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THE SHARP BOUND OF THE THIRD HANKEL DETERMINANTS FOR INVERSE OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

B. Rath, K. S. Kumar, D. V. Krishna, G. K. S. Viswanadh. *The sharp bound of the third Hankel determinants for inverse of starlike functions with respect to symmetric points*, Mat. Stud. **58** (2022), 45–50.

We study the sharp bound for the third Hankel determinant for the inverse function f, when it belongs to of the class of starlike functions with respect to symmetric points.

Let $\mathcal{S}_s^*$ be the class of starlike functions with respect to symmetric points. We prove the following statements (Theorem): If $f \in \mathcal{S}_s^*$ then

$$\left|H_{3,1}(f^{-1})\right| \leq 1,$$

and the result is sharp for $f(z) = z/(1-z^2)$.

1. Introduction. Let $\mathcal{A}$ be the family of all analytic normalized mappings f of the form

$$f(z) = \sum_{n=1}^{+\infty} a_t z^n, \quad a_1 = 1,$$

in the open unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. Pommerenke [9] characterized the r^{th}-Hankel determinant of order n, for f with $r,\ n \in \mathbb{N} = \{1, 2, 3, ...\}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{1}$$

In recent years, research on the estimation of an upper bound of the second and third order Hankel determinant is investigated by many authors. Particularly, the problem of estimating $H_{3,1}(f)$ is technically much more difficult [2, 4, 6, 7, 11, 14, 16], and only few sharp bounds have been obtained.

The class of starlike functions with respect to symmetric points is introduced by Sakaguchi [12] and is denoted as $\mathcal{S}_s^*$. These functions satisfy the analytic condition

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0 \quad z \in \mathbb{D}. \tag{2}$$

2010 *Mathematics Subject Classification:* 30C45, 30C50.
Keywords: analytic function; upper bound; Hankel determinant; Carathéodory function.
doi:10.30970/ms.58.1.45-50

Recently, when $f \in \mathcal{S}_s^*$, Virendra et al. [15] estimated bounds for the third Hankel determinant, namely $H_{3,1}(f)$ obtained for $r = 3, n = 1$ in (1).
For $f \in \mathcal{S}$ its inverse function f^{-1} is given by

$$f^{-1}(w) = w + \sum_{n=2}^{+\infty} t_n w^n, \quad |w| < r_0(f) \quad \left(r_0(f) \geq \frac{1}{4}\right).$$

Ali [1] determined sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő coefficient functional of the inverse functions which belong to the class of strongly starlike functions denoted by $\mathcal{SS}^*(\alpha)$ defined as $|\arg(zf'(z)/f(z))| < \pi\alpha/2$, $(0 < \alpha \leq 1)$. Recently, Sim et al. [14] obtained sharp bound of $|H_{2,2}(f^{-1})|$ for the class of strongly Ozaki functions denoted by $\mathcal{F}_o(\lambda)$ is defined as

$$\operatorname{Re}\{1 + (zf''(z)/f'(z))\} < (1 - 2\lambda)/2 \quad (1/2 \leq \lambda \leq 1).$$

Motivated by the results obtained by the authors mentioned above, in this paper we are making an attempt to estimate sharp bound for the third Hankel determinant namely $|H_{3,1}(f^{-1})|$, when f belongs to the class of $\mathcal{S}_s^*$.

Let $\mathcal{P}$ be a class of all functions p having a positive real part in $\mathbb{D}$:

$$p(z) = 1 + \sum_{n=1}^{+\infty} c_n z^n. \tag{3}$$

Every such a function is called the Carathéodory function. In view of (2) and (3), the coefficients of functions in $\mathcal{S}_s^*$ have suitable representation expressed by coefficients of functions in $\mathcal{P}$. Hence, to estimate the upper bound of $|H_{3,1}(f^{-1})|$, we build our computation on the well known formulas on coefficients c_2 (see [9, p. 166]), c_3 (see [8]) and c_4 can be found in [11].

The foundation for proof of our main result are the following lemmas and we adopt some ideas from Libera and Złotkiewicz [8].

Lemma 1 ([5]). *If $p \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$, satisfies for the values $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0,1]$, which is same as $|c_{n+k} - \mu c_n c_k| \leq 2$, for $n, k \in \mathbb{N}$, with $\mu \in [0,1]$.*

Lemma 2 ([9]). *For $p \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function*

$$p_0 = \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

Lemma 3. *If $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + t\zeta, \quad 4c_3 = c_1^3 + 2c_1 t\zeta - c_1 t\zeta^2 + 2t\left(1 - |\zeta|^2\right)\eta,$$

and

$$8c_4 = c_1^4 + 3c_1^2 t\zeta + \left(4 - 3c_1^2\right)t\zeta^2 + c_1^2 t\zeta^3 + 4t\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\xi + \\ +4t\left(1 - |\zeta|^2\right)\left(c_1\eta - c\zeta\eta - \bar{\zeta}\eta^2\right),$$

where $t := 4 - c_1^2$, for some ζ, η and ξ such that $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$.

2. Bound for inverse of $\mathcal{S}_s^*$. We now prove the main theorem of this paper.

Theorem. *If $f \in \mathcal{S}_s^*$ then*

$$\left|H_{3,1}(f^{-1})\right| \leq 1,$$

and the result is sharp for $f(z) = z/(1 - z^2)$.

Proof. For $f \in \mathcal{S}_s^*$, there exists an analytic function $p \in \mathcal{P}$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = p(z) \Longleftrightarrow 2zf'(z) = p(z)\left\{f(z)-f(-z)\right\}. \tag{4}$$

Using the series representation for f and p in (4), a simple calculation gives

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{c_2}{2}, \quad a_4 = \frac{c_1c_2+2c_3}{8} \quad \text{and} \quad a_5 = \frac{c_2^2+2c_4}{8}. \tag{5}$$

Now from the definition (1), we have

$$w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n(f^{-1}(w))^n = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n\Big(w + \sum_{n=2}^{\infty} t_n w^n\Big)^n.$$

Upon simplification, we obtain

$$\begin{aligned}(t_2+a_2)w^2 + (t_3+2a_2t_2+a_3)w^3 + (t_4+2a_2t_3+a_2t_2^2+3a_3t_2+a_4)w^4 \\ +(t_5+2a_2t_4+2a_2t_2t_3+3a_3t_3+3a_3t_2^2+4a_4t_2+a_5)w^5+\ldots = 0.\end{aligned} \tag{6}$$

Equating the coefficients in powers of w from (6), after simplifying, we get

$$\begin{aligned}&t_2 = -a_2; t_3 = -a_3+2a_2^2; t_4 = -a_4+5a_2a_3-5a_2^3;\\ &t_5 = -a_5+6a_2a_4-21a_2^2a_3+3a_3^2+14a_2^4.\end{aligned} \tag{7}$$

From (5) in (7), upon simplification, we obtain

$$\begin{aligned}&t_2 = -\frac{c_1}{2},\ t_3 = \frac{1}{2}\left(c_1^2-c_2\right),\ t_4 = \frac{1}{8} = \left(-5c_1^3+9c_1c_2-2c_3\right)\\ &t_5 = \frac{1}{8}\left(7c_1^4-18c_1^2c_2+5c_2^2+6c_1c_3-2c_4\right).\end{aligned} \tag{8}$$

Now, in view of (1) with $r=3$ and $n=1$, we have

$$H_{3,1}(f^{-1}) = \begin{vmatrix} t_1=1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix}, \tag{9}$$

Using the values of t_j, $(j \in \{2,3,4,5\})$ from (8) in (9), we obtain

$$H_{3,1}(f^{-1}) = \frac{1}{64}\left(c_1^6-6c_1^4c_2+13c_1^2c_2^2-12c_2^3+4c_1c_2c_3-4c_3^2-4c_1^2c_4+8c_2c_4\right). \tag{10}$$

Substituting the values of c_2, c_3 and c_4 from Lemma 1.3 and taking into account that $t=(4-c_1^2)$ in (10), after simplifying, we get

$$\begin{aligned}H_{3,1}(f^{-1}) = \frac{(4-c_1^2)^2}{64}\Bigg(&\frac{1}{4}c_1^2\zeta^2 + \frac{1}{4}c_1^2\zeta^4 + \frac{1}{4}\left(1-|\zeta|^2\right)\left(4\zeta c_1 - 4\zeta^2 c_1\right)\eta + \\ &+\left(1-|\zeta|^2\right)\left(-1-|\zeta|^2\right)\eta^2 - \left(4-\frac{c_1^2}{2}\right)\zeta^3 + 2\left(1-|\zeta|^2\right)\left(1-|\eta|^2\right)\zeta\xi\Bigg).\end{aligned} \tag{11}$$

Taking modulus on either side of the above expression, since $|\xi| \le 1$, using $|\zeta| := x \in [0,1]$, $|\eta| := y \in [0,1]$ and $c_1 := c \in [0,2]$ in (11), we obtain

$$\left| H_{3,1}(f^{-1}) \right| \le \frac{F(c,x,y)}{64}, \tag{12}$$

where $F\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$F(c,x,y) = \left(4-c^2\right)^2 \left(\frac{c^2x^2}{4} + \left(4 - \frac{c^2}{2}\right) x^3 + \frac{c^2x^4}{4} + \frac{1}{4}\left(1-x^2\right)\left(4cx+4cx^2\right) y + \right.$$
$$\left. + \left(1+x^2\right)\left(1-x^2\right) y^2 + 2x\left(1-x^2\right)\left(1-y^2\right) \right) \tag{13}$$

Now we will maximize the function $F(c,x,y)$ in the region $\Omega := [0,2] \times [0,1] \times [0,1]$.

A. On the vertices of Ω, from (13), we have

$$F(0,0,0) = 0, \quad F(0,1,0) = F(0,1,1) = 64, \quad F(0,0,1) = 16,$$
$$F(2,0,0) = F(2,0,1) = F(2,1,0) = F(2,1,1) = 0.$$

B. On the edges of Ω from (13), we have

(a) $F(0,0,y) = 16y^2 \le 16$ for $y \in (0,1)$.
(b) $F(0,1,y) = 64$ for $y \in (0,1)$,
(c) $F(0,x,0) = 32x + 32x^3 \le 64$ for $x \in (0,1)$,
(d) $F(0,x,1) = 16 + 64x^3 - 16x^4$ for $x \in (0,1)$, is an increasing function of x. Therefore,

$$F(0,x,1) \le F(0,1,1) = 64.$$

(e) $F(c,0,1) = \left(4-c^2\right)^2 \le 16$, for $c \in (0,2)$,
(f) $x = 1$ and $y = 1 \vee x = 1$ and $y = 0$, then $F(c,1,y) = 4(4-c^2)^2 \le 64$.
(g) $F(2,x,0) = F(2,x,1) = F(2,0,y) = F(2,1,y) = F(c,0,0) = 0$ for $c \in (0,2)$, $x \in (0,1)$ and $y \in (0,1)$.

C. Considering the edges of Ω, from (13), we get

(a) $F(2,x,y) = 0$ for $x \in (0,1),\ y \in (0,1)$.
(b) If $x \in (0,1),\ y \in (0,1)$ then

$$F(0,x,y) = 16(4x^3 + (1-x^2)(1+x^2)y^2 + 2x(1-x^2)(1-y^2)) = 32x + 32x^3 +$$
$$+(16 - 32x + 32x^3 - 16x^4)y^2 = 32x + 32x^3 + 16(1-x)^3(1+x)y^2 := G_1(x,y)$$

for $x \in (0,1)$ and $y \in (0,1)$; $G_1(x,y)$ is an increasing function of y and hence

$$G_1(x,y) \le G_1(x,1) = 16 + 64x^3 - 16x^4,$$

then from B(d), we have $F(0,x,y) \le 64$.

(c) $F(c,0,y) = (4-c^2)^2y^2 \le (4-c^2)^2 \le 16$ for $c \in (0,2),\ y \in (0,1)$.

(d) For the edge $x = 1$, we observe that $F(c,1,y)$ is independent of y, so it is same as B(f), i.e

$$F(c,1,y) \le 64, \quad c \in (0,2) \text{ and } y \in (0,1).$$

(e) For $c \in (0,2),\ x \in (0,1)$

$$F(c,x,0) = \left(4-c^2\right)^2 \left(\frac{c^2x^2}{4} + \left(4 - \frac{c^2}{2}\right) x^3 + \frac{c^2x^4}{4} + 2x\left(1-x^2\right) \right) =$$
$$= (4-c^2)^2 \left\{ 2x + 2x^3 + \left(\frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{4} \right) c^2 \right\} \le (4-c^2)^2 \left\{ 4 + \frac{c^2}{64} \right\} =$$
$$= 64 - \frac{c^2}{64}(1040 - c^4 + 248(4-c^2)) \le 64.$$

(f) For $c \in (0,2),\ x \in (0,1)$

$$F(c,x,1) = \left(4-c^2\right)^2 \left(\frac{c^2x^2}{4} + \left(4-\frac{c^2}{2}\right)x^3 + \frac{c^2x^4}{4} + \frac{1}{4}\left(1-x^2\right)\left(4cx+4cx^2\right) + \right.$$
$$\left. + \left(1+x^2\right)\left(1-x^2\right)\right) =$$
$$= \left(4-c^2\right)^2 \left(1+4x^3-x^4+c(x+x^2-x^3-x^4)+c^2\left(\frac{x^2}{4}-\frac{x^3}{2}+\frac{x^4}{4}\right)\right) =$$
$$= \left(4-c^2\right)^2 \left(1+4x^3-x^4+2(x+x^2-x^3-x^4)+4\left(\frac{x^2}{4}-\frac{x^3}{2}+\frac{x^4}{4}\right)\right) \le$$
$$\le \left(4-c^2\right)^2 \left(1+2x+3x^2-2x^4\right) \le 16 \times 4 = 64.$$

D. Now we consider the interior of region Ω.

Differentiate $F(c,x,y)$ given in (13) partially with respect to y, we get

$$\frac{\partial F}{\partial y} = 16cx - 8c^3x + c^5x + 16cx^2 - 8c^3x^2 + c^5x^2 - 16cx^3 + 8c^3x^3 - c^5x^3 -$$
$$-16cx^4 + 8c^3x^4 - c^5x^4 + 32y - 16c^2y + 2c^4y - 64xy + 32c^2xy -$$
$$-4c^4xy + 64x^3y - 32c^2x^3y + 4c^4x^3y - 32x^4y + 16c^2x^4y - 2c^4x^4y.$$

Since $\frac{\partial F}{\partial y} = 0$, only for $y = -\frac{cx(1+x)}{2(-1+x)^2} := y_0$ and $y_0 < 0$ for $x \in (0,1)$, we conclude that $F(c,x,y)$ has no critical point in the interior of Ω.

In review of cases **A**, **B**, **C** and **D**, we obtain

$$\max\left\{F(c,x,y) = 64\colon c \in [0,2], x \in [0,1] \text{ and } y \in [0,1]\right\}. \tag{14}$$

From expression (12) and (14), we get $\left|H_{3,1}(f^{-1})\right| \le 1$.

The result is sharp and equality is attained by the function

$$f(z) = f_0(z) := \frac{z}{1-z^2}, \quad z \in \mathbb{D},$$

which belongs to $\mathcal{S}_s^*$ having the coefficients $a_2 = a_4 = 0$ and $a_3 = a_5 = 1$ from which, we obtain $t_2 = t_4 = 0$, $t_3 = -1$ and $t_4 = 2$. □

Acknowledgement. The authors are highly grateful to the esteemed Referee(s) for a comprehensive reading of the manuscript and making valuable suggestions, leading to a better paper presentation.

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Received 27.07.2022
Revised 23.10.2022