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QUASI-MONOMIALS WITH RESPECT TO SUBGROUPS OF THE PLANE AFFINE GROUP

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Let H be a subgroup of the plane affine group $\mathrm{Aff}(2)$ considered with the natural action on the vector space of two-variable polynomials. The polynomial family $\{B_{m,n}(x,y)\}$ is called quasi-monomial with respect to H if the group operators in two different bases $\{x^m y^n\}$ and $\{B_{m,n}(x,y)\}$ have *identical* matrices. We obtain a criterion of quasi-monomiality for the case when the group H is generated by rotations and translations in terms of exponential generating function for the polynomial family $\{B_{m,n}(x,y)\}$.

1. Introduction. An important area of application of the group representation theory is the analysis of 2D and 3D images. For recognizing and classifying images using the machine learning algorithms it is necessary to construct such *features* of images that remain invariant for those geometric transformations of the plane which do not distort the scene of an image. For 2D images, such transformations are rotations, translations, scaling and the composition of these transformations. The corresponding invariant features were first presented in [1] and are called *moment invariants*. If we identify a halftone image with a limited function of two variables $f\colon \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^2$, then the value

$$\boldsymbol{m}_{p,q}(f(x,y)) = \boldsymbol{m}_{pq} = \iint\limits_{\Omega} f(x,y)\pi_{m,n}(x,y)dxdy,$$

is called the π-moment of the image, of order $p+q$, where the family of polynomials $\{\pi_{m,n}(x,y)\}$ is the basis of an infinite-dimensional vector $\mathbb{R}$-space of polynomials in two variables.

Since the 1960s the moment invariants have been actively used in the image analysis, see [2]–[4]. Depending on the choice of basis $\{\pi_{m,n}(x,y)\}$ different systems of moments are considered. For the simplest case $\pi_{m,n}(x,y) = x^m y^n$ the corresponding moments are called *geometric moments.* The real plane affine group $\mathrm{Aff}(2)$ and its subgroups act naturally on the geometric moments and as a result the corresponding *algebras of moment invariants* arise. Of particular interest in applications are the moment invariants with respect to the action of the group that is a semi-direct product of the plane translation group $T(2)$, the direct product of the *complex* plane rotation group $SO(2)$ and the uniform scaling group $\mathbb{R}^*$. The algebra of moment invariants of this group is well studied, in particular, there is a well-known explicit description of its generating elements [5], [6]. However, the practical use

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of geometric moments causes difficulties due to their numerical instability when working in discrete domains, because the values $x^m y^m$ increase rapidly with increasing the image size. To avoid this problem one consider the so-called *orthogonal moments*, which are generated by the basis $\pi_{m,n}(x,y) = F_m(x)F_n(y)$ where $\{F_n(x)\}$ is a family of orthogonal polynomials in one variable. But there is a problem of calculation of moment invariants which now need to be calculated in a new basis. The change of the basis encounters great technical difficulties and is satisfactorily solved only for Legendre moments, and only for simple transformations of the plane [7], which do not include rotations.

A fundamentally different approach was used in the paper [10]. The authors discover an interesting and unexpected fact: it turns out that for the basis $\pi_{m,n}(x,y) = H_m(x)H_n(y)$, where $\{H_n(x)\}$ are classical Hermitian polynomials, the form of invariant moments with respect to the group $SO(2)$ is the same as for geometric moments. This follows from the fact that the matrix of the linear operator T_θ of rotation angle θ in the basis $\{x^m y^n\}$ is the same, see [10], as in the basis $\{H_m(x)H_n(y)\}$. Thus, this nice property of Hermitian polynomials allowed us to efficiently calculate the $SO(2)$-invariant Hermitian orthogonal moments.

In [8] these ideas were developed and a complete description of all polynomials which this property was given. The property was called the *quasi-monomial* property. For completeness, we give definitions and the main results about the quasi-monomial polynomials with respect to the group of rotations $SO(2)$. The group $SO(2)$ acts on functions in two variables by rotations T_θ :

$$T_\theta(f(x,y)) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta), \theta \in [0, 2\pi].$$

In particular, T_θ acts on the basis vectors as follows

$$T_\theta(x^m \cdot y^n) = (x\cos\theta - y\sin\theta)^m \cdot (x\sin\theta + y\cos\theta)^n =$$
$$= \sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^j \binom{m}{j}\binom{n}{k}(\cos\theta)^{m-j+k}(\sin\theta)^{n-k+j}x^{m+n-j-k}y^{j+k}.$$

Let $\{B_{m,n}(x,y)\}$ be another basis, $\deg_x B_{m,n}(x,y) = m, \deg_y B_{m,n}(x,y) = n$. We are interested is such a basis which transformed under rotation T_θ in the same way as are the monomials $x^m y^n$.

Definition 1 ([8])**.** The polynomial family $\{B_{m,n}(x,y)\}$ is called *quasi-monomial with respect to the rotation group* $SO(2)$ if the following identity holds

$$T_\theta(B_{m,n}(x,y)) = \sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^j \binom{m}{j}\binom{n}{k}(\cos\theta)^{m-j+k}(\sin\theta)^{n-k+j}B_{m+n-j-k,j+k}(x,y),$$

for all $m, n \in \mathbb{N}$.

In other words, the linear operator T_θ in these two different bases $\{x^m y^n\}$ and $\{B_{m,n}(x,y)\}$ has *identical* matrices. It turns out that there is a simple criterion for the quasi-monomiality of a polynomial family in terms of its exponential generating function.

Theorem 1 ([8])**.** *The polynomial family* $\{B_{m,n}(x,y)\}$ *defined by the exponential generating function* $G = \sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}$ *is quasi-polynomial with respect to the rotation group* $SO(2)$ *if and only if* G *is a function of the variables* $ux+vy, x^2+y^2$ *and* u^2+v^2,

$$G = G(ux+vy, x^2+y^2, u^2+v^2).$$

The quasi-monomial polynomials with respect to $SO(2)$ also allow a description in the language of differential equations

Theorem 2 ([8]). *The polynomial family $\{B_{m,n}(x,y)\}$ is a quasi-monomial with respect to the group of rotations if it satisfies the differential equation*

$$x\frac{\partial B_{m,n}(x,y)}{\partial y} - y\frac{\partial B_{m,n}(x,y)}{\partial x} = nB_{m+1,n-1}(x,y) - mB_{m-1,n+1}(x,y).$$

Also, in this case, the exponential generating function G for $B_{m,n}(x,y)$ satisfies the following differential equation

$$x\frac{\partial G}{\partial y} - y\frac{\partial G}{\partial x} = v\frac{\partial G}{\partial u} - u\frac{\partial G}{\partial v}.$$

In this paper we introduce a notion of quasi-monomiality with respect to an arbitrary subgroup of the plane affine group Aff(2).

Definition 2. The family of polynomials $\{B_{m,n}(x,y)\}$ is called *quasi-monomial with respect to a subgroup H of the group* Aff(2), if they change under the action of H in the same way as are the polynomials $\{x^m y^n\}_{m,n=0}^{\infty}$.

Knowledge of such quasi-monomials with respect to the groups of plane and space transformations is important for fast and stable calculation of the corresponding image moment invariants.

This paper gives a similar description of quasi-monomial polynomials with respect to continuous subgroups of transformations of the plane affine group, namely the group of translations and the scaling group (uniform and non-uniform). The quasi-monomials with respect to the groups generated by pairwise joint transformations of these groups and the rotation group are also described.

2. The quasi-monomials with respect to the scaling group. The two-parametric plane scaling group acts on a function as follows

$$T_{s,t}(f(x,y)) = f(sx,ty), s,t \in \mathbb{R}.$$

Definition 3. A system of polynomials $\{B_{m,n}\}$ is called *quasi-polynomials* with respect to the group of plane stretches if the action of the group coincides with the action of the group on monomials, i.e.,

$$T_{s,t}(B_{m,n}(x,y)) = t^m s^n B_{m,n}(x,y), \tag{1}$$

for all $s,t \in \mathbb{R}$.

The following theorem presents a simple criterion for the quasi-monomiality of a polynomial family in terms of its generating function

Theorem 3. *The polynomial family $\{B_{m,n}(x,y)\}$ defined by the exponential generating function*

$$G = \sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}$$

is quasi-monomial with respect to the scaling group if and only if G is a function of the two variables xu, yv, $G = G(xu,yv)$.

Proof. ($\Longrightarrow$) Assume that the family of polynomials $\{B_{m,n}(x,y)\}$ satisfies the condition (1). We first prove that the polynomials $\{B_{m,n}(x,y)\}$ satisfy the following system of differential equations $x\frac{\partial B_{m,n}(x,y)}{\partial x} = mB_{m,n}(x,y)$, $y\frac{\partial B_{m,n}(x,y)}{\partial y} = nB_{m,n}(x,y)$, for all integer indices m,n.

In fact, let us differenting the identity (1) by s

$$x\frac{\partial B_{m,n}(sx,ty)}{\partial x} = ms^{m-1}t^n B_{m,n}(x,y).$$

Putting $s=1, t=1$ we get

$$x\frac{\partial B_{m,n}(x,y)}{\partial x} = mB_{m,n}(x,y).$$

Similarly, differentiation by t we obtain the second identity.

Taking into account the first identity we have

$$x\frac{\partial G}{\partial x} = \sum_{m,n=0}^{\infty}\left(x\frac{\partial B_{m,n}(x,y)}{\partial x}\right)\frac{u^m}{m!}\frac{v^n}{n!} = \sum_{m,n=0}^{\infty} mB_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!} =$$
$$= u\sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^{m-1}}{(m-1)!}\frac{v^n}{n!} = u\frac{\partial G}{\partial u}.$$

Similarly, we find that $y\frac{\partial G}{\partial y} = v\frac{\partial G}{\partial v}$. Thus, the generating function G satisfied the system of partial differential equations $x\frac{\partial G}{\partial x} = u\frac{\partial G}{\partial u}$, $y\frac{\partial G}{\partial y} = v\frac{\partial G}{\partial u}$.

A system of two differential equations that contains a function of four variables cannot have more than two functionally independent solutions, see [9]. However, xu and yv are obviously its solutions and are independent. Hence, G must be a function of xu and yv only.

($\Longleftarrow$) Now let us prove the reverse implication. We have $G(x,y,u,v) = G(xu,yv)$. Let us prove that $B_{m,n}(x,y)$ is a homogeneous polynomial. Note that, since $G(sxu,tyv) = G(x(su),y(tv))$, the function G satisfies the identity $G(sx,ty,u,v) = G(x,y,su,tv)$.

Now we have, on the one hand

$$G(sx,ty,u,v) = \sum_{m,n=0}^{\infty} B_{m,n}(sx,ty)\frac{u^m}{m!}\frac{v^n}{n!},$$

and on other hand we get

$$G(sx,ty,u,v) = G(x,y,su,tv) = \sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{(su)^m}{m!}\frac{(tv)^n}{n!}.$$

Comparing the right-hand sides, we obtain

$$\sum_{m,n=0}^{\infty} B_{m,n}(sx,ty)\frac{u^m}{m!}\frac{v^n}{n!} = \sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{(su)^m}{m!}\frac{(tv)^n}{n!}.$$

Equating the coefficients at the same powers of u and v, we get $B_{m,n}(sx,ty) = s^m t^n B_{m,n}(x,y)$, as required. □

Consider the group of *uniform* scaling, i.e., the following transformations of the plane

$$\begin{cases} x' = sx, \\ y' = sy. \end{cases}$$

The polynomial family $\{B_{m,n}(x,y)\}$ is said to be the quasi-monomials with respect to the group of uniform scaling if $B_{m,n}(sx,sy)=s^{m+n}B_{m,n}(x,y)$.

The following theorem gives a complete description of such polynomials in terms of its exponential generating functions.

Theorem 4. *The polynomial family $\{B_{m,n}(x,y)\}$ defined by the exponential generating function*

$$G=\sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}$$

is quasi-monomial with respect to a group of uniform scaling if and only if G is a function of the three variables $\frac{y}{x}, ux$ and vx, $G=G\left(\frac{y}{x},ux,vx\right)$.

Proof. ($\Longrightarrow$) Differentiating the identity $B_{m,n}(sx,sy)=s^{m+n}B_{m,n}(x,y)$, by s at $s=1$ we get the following equation

$$x\frac{\partial B_{m,n}(x,y)}{\partial x}+y\frac{\partial B_{m,n}(x,y)}{\partial y}=(m+n)B_{m,n}(x,y).$$

Then, similarly to the proof of Theorem 1, we find that the generating function satisfies such equation $x\frac{\partial G}{\partial x}+y\frac{\partial G}{\partial y}=u\frac{\partial G}{\partial u}+v\frac{\partial G}{\partial v}$. The equation cannot have more than three functionally independent solutions, which we can indicate explicitly $\frac{y}{x}, ux, vx$. Therefore, the generating function is a function of the variables $\frac{y}{x}, ux, vx$.

Sufficiency is proved in the same way as in Theorem 3. □

3. Quasi-polynomials with respect to the plane translation group. The two-parametric group of plane translations is generated by transformations of the form

$$\begin{cases} x'=x+a,\\ y'=y+b.\end{cases}$$

The group act on functions in the following way

$$f(x',y'))=f(x+a,x+b), a,b\in\mathbb{R}.$$

Since

$$(x+a)^m(y+b)^n=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}x^iy^ja^{m-i}b^{n-j},$$

then we come to such definition.

Definition 4. The quasi-monomial family $\{B_{m,n}(x,y)\}$ is called *quasi-monomial* with respect to the translation group if the following identity holds

$$B_{m,n}(x+a,y+b)=\sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}a^{m-s}b^{n-k}B_{s,k}(x,y), \tag{2}$$

for all $m,n\in\mathbb{N}$.

The following theorem presents a simple criterion for the quasi-monomiality of a polynomial family in terms of its exponential generating function.

Theorem 5. *The polynomial family $\{B_{m,n}(x,y)\}$ is a quasi-monomial family with respect to the translation group if and only if its exponential generating function has the form*

$$G = C(u,v)e^{xu+yv},$$

where $C(u,v)$ is an arbitrary power series in variables u,v.

Proof. ($\Longrightarrow$) We first differentiate (2) by a at $a=0$ and $b=0$. We obtain the differential equation on $B_{m,n}(x,y)$

$$\frac{\partial B_{m,n}(x,y)}{\partial x} = mB_{m-1,n}(x,y).$$

Similarly, differentiating by b we obtain another differential equation on $B_{m,n}(x,y)$

$$\frac{\partial B_{m,n}(x,y)}{\partial y} = nB_{m,n-1}(x,y).$$

Taking into account the first identity we have

$$\frac{\partial G}{\partial x} = \sum_{m,n=0}^{\infty}\left(\frac{\partial B_{m,n}(x,y)}{\partial x}\right)\frac{u^m}{m!}\frac{v^n}{n!} = \sum_{m,n=0}^{\infty} mB_{m-1,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!} =$$
$$= u\sum_{m,n=0}^{\infty} B_{m-1,n}(x,y)\frac{u^{m-1}}{(m-1)!}\frac{v^n}{n!} = u\sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!} = uG.$$

Similarly, we obtain the identity $\frac{\partial G}{\partial y} = vG$. Therefore G satisfies the following system of differential equations $\frac{\partial G}{\partial x} = uG$, $\frac{\partial G}{\partial y} = vG$. This system of first-order differential equations has a solution $G = C(u,v)e^{xu+yv}$, where C is a function of u,v. We can assume that $C(u,v)$ is a power series in the variables u,v.

($\Longleftarrow$) Suppose now that the generating function for a polynomial family $\{B_{m,n}(x,y)\}$ has the form $G = C(u,v)e^{xu+yv}$.

Consider the shift operator $T_{a,b}$

$$T_{a,b}(x) = x+a, T_{a,b}(y) = y+b.$$

Then $T_{a,b}(xu+yv) = au+vb+ux+vy$, and we have on the one hand

$$T_{a,b}(G) = T_{a,b}\left(\sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}\right) = \sum_{m,n=0}^{\infty} T_{a,b}(B_{m,n}(x,y))\frac{u^m}{m!}\frac{v^n}{n!}.$$

On the other hand

$$T_{a,b}(G) = C(u,v)T_{a,b}\left(e^{xu+yv}\right) = C(u,v)e^{T_{a,b}(xu+yv)} = C(u,v)e^{au+vb+ux+vy} =$$
$$= e^{au+vb}G = \left(\sum_{k,s=0}^{\infty} a^s b^k \frac{u^s}{s!}\frac{v^k}{k!}\right)\left(\sum_{m,n=0}^{\infty} B_{m,n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!}\right) =$$
$$= \sum_{m,n=0}^{\infty}\left(\sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}a^{m-s}b^{n-k}B_{s,k}(x,y)\right)\frac{u^m}{m!}\frac{v^n}{n!}.$$

By equating the coefficients of the same powers of u and v we get that

$$T_{a,b}(B_{m,n}(x,y)) = \sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}a^{m-s}b^{n-k}B_{s,k}(x,y).$$

Therefore, the polynomials $B_{m,n}(x, y)$ are a quasi-monomials with respect to the translation group of the plane. □

The property of quasi-mononomiality can be lost if the polynomials are normalized, i.e. multiplied by some constants. Normalization is often used to limit the allowable range of polynomial values in calculations. The following theorem explores what kind of normalization preserves the quasi-monomiality property.

Theorem 6. *Let $\{B_{m,n}(x, y)\}$ be a quasi-monomial family with respect to a group of translations. The polynomial family $\{\tilde{B}_{m,n}(x, y)\}$, where $\tilde{B}_{m,n}(x, y) = \alpha_{m,n} B_{m,n}(x, y)$, is a quasi-monomial with respect to a group of translations if and only if each the coefficient $\alpha_{m,n}$ is a function ϕ which satisfies the recurrence relation $\phi(m + n) = \phi(m + n - 1)$.*

Proof. ($\Longrightarrow$) Since

$$\frac{\partial \tilde{B}_{m,n}(x, y)}{\partial x} = m\tilde{B}_{m-1,n-1}(x, y), \quad \frac{\partial \tilde{B}_{m,n}(x, y)}{\partial y} = n\tilde{B}_{m,n-1}(x, y),$$

we have

$$\alpha_{m,n} \frac{\partial B_{m,n}(x, y)}{\partial x} = m\alpha_{m-1,n} B_{m+1,n}(x, y), \quad \alpha_{m,n} \frac{\partial B_{m,n}(x, y)}{\partial y} = n\alpha_{m,n-1} B_{m,n-1}(x, y).$$

We obtain the system of recurrence equations for the sequence $\alpha_{m,n}$:

$$\alpha_{m-1,n} = \alpha_{m,n}, \quad \alpha_{m,n-1} = \alpha_{m,n}.$$

The solution is $\alpha_{m,n} = \phi(m + n - 1)$ where ϕ is an arbitrary function.

($\Longleftarrow$) Now let us prove the reverse implication. If $\alpha_{m,n} = \phi(m + n - 1)$, then obviously

$$\alpha_{m-1,n} = \alpha_{m,n}, \quad \alpha_{m,n-1} = \alpha_{m,n}.$$

Further

$$\frac{\partial \tilde{B}_{m,n}(x, y)}{\partial x} = \alpha_{m,n} \frac{\partial B_{m,n}(x, y)}{\partial x} = m\alpha_{m-1,n} B_{m-1,n}(x, y) = m\tilde{B}_{m-1,n}(x, y).$$

Similarly, we get that

$$\frac{\partial \tilde{B}_{m,n}(x, y)}{\partial y} = n\tilde{B}_{m,n-1}(x, y).$$

Therefore, the generating function for polynomials $\tilde{B}_{m,n-1}(x, y)$ satisfies the conditions of Theorem 5 and the family of polynomials $\tilde{B}_{m,n}$ will be quasi-monomial with respect to the translation group. □

Knowing the generating function for a system of polynomials, we can get their explicit form. For example, consider the following generating function

$$G = \frac{1}{1 - (u^2 + v^2)} e^{xu+yv} = \sum_{m,n=0}^{\infty} B_{m,n} \frac{u^m}{m!} \frac{v^n}{n!}.$$

For small m, n we get

$$B_{0,0}\,(x,y)=1, B_{1,0}\,(x,y)=x, B_{0,1}\,(x,y)=y,$$
$$B_{2,0}\,(x,y)=x^2+2, B_{1,1}\,(x,y)=xy, B_{0,2}\,(x,y)=y^2+2,$$
$$B_{3,0}\,(x,y)=x^3+6\,x, B_{2,1}\,(x,y)=x^2y+2\,y,$$
$$B_{1,2}\,(x,y)=xy^2+2\,x, B_{0,3}(x,y)=y^3+6\,y.$$

Find an explicit formula for these polynomials. We have

$$\frac{1}{1-(u^2+v^2)}=\sum_{i=0}^{\infty}(u^2+v^2)^i=\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}u^{2k}v^{2(i-k)}=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{i,k}u^iv^k,$$

where $a_{i,k}=\begin{cases}\binom{\frac{i+k}{2}}{\frac{i}{2}}, & \text{if } i,k \text{ are even;}\\ 0. & \end{cases}$

Since

$$e^{xu+yv}=\sum_{j=0}^{\infty}(xu+yv)^j\frac{1}{j!}=\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{j}{s}(xu)^s(yv)^{j-s}\frac{1}{j!}=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^jy^s}{j!s!}u^jv^s,$$

we have

$$\frac{1}{1-(u^2+v^2)}\,e^{xu+yv}=\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{i,k}u^iv^k\right)\left(\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\frac{x^jy^s}{j!s!}u^jv^s\right)=$$
$$=\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}\left(r!t!\sum_{i+j=r,k+s=t}a_{i,k}\frac{x^jy^s}{j!s!}\right)\frac{u^r}{r!}\frac{v^t}{t!}$$

Hence

$$B_{r,t}(x,y)=r!t!\sum_{\substack{i+j=r\\k+s=t}}a_{i,k}\frac{x^jy^s}{j!s!}=r!t!\sum_{\substack{2i+j=r\\2k+s=t}}a_{2i,2k}\frac{x^jy^s}{j!s!}=r!t!\sum_{\substack{2i+j=r\\2k+s=t}}\binom{i+k}{i}\frac{x^jy^s}{j!s!}=$$
$$=r!t!\sum_{i=0}^{r/2}\sum_{k=0}^{t/2}\binom{i+k}{i}\frac{x^{r-2i}y^{t-2k}}{(r-2i)!(t-2k)!}.$$

The polynomials $B_{m,n}(x,y)$ satisfy the recurrence relations, which we give without proof.

Theorem 7. *The polynomials $B_{m,n}(x,y)$ satisfy the following recurrence relations*

$$B_{m+1,n}(x,y)=xB_{m,n}(x,y)+2\sum_{k=0}^{\lceil(n-1)/2\rceil}\sum_{s=0}^{(m-1)/2}h_{s,k}^{(m,n)}B_{m-2s-1,n-2k}(x,y),$$
$$B_{m,n+1}(x,y)=yB_{m,n}(x,y)+2\sum_{k=0}^{\lceil(n-1)/2\rceil}\sum_{s=0}^{(m-1)/2}h_{k,s}^{(n,m)}B_{m-2s,n-2k-1}(x,y),$$

where

$$h_{s,k}^{(m,n)}=(2s+1)!(2k)!\binom{m}{2s+1}\binom{n}{2k}\binom{s+k}{k}.$$

For the partial case of uniform translations

$$T_a(x) = x + a, \quad T_a(y) = y + a.$$

the following statement is true.

Theorem 8. *The polynomial family $\{B_{m,n}(x,y)\}$ is quasi-monomial family with respect to the group of uniform translations if and only if its exponential generating function has the form $G = C(x-y,u,v)e^{xu+yv}$ where C is an arbitrary power series of variables $x-y,u,v$.*

The proof is similar to that of Theorem 5.

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