\[ F(0, x, y) = 16(4x^3 + (1 - x^2)(1 + x^2)y^2 + 2x(1 - x^2)(1 - y^2)) = 32x + 32x^3 + (16 - 32x + 32x^3 - 16x^4)y^2 = 32x + 32x^3 + 16(1 - x^3)(1 + x)y^2 := G_1(x, y) \]
for $x \in (0, 1)$ and $y \in (0, 1)$; $G_1(x, y)$ is an increasing function of $y$ and hence
\[ G_1(x, y) \leq G_1(x, 1) = 16 + 64x^3 - 16x^4, \]
then from B(d), we have $F(0, x, y) \leq 64$.
- (c) $F(c, 0, y) = (4 - c^2)^2 y^2 \leq (4 - c^2)^2 \leq 16$ for $c \in (0, 2)$, $y \in (0, 1)$.
- (d) For the edge $x = 1$, we observe that $F(c, 1, y)$ is independent of $y$, so it is same as
\[ B(f) \]
i.e
\[ F(c, 1, y) \leq 64, \quad c \in (0, 2) \text{ and } y \in (0, 1). \]
- (e) For $c \in (0, 2)$, $x \in (0, 1)$
\[ \begin{align*}
F(c, x, 0) &= \left(4 - c^2\right)^2 \left(\frac{c^2 x^2}{4} + \left(4 - \frac{c^2}{2}\right) x^3 + \frac{c^2 x^4}{4} + 2x \left(1 - x^2\right)\right) \\
&= (4 - c^2)^2 \left(2x + x^3 + \left(\frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{4}\right) c^2\right) \leq (4 - c^2)^2 \left(4 + \frac{c^2}{64}\right) \\
&= 64 - \frac{c^2}{64} (1040 - c^4 + 248(4 - c^2)) \leq 64.
\end{align*} \]
(f) For $c \in (0, 2), \ x \in (0, 1)$

\[
F(c, x, 1) = (4 - c^2)^2 \left( \frac{c^2 x^2}{4} + \left( 4 - \frac{c^2}{2} \right) x^3 + \frac{c^2 x^4}{4} + \frac{1}{4}(1 - x^2)(4cx + 4cx^2) + \right. \\
\left. + (1 + x^2) (1 - x^2) \right) = \\
= (4 - c^2)^2 \left( 1 + 4x^3 - x^4 + c(x + x^2 - x^3 - x^4) + c^2 \left( \frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{4} \right) \right) = \\
= (4 - c^2)^2 \left( 1 + 4x^3 - x^4 + 2(x + x^2 - x^3 - x^4) + 4 \left( \frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{4} \right) \right) \leq \\
\leq (4 - c^2)^2 \left( 1 + 2x + 3x^2 - 2x^4 \right) \leq 16 \times 4 = 64.
\]

D. Now we consider the interior of region $\Omega$.

Differentiate $F(c, x, y)$ given in (13) partially with respect to $y$, we get

\[
\frac{\partial F}{\partial y} = 16cx - 8c^3x + c^5x + 16cx^2 - 8c^3x^2 + c^5x^2 - 16cx^3 + 8c^3x^3 - c^5x^3 - \\
-16cx^4 + 8c^3x^4 - c^5x^4 + 32y - 16c^2y + 2c^4y - 64xy + 32c^2xy - \\
-4c^4xy + 64c^3y - 32c^2x^3y + 4c^4x^3y - 32c^4y + 16c^2x^4y - 2c^4x^4y.
\]

Since $\frac{\partial F}{\partial y} = 0$, only for $y = -\frac{cx(1+x)}{2(1-x^2)^2} := y_0$ and $y_0 < 0$ for $x \in (0, 1)$. Hence, we conclude that $F(c, x, y)$ has no critical point in the interior of $\Omega$.

In review of cases A, B, C and D, we obtain

\[
\max \left\{ F(c, x, y) = 64 : c \in [0, 2], x \in [0, 1] \text{ and } y \in [0, 1] \right\}. \quad (14)
\]

From expression (12) and (14), we get $|H_{3,1}(f^{-1})| \leq 1$.

The result is sharp and equality is attained by the function

\[
f(z) = f_0(z) := \frac{z}{1 - z^2}, \quad z \in \mathbb{D},
\]

which belongs to $\mathcal{S}^*_s$ having the coefficients $a_2 = a_4 = 0$ and $a_3 = a_5 = 1$ from which, we obtain $t_2 = t_4 = 0$, $t_3 = -1$ and $t_4 = 2$.

\[\Box\]

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Department of Mathematics, Gitam Institute of Science, GITAM University, Visakhapatnam- 530 045, A.P., India
brath@gitam.edu
skarri9@gitam.in
vamsheekrishna1972@gmail.com
svsu06@gmail.com

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