УДК 517.53

I. V. Andrusyak, P. V. Filevych, O. H. Oryshchyn

MINIMAL GROWTH OF ENTIRE FUNCTIONS WITH PRESCRIBED ZEROS OUTSIDE EXCEPTIONAL SETS

I. V. Andrusyak, P. V. Filevych, O. H. Oryshchyn. *Minimal growth of entire functions with prescribed zeros outside exceptional sets*, Mat. Stud. **58** (2022), 51–57.

Let h be a positive continuous increasing to $+\infty$ function on $\mathbb{R}$. It is proved that for an arbitrary complex sequence (ζ_n) such that $0<|\zeta_1|\le|\zeta_2|\le\ldots$ and $\zeta_n\to\infty$ as $n\to\infty$, there exists an entire function f whose zeros are the ζ_n, with multiplicities taken into account, for which

$$\ln m_2(r,f)=o(N(r)),\quad r\notin E,\ r\to+\infty.$$

with a set E satisfying $\int_{E\cap(1,+\infty)}h(r)dr<+\infty$, if and only if $\ln h(r)=O(\ln r)$ as $r\to+\infty$. Here $N(r)$ is the integrated counting function of the sequence (ζ_n) and

$$m_2(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|\ln|f(re^{i\theta})||^2d\theta\right)^{1/2}.$$

1. Introduction and results. Let $\mathcal{Z}$ be the class of all complex sequences $\zeta=(\zeta_n)$ such that $0<|\zeta_1|\le|\zeta_2|\le\ldots$ and $\zeta_n\to\infty$ as $n\to\infty$. For every sequence $\zeta=(\zeta_n)$ from the class $\mathcal{Z}$, we denote by $\mathcal{E}(\zeta)$ the class of all entire functions whose zeros are precisely the ζ_n. Here a complex number that occurs m times in the sequence ζ corresponds to a zero of multiplicity m, and for each $r\ge0$ we put

$$n(r,\zeta)=\sum_{|\zeta_n|\le r}1,\qquad N(r,\zeta)=\int_0^r\frac{n(t,\zeta)}{t}dt.$$

Let us set $\mathbb{D}_R=\{z\in\mathbb{C}:|z|<R\}$ for every $R>0$. If $R>0$, then for an arbitrary meromorphic function f in $\mathbb{D}_R$ and all $r\in[0,R)$ we denote by $T(r,f)$ the Nevanlinna characteristic function, and

$$m_q(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|\ln|f(re^{i\theta})||^qd\theta\right)^{1/q},\quad q\ge1.$$

For an arbitrary entire function f and each $r\ge0$, we put $M(r,f)=\max\{|f(z)|:|z|=r\}$.

By L denote the class of all positive continuous increasing to $+\infty$ functions on $\mathbb{R}$.

A. A. Goldberg [1] proved the following two theorems.

Theorem A ([1]). *Let $\zeta\in\mathcal{Z}$ be an arbitrary sequence. Then there exists an entire function $f\in\mathcal{E}(\zeta)$ such that*

$$\ln\ln M(r,f)=o(N(r,\zeta)),\quad r\notin E,\ r\to+\infty,\tag{1}$$

where E is an exceptional set of finite logarithmic measure, i.e., $\int_{E\cap(1,+\infty)}d\ln r<+\infty$.

2010 *Mathematics Subject Classification:* 30D15, 30D20, 30D35.
Keywords: entire function; zeros; maximum modulus; Nevanlinna characteristic function.
doi:10.30970/ms.58.1.51-57

Theorem B ([1]). *Let $\psi \in L$. If $\psi(x) = o(x)$ as $x \to +\infty$, then there exist a sequence $\zeta \in \mathcal{Z}$ and a set F of upper linear density 1, i.e.,*

$$\overline{\lim_{r\to+\infty}} \frac{1}{r} \int_{F\cap(0,r)} dr = 1,$$

such that for any entire function $f \in \mathcal{E}(\zeta)$ we have

$$\psi(N(r,\zeta)) = o(\ln\ln M(r,f)), \quad r \in F,\ r \to +\infty. \tag{2}$$

The following two theorems show that we can make more precise conclusions about the sizes of the sets E and F in Theorems A and B.

Theorem C ([2]). *Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. Then there exist an entire function $f \in \mathcal{E}(\zeta)$ and a function $\alpha \in L$ such that (1) holds with an exceptional set E satisfying*

$$\int_{E\cap(1,+\infty)} r^{\alpha(r)} dr < +\infty. \tag{3}$$

Theorem D ([2]). *Let $\psi \in L$. If $\varliminf\limits_{x\to+\infty} \frac{\psi(x)}{x} = 0$, then there exist a sequence $\zeta \in \mathcal{Z}$ and a set $F = \cup_{n=0}^{\infty}(x_n; y_n)$ satisfying*

$$1 < x_0 < y_0 < x_1 < y_1 < \ldots, \qquad \lim_{n\to\infty} \frac{\ln y_n}{\ln x_n} = +\infty,$$

such that for any entire function $f \in \mathcal{E}(\zeta)$ we have (2).

Note that Theorem C is also true for the relation

$$\ln T(r,f) = o(N(r,\zeta)), \quad r \notin E,\ r \to +\infty,$$

instead of (1), because for an arbitrary entire function f and every $r \geq 0$ we obtain

$$T(r,f) := \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta \leq \ln^+ M(r,f).$$

The proof of Theorem D, given in [2], shows that this theorem is true for the relation

$$\psi(N(r,\zeta)) = o(\ln T(r,f)), \quad r \in F,\ r \to +\infty,$$

instead of (2). Therefore, Theorem D is also true for the relation

$$\psi(N(r,\zeta)) = o(\ln m_2(r,f)), \quad r \in F,\ r \to +\infty,$$

instead of (2), because for any function $f \in \mathcal{E}(\zeta)$ and each $r \geq 0$ we have

$$m_2(r,f) \geq m_1(r,f) = 2T(r,f) - N(r,\zeta) - \ln|f(0)|.$$

In this paper, we will prove the following two theorems.

Theorem 1. *Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. Then there exist an entire function $f \in \mathcal{E}(\zeta)$ and a function $\alpha \in L$ such that*

$$\ln m_2(r,f) = o(N(r,\zeta)), \quad r \notin E,\ r \to +\infty, \tag{4}$$

where E is an exceptional set satisfying (3).

Theorem 2. *Let $h \in L$. If*

$$\varlimsup_{r\to+\infty} \frac{\ln h(r)}{\ln r} = +\infty, \tag{5}$$

then there exists a sequence $\zeta \in \mathcal{Z}$ such that for any function $f \in \mathcal{E}(\zeta)$ we get

$$N(r,\zeta) = o(\ln m_2(r,f)), \quad r \in F(f),\ r \to +\infty, \tag{6}$$

where $F(f)$ is a set satisfying

$$\int_{F(f)\cap(1,+\infty)} h(r)dr = +\infty. \tag{7}$$

Theorem 2 shows that the function $\alpha \in L$ in Theorem 1 depends on ζ in general. Therefore, estimate (3) for the size of the exceptional set E in Theorem 1 is exact in a certain sense.

In connection with the above results, the following question arises: *is* (3) *an exact estimate for the size of the exceptional set E in Theorem C?* In other words, *is it possible to replace* $\ln m_2(r,f)$ *by* $\ln\ln M(r,f)$ *in Theorem 2?* This question remains open.

At the end of the introductory part, we note that some other problems concerning comparisons of the growth of an entire function f to the distribution of its zeros were considered, in particular, in [3]–[10]. We also note that questions regarding the sizes of exceptional sets in various asymptotic relations between characteristics of entire functions were investigated, for example, in [12]–[19].

2. Auxiliary results. We will deduce Theorem 1 from Theorem C by using the following two lemmas.

Lemma 1 ([20])**.** *Let $0 < r < R < \varrho$, and let f be a meromorphic function in $\mathbb{D}_\varrho$, with $f(0) = 1$. Then*

$$m_2(r,f) \le (1 + 8/\sqrt{\log_2(R/r)})T(R,f).$$

Lemma 2 ([21])**.** *Let $-\infty < x_0 < a \le +\infty$, and functions $H(x)$, $u(x)$ and $\varphi(u)$ satisfy the following conditions:*

1) H is continuous increasing to $+\infty$ on $[x_0,a)$;

2) u is non-decreasing unbounded on $[x_0,a)$;

3) φ is positive non-decreasing unbounded on $[u_0,+\infty)$ and $\int_{u_0}^{+\infty} \frac{du}{\varphi(u)} < +\infty$, where $u_0 = u(x_0)$.

Then for the set

$$E = \left\{ x \in [x_0,a) : u\left(H^{-1}\left(H(x) + \frac{1}{\varphi(u(x))}\right)\right) \ge u(x) + 1 \right\}$$

we have $\int_E dH(x) < +\infty$.

Note that Lemma 2 is a version of the classical Borel-Nevanlinna theorem (see, for example, [11], p. 120) and is easily deduced from this theorem.

3. Proof of Theorems. *Proof of Theorem 1.* Let $\zeta \in \mathcal{Z}$ be an arbitrary sequence. By Theorem C, there exist an entire function $f \in \mathcal{E}(\zeta)$ and a function $\beta \in L$ such that

$$\ln T(r,f) = o(N(r,\zeta)), \quad r \notin E_1,\ r \to +\infty, \tag{8}$$

where E_1 is an exceptional set satisfying $\int_{E_1\cap(1,+\infty)} r^{\beta(r)}dr < +\infty$. Clearly, we can assume that $f(0)=1$. Let us prove that there exists a function $\alpha\in L$ such that for the function f we have (4) with an exceptional set E satisfying (3).

Since $\ln r = o(N(r,\zeta))$ as $r\to+\infty$, there exists a function $\eta\in L$ for which

$$\eta(r)\ln r = o(N(r,\zeta)),\quad r\to+\infty. \tag{9}$$

We choose some $r_0>1$ such that $T(r_0,f)>1$, and consider the set

$$E_2=\{r>r_0 : \ln m_2(r,f) > \eta(r)\ln r + 2\ln T(r,f)\}.$$

Fix an arbitrary integer $k\ge 1$ and prove that $\int_{E_2} r^k dr < +\infty$. For each $r\ge r_0$, we put

$$R(r)=\left(r^{k+1}+\frac{1}{T^2(r,f)}\right)^{1/(k+1)}.$$

Note that

$$\ln\frac{R(r)}{r}=\frac{1}{k+1}\ln\left(1+\frac{1}{r^{k+1}T^2(r,f)}\right)\sim\frac{1}{(k+1)r^{k+1}T^2(r,f)},\quad r\to+\infty. \tag{10}$$

Let $H(r)=r^{k+1}$ and $u(r)=T(r,f)$ for all $r\ge r_0$, and let $\varphi(u)=u^2$ for all $u\ge1$. Using Lemma 2, we see that, for the set $F=\{r\ge r_0 : T(R(r),f)>T(r,f)+1\}$, the estimate $\int_F r^k dr<+\infty$ holds.

Further, for all sufficiently large $r\notin F$, say for $r\ge r_1$, by Lemma 1 and (10) we have

$$\ln m_2(r,f)\le \ln\left(1+\frac{8\sqrt{\ln 2}}{\sqrt{\ln(R(r)/r)}}\right)+\ln T(R(r),f)\le \eta(r)\ln r+2\ln T(r,f),$$

that is, $r\notin E_2$. Therefore, $E_2\subset F\cup[r_0,r_1]$, and hence $c_k:=\int_{E_2} r^k dr<+\infty$.

We choose a sequence (s_k) increasing to $+\infty$ such that $s_1\ge r_0$ and $s_k\ge 2^k c_{k+1}$ for every integer $k\ge1$. It is easy to see that there exists a function $\gamma\in L$ such that $\gamma(r)\le k$ for all $r\in[s_k,s_{k+1})$ and every integer $k\ge1$. Then

$$\int_{E_2} r^{\gamma(r)}dr=\int_{E_2\cap[r_0,s_1)} r^{\gamma(r)}dr+\sum_{k=1}^{\infty}\int_{E_2\cap[s_k,s_{k+1})} r^{\gamma(r)}dr\le$$
$$\le\int_{r_0}^{s_1} r^{\gamma(r)}dr+\sum_{k=1}^{\infty}\int_{E_2\cap[s_k,s_{k+1})} r^k dr\le\int_{r_0}^{s_1} r^{\gamma(r)}dr+\sum_{k=1}^{\infty}\frac{1}{s_k}\int_{E_2\cap[s_k,s_{k+1})} r^{k+1}dr\le$$
$$\le\int_{r_0}^{s_1} r^{\gamma(r)}dr+\sum_{k=1}^{\infty}\frac{c_{k+1}}{s_k}\le\int_{r_0}^{s_1} r^{\gamma(r)}dr+\sum_{k=1}^{\infty}\frac{1}{2^k}<+\infty.$$

We set $\alpha(r)=\min\{\beta(r),\gamma(r)\}$ for all $r\in\mathbb{R}$, and let $E=E_1\cup E_2$. Then $\alpha\in L$ and

$$\int_E r^{\alpha(r)}dr\le\int_{E_1} r^{\beta(r)}dr+\int_{E_2} r^{\gamma(r)}dr<+\infty,$$

that is, the set E satisfies (3). In addition, from the definition of the set E_2, (9), and (8) we see that relation (4) holds. □

Proof of Theorem 2. Le $h \in L$ be a function that satisfies (5). We set $l(x) = h(x/e)$ for all $x \in \mathbb{R}$. Then $l \in L$ and

$$\overline{\lim_{r\to+\infty}} \frac{\ln l(r)}{\ln r} = +\infty. \tag{11}$$

It follows from (11) that there exists a sequence (r_k) increasing to $+\infty$ such that $r_1 > 1$, $l(r_1) > 1$, and for every integer $k \geq 2$ we have

$$r_k > e^2 r_{k-1}, \qquad \ln([l(r_k)] - [l(r_{k-1})]) > kl(r_{k-1}) \ln r_k. \tag{12}$$

Here and further, for a number $x \in \mathbb{R}$, $[x]$ denotes the largest integer not greater than x.

For all integers $k \geq 1$, we put $n_k = [l(r_k)]$. It is clear that (n_k) is an increasing sequence of positive integers. Let $m_1 = n_1$, and let $m_k = n_k - n_{k-1}$ for each integer $k \geq 2$. Note that $\sum_{j=1}^{k} m_j = n_k$ for an arbitrary integer $k \geq 1$.

Let us form the sequence $\zeta = (\zeta_n)$ as follows $\underbrace{r_1, \ldots, r_1}_{m_1 \text{ times}}, \underbrace{r_2, \ldots, r_2}_{m_2 \text{ times}}, \ldots, \underbrace{r_k, \ldots, r_k}_{m_k \text{ times}}, \ldots$, that is, we set $\zeta_n = r_k$ for all integers $n \in (n_k - m_k, n_k]$ and $k \geq 1$. Then $n(r, \zeta) = 0$ if $r \in [0, r_1)$, and $n_\zeta(r) = n_k$ if $r \in [r_k, r_{k+1})$ for some integer $k \geq 1$.

Consider a function $f \in \mathcal{E}(\zeta)$ and prove that this function satisfies (6) with a set $F(f)$ satisfying (7).

The function f has no zeros in the disk $\mathbb{D}_{r_1}$, and therefore there exists an analytic function

$$g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

in $\mathbb{D}_{r_1}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{D}_{r_1}$. Let $r > 0$, and let $c_p(r)$ be the p-th Fourier coefficient of the function $\ln |f(re^{i\theta})|$, i.e.

$$c_p(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\theta} \ln |f(re^{i\theta})| d\theta, \quad p \in \mathbb{Z}.$$

Then, since all ζ_n are positive, for each integer $p \geq 1$, according to the Poisson-Jensen formula (see, for example, [11, p. 16–17]), we have

$$c_p(r) = \frac{1}{2}\alpha_p r^p + \frac{1}{2p} \sum_{|\zeta_n|<r} \left(\left(\frac{r}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{r}\right)^p \right). \tag{13}$$

Using (13), for $R > r > 0$ we obtain the following equality

$$c_p(R) - \left(\frac{R}{r}\right)^p c_p(r) = \frac{1}{2p} \sum_{r\leq|\zeta_n|<R} \left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{R}\right)^p \right) + \frac{1}{2p} \sum_{|\zeta_n|<r} \left(\left(\frac{\zeta_n R}{r^2}\right)^p - \left(\frac{\zeta_n}{R}\right)^p \right).$$

Both terms on the right-hand side of this equality are non-negative, and so we have

$$|c_p(R)| + \left(\frac{R}{r}\right)^p |c_p(r)| \geq \frac{1}{2p} \sum_{r\leq|\zeta_n|<R} \left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{R}\right)^p \right).$$

Since $x^2 + y^2 \geq (x + y)^2/2$ for arbitrary real x and y, we get

$$|c_p(R)|^2 + \left(\frac{R}{r}\right)^{2p} |c_p(r)|^2 \geq \frac{1}{8p^2} \left(\sum_{r\leq|\zeta_n|<R} \left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{R}\right)^p \right) \right)^2. \tag{14}$$

We now denote by K the set of all integers $k \geq 1$ such that $m_2(r,f) \geq \sqrt[4]{m_k}$ for all $r \in [r_k \exp(-1/m_k), r_k]$.

Let us first consider the case when the set K is infinite. In this case, we put

$$F(f) = \bigcup_{k \in K} [r_k \exp(-1/m_k), r_k].$$

Since $h(r_k \exp(-1/m_k)) \geq l(r_k) \geq n_k > m_k$ for each integer $k \geq 1$, we have

$$\int_{r_k \exp(-1/m_k)}^{r_k} h(r)dr \geq m_k r_k (1 - e^{-1/m_k}) = (1 + o(1)) r_k, \quad k \to +\infty.$$

Therefore, for the set $F(f)$ estimate (7) holds. In addition, if $k \in K$ and $k \geq 2$, then, using the second of inequalities (12), for an arbitrary $r \in [r_k \exp(-1/m_k), r_k]$ we obtain

$$N(r, \zeta) = \int_{r_1}^{r} \frac{n(t,\zeta)}{t} dt \leq n_{k-1} \ln \frac{r}{r_1} < \frac{1}{k} \ln m_k \leq \frac{4}{k} \ln m_2(r,f),$$

and this implies (6).

Let us now consider the case when the set K is finite. Then for each integer $k \geq k_1$ there exists a point $s_k \in [r_k \exp(-1/m_k), r_k]$ such that $m_2(s_k, f) < \sqrt[4]{m_k}$. Put

$$F(f) = \cup_{k \geq 1} [r_k \exp(1/m_k), r_k \exp(2/m_k)].$$

Since $e^x - 1 > x$ for all $x > 0$, for each integer $k \geq 1$ we have

$$\int_{r_k \exp(1/m_k)}^{r_k \exp(2/m_k)} h(r)dr \geq m_k r_k (e^{2/m_k} - e^{1/m_k}) > r_k,$$

and therefore, for the set $F(f)$ estimate (7) holds. By (12) we obtain $r_k \exp(2/m_k) < r_{k+1}$ for each integer $k \geq 1$, and in addition $n_k \sim m_k$ as $k \to \infty$. So, for an arbitrary integer $k \geq k_2$, we get

$$N(r_k e^{2/m_k}, \zeta) = \int_{r_1}^{r_k} \frac{n(t,\zeta)}{t} dt + \int_{r_k}^{r_k \exp(2/m_k)} \frac{n(t,\zeta)}{t} dt \leq n_{k-1} \ln \frac{r_k}{r_1} + \frac{2n_k}{m_k} \leq \frac{1}{k} \ln m_k + 3. \tag{15}$$

Using (14) with $r = s_k$ and taking into account that $e^x - e^{-x} > 2x$ for all $x > 0$, for arbitrary integers $k \geq 1$ and $p \in [1, m_k]$, and for each $R \in [r_k \exp(1/m_k), r_k \exp(2/m_k)]$ we have

$$|c_p(R)|^2 + e^6 |c_p(s_k)|^2 \geq |c_p(R)|^2 + \left(\frac{R}{s_k}\right)^{2p} |c_p(s_k)|^2 \geq$$

$$\geq \frac{1}{8p^2} \left(\sum_{s_k \leq |\zeta_n| < R} \left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{R}\right)^p \right) \right)^2 = \frac{m_k^2}{8p^2} \left(\left(\frac{R}{r_k}\right)^p - \left(\frac{r_k}{R}\right)^p \right)^2 \geq$$

$$\geq \frac{m_k^2}{8p^2} \left(2p \ln \frac{R}{r_k} \right)^2 = \frac{m_k^2}{2} \ln^2 \frac{R}{r_k} \geq \frac{1}{2}.$$

Therefore, for each $R \in [r_k \exp(1/m_k), r_k \exp(2/m_k)]$ and all integers $k \geq k_3$ we obtain

$$m_2^2(R,f) = |c_0(R)|^2 + 2 \sum_{p=1}^{\infty} |c_p(R)|^2 \geq 2 \sum_{p=1}^{m_k} |c_p(R)|^2 \geq 2 \sum_{p=1}^{m_k} \left(\frac{1}{2} - e^6 |c_p(s_k)|^2 \right) \geq$$

$$\geq m_k - e^6 m_2^2(s_k, f) > m_k - e^6 \sqrt{m_k} > m_k/2.$$

This and (15) imply (6). □

REFERENCES

1. A.A. Gol'dberg, *The representation of a meromorphic function in the form of a quotient of entire functions*, Izv. Vyssh. Uchebn. Zaved. Mat., **10** (1972), 13–17. (in Russian)
2. I.V. Andrusyak, P.V. Filevych, *The growth of an entire function with a given sequence of zeros*, Mat. Stud., **30** (2008), №2, 115–124.
3. W. Bergweiler, *Canonical products of infinite order*, J. Reine Angew. Math., **430** (1992), 85–107. doi.org/10.1515/crll.1992.430.85
4. W. Bergweiler, *A question of Gol'dberg concerning entire functions with prescribed zeros*, J. Anal. Math., **63** (1994), №1, 121–129. doi.org/10.1007/BF03008421
5. J. Miles, *On the growth of entire functions with zero sets having infinite exponent of convergence*, Ann. Acad. Sci. Fenn. Math., **27** (2002), 69–90.
6. M.M. Sheremeta, *A remark to the construction of canonical products of minimal growth*, Mat. Fiz. Anal. Geom., **11** (2004), №2, 243–248.
7. I.V. Andrusyak, P.V. Filevych, *The minimal growth of entire function with given zeros*, Nauk. Visn. Chernivets'kogo Univ. Mat., **421** (2008), 13–19. (in Ukrainian)
8. I.V. Andrusyak, P.V. Filevych, *The growth of entire function with zero sets having integer-valued exponent of convergence*, Mat. Stud., **32** (2009), №1, 12–20. (in Ukrainian)
9. I.V. Andrusyak, P.V. Filevych, *The minimal growth of entire functions with given zeros along unbounded sets*, Mat. Stud., **54** (2020), №2, 146–153. doi.org/10.30970/ms.54.2.146-153
10. A.A. Kondratyuk, Y.V. Vasyl'kiv, *Growth majorants and quotient representations of meromorphic functions*, Comput. Methods Funct. Theory, **1** (2001), №2, 595–606. doi.org/10.1007/BF03321007
11. A.A. Gol'dberg, I.V. Ostrovskii, Value distribution of meromorphic functions, Nauka, Moscow, 1970. (in Russian)
12. W. Bergweiler, *On meromorphic functions that share three values and on the exceptional set in Wiman-Valiron theory*, Kodai Math. J., **13** (1990), №1, 1–9. doi.org/10.2996/kmj/1138039154
13. P.V. Filevych, *On the London theorem concerning the Borel relation for entire functions*, Ukr. Math. J., **50** (1998), №11, 1801–1804. doi.org/10.1007/BF02524490
14. O.B. Skaskiv, P.V. Filevych, *On the size of an exceptional set in the Wiman theorem*, Mat. Stud., **12** (1999), №1, 31–36.
15. P.V. Filevych, *On an estimate of the size of the exceptional set in the lemma on the logarithmic derivative*, Math. Notes, **67** (2000), №4, 512–515. doi.org/10.1007/BF02676408
16. P.V. Filevych, *An exact estimate for the measure of the exceptional set in the Borel relation for entire functions*, Ukr. Math. J., **53** (2001), №2, 328–332. doi.org/10.1023/A:1010489609188
17. O.B. Skaskiv, T.M. Salo, *Entire Dirichlet series of rapid growth and new estimates for the measure of exceptional sets in theorems of the Wiman-Valiron type*, Ukr. Math. J., **53** (2001), №6, 978–991. doi.org/10.1023/A:1013308103502
18. P.V. Filevych, *Asymptotic relations between maximums of absolute values and maximums of real parts of entire functions*, Math. Notes, **75** (2004), №3-4, 410–417. doi.org/10.1023/B:MATN.0000023320.27440.57
19. O.B. Skaskiv, T.M. Salo, *Minimum modulus of lacunary power series and h-measure of exceptional sets*, Ufa Math. J., **9** (2017), №4, 135–144. doi.org/10.13108/2017-9-4-135
20. J. Miles, D.F. Shea, *On the growth of meromorphic functions having at least one deficient value*, Duke Math. J., **43** (1976), №1, 171–186. doi.org/10.1215/S0012-7094-76-04315-5
21. M. Magola, P. Filevych, *The distribution of values of random analytic functions*, Mat. Visn. Nauk. Tov. Im. Shevchenka, **9** (2012), 180–215. (in Ukrainian)

Department of Mathematics
Lviv Politechnic National University
Lviv, Ukraine
andrusyak.ivanna@gmail.com
p.v.filevych@gmail.com
oksana.orushchun@gmail.com

Received 12.07.2022
Revised 19.09.2022