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NON-PERIODIC GROUPS WITH THE RESTRICTIONS ON THE NORM OF CYCLIC SUBGROUPS OF NON-PRIME ORDER

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One of the main directions in group theory is the study of the impact of characteristic subgroups on the structure of the whole group. Such characteristic subgroups include different Σ-norms of a group. A Σ-norm is the intersection of the normalizers of all subgroups of a system Σ. The authors study non-periodic groups with the restrictions on such a Σ-norm, the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order, which is the intersection of the normalizers of all cyclic subgroups of composite or infinite order of G. It was proved that if G is a mixed non-periodic group, then its norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is either Abelian (torsion or non-periodic) or non-periodic non-Abelian. Moreover, a non-periodic group G has the non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order if and only if G is non-Abelian and every cyclic subgroup of non-prime order of a group G is normal in it, and $G = N_G(C_{\bar{p}})$. Additionally the relations between the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order and the norm $N_G(C_\infty)$ of infinite cyclic subgroups, which is the intersection of the normalizers of all infinite cyclic subgroups, in non-periodic groups are studied. It was found that in a non-periodic group G with the non-Abelian norm $N_G(C_\infty)$ of infinite cyclic subgroups norms $N_G(C_\infty)$ and $N_G(C_{\bar{p}})$ coincide if and only if $N_G(C_\infty)$ contains all elements of composite order of a group G and does not contain non-normal cyclic subgroups of order 4. In this case $N_G(C_{\bar{p}}) = N_G(C_\infty) = G$.

1. Introduction. One of the main directions in group theory is the study of the impact of characteristic subgroups on the structure of the whole group. Such characteristic subgroups include different Σ-norms of the group. A Σ-norm is the intersection of the normalizers of all subgroups of a system Σ (assuming that the system Σ is non-empty). It is clear that when the Σ-norm coincides with a group, then all subgroups of the system Σ are normal in the last one.

For the first time, R. Baer [1] considered the Σ-norm as a proper subgroup of a group in 1935 for the system of all subgroups of this group. He called it the norm of a group and denoted by $N(G)$. Narrowing the system of subgroups one can get different Σ-norms which can be considered as generalizations of the norm $N(G)$. Recently the interest to study the Σ-norms does not decrease in view of series of papers [2, 3, 4, 7, 8, 9, 10, 11, 12, 13].

The authors investigate the generalized norm of a group, which is closely related to the properties of some systems of cyclic subgroups of a group. Let us note that the Baer norm $N(G)$ is the intersection of the normalizers of all cyclic subgroups of a group. That is why the natural question arises, to investigate Σ-norms of a group for systems Σ consisting of

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some subsystems of cyclic subgroups, in particular, consider the case when the system Σ contains only cyclic subgroups of infinite or composite order. The corresponding Σ-norm is called the norm of cyclic subgroups of non-prime order of G and denoted by $N_G(C_{\bar{p}})$.

The authors focused on the study of the properties of the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order in non-periodic groups, its impact on group properties and relations with the norm $N_G(C_\infty)$ of infinite cyclic subgroups, which is the intersection of normalizers of all infinite cyclic subgroups of a group G (see, [9, 12]).

As will be shown below, a non-periodic group G with the non-Abelian norm $N_G(C_{\bar{p}})$ coincides with this norm. In this case, a group G is the semi-direct product of a normal Abelian subgroup A, which contains all elements of non-prime order of this group, and a cyclic subgroup of order 2, which induces an irreducible automorphism of order 2 on A.

Let us note that some results concerning the properties of the norm $N_G(C_{\bar{p}})$ were announced in [2, 11, 12].

2. Preliminary results.

Definition 1. The norm of cyclic subgroups of non-prime order of non-periodic group G is the intersection of the normalizers of all cyclic subgroups of composite or infinite order of G and is denoted by $N_G(C_{\bar{p}})$.

It is clear that in a non-periodic group G coinciding with its norm $N_G(C_{\bar{p}})$ all cyclic subgroups of composite or infinite order are normal. Such non-Dedekind groups were studied in [6] and were called almost Dedekind groups. The structure of non-periodic almost Dedekind groups is described in the following proposition.

Proposition 1. *A non-periodic group G is almost Dedekind if and only if $G = C \rtimes \langle b \rangle$, where C is a non-periodic Abelian group, $|b| = 2$, $b^{-1}cb = c^{-1}$ for any element $c \in C$.*

Corollary 1. *The center of a non-periodic almost Dedekind group is an elementary Abelian (in particular, identity) 2-group.*

The following result follows directly from Proposition 1.

Corollary 2. *Any non-periodic group without involutions, in which each cyclic subgroup of infinite or composite order is normal, is Abelian.*

Further we will consider non-periodic groups G, in which the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is some (usually proper) subgroup of a group.

Let us formulate some statements characterizing the properties of the norm $N_G(C_{\bar{p}})$. We will be used them actively in this section.

Lemma 1. *Let G be a non-periodic group and $N_G(C_{\bar{p}})$ be its norm of cyclic subgroups of non-prime order. Then the following statements take place:*
1) $N_G(C_{\bar{p}}) \supseteq N(G) \supseteq Z(G)$, *where $N(G)$ is the norm of G;*
2) *if the subgroup $N_G(C_{\bar{p}})$ is non-periodic, then $N_G(C_{\bar{p}}) = N_{N_G(C_{\bar{p}})}(C_{\bar{p}})$;*
3) *every cyclic subgroup of infinite or composite order of the group $N_G(C_{\bar{p}})$ is normal in it;*
4) *if the norm $N_G(C_{\bar{p}})$ is non-periodic, then it is either Abelian or almost Dedekind;*
5) *if H is non-periodic subgroup of G, which contains the norm $N_G(C_{\bar{p}})$, then*
$$N_G(C_{\bar{p}}) \subseteq N_H(C_{\bar{p}});$$
6) *if $H \subseteq C_G(N_G(C_{\bar{p}}))$ and the group $G_1 = H \cdot N_G(C_{\bar{p}})$ is non-periodic, then $G_1 = N_{G_1}(C_{\bar{p}})$.*

The proof of Lemma 1 follows directly from the definition of the norm of cyclic subgroups of non-prime order.

Lemma 2. *If the norm $N_G(C_{\bar{p}})$ of a non-periodic group G does not contain elements of infinite order, then it is Abelian.*

Proof. Let us suppose that, contrary to the lemma, the norm $N_G(C_{\bar{p}})$ is a periodic non-Abelian group. Then for an arbitrary element $x \in G$, $|x| = \infty$ the subgroup $\langle x \rangle$ is $N_G(C_{\bar{p}})$-admissible, so, $[\langle x \rangle, N_G(C_{\bar{p}})] \subseteq \langle x \rangle \cap N_G(C_{\bar{p}}) = E$ and $\langle x \rangle \subseteq C_G(N_G(C_{\bar{p}}))$.

By Lemma 1 $G_1 = \langle x \rangle \cdot N_G(C_{\bar{p}}) = \langle x \rangle \times N_G(C_{\bar{p}})$ is non-periodic almost Dedekind. Taking into account that the center of such a group is elementary Abelian 2-group by Corollary 1. It contradicts the condition $x \in Z(G_1)$. Thus, the assumption is false and the norm $N_G(C_{\bar{p}})$ is Abelian. □

Thus, if the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is torsion, then it is Abelian. Let us prove another sufficient condition for the Abelianity of this norm.

Lemma 3. *If a non-periodic group G contains such a cyclic subgroup $\langle x \rangle$ of infinite or composite order that $\langle x \rangle \cap N_G(C_{\bar{p}}) = E$, then the norm $N_G(C_{\bar{p}})$ is Abelian.*

Proof. Let the subgroup $\langle x \rangle$ satisfy conditions of the lemma, but the norm $N_G(C_{\bar{p}})$ be non-Abelian. Then $N_G(C_{\bar{p}})$ is non-periodic non-Abelian group by Lemma 2. Since subgroups $\langle x \rangle$ and $N_G(C_{\bar{p}})$ are normal in the group $G_1 = \langle x \rangle\, N_G(C_{\bar{p}})$, then $G_1 = \langle x \rangle \times N_G(C_{\bar{p}})$ and $x \in Z(G_1)$. By Lemma 1, $G_1 = N_{G_1}(C_{\bar{p}})$ and G_1 is almost Dedekind. By Corollary 1, the center of an almost Dedekind group is an elementary Abelian 2-group. It contradicts the condition $x \in Z(G_1)$. Thus, the group $N_G(C_{\bar{p}})$ is Abelian and the lemma is proved. □

Corollary 3. *In a non-periodic group G with the non-Abelian norm $N_G(C_{\bar{p}})$ every cyclic subgroup $\langle x \rangle$ of infinite or composite order has non-identity intersection with the norm $N_G(C_{\bar{p}})$.*

By Lemma 1 and Lemma 2, the norm $N_G(C_{\bar{p}})$ is either Abelian or almost Dedekind (provided that the system of cyclic subgroups of composite or infinite order in $N_G(C_{\bar{p}})$ is non-empty). On the other hand, it is possible when $N_G(C_{\bar{p}})$ does not contain cyclic subgroups of composite or infinite order. In particular, such a property is inherent in periodic Olshansky groups [15]. In periodic Olshansky groups all proper subgroups are cyclic and have prime order.

Let us show that in the latter case the norm $N_G(C_{\bar{p}})$ of a non-periodic group G is an elementary Abelian p-group.

Theorem 1. *If the norm $N_G(C_{\bar{p}})$ of a non-periodic group G is a non-identity subgroup and does not contain cyclic subgroups of infinite and composite order, then $N_G(C_{\bar{p}})$ is an elementary Abelian p-group (p is prime).*

Proof. By the condition of the theorem, $N_G(C_{\bar{p}})$ is torsion, all elements of which are of prime order. By Lemma 2, the norm $N_G(C_{\bar{p}})$ is Abelian. Taking into account that it does not contain elements of composite order, we get that $N_G(C_{\bar{p}})$ is an elementary Abelian p-group, where p is prime. The theorem is proved. □

The following example confirms the existence of non-periodic groups in which the norm $N_G(C_{\bar{p}})$ satisfies the conditions of Theorem 1.

Example 1. $G = (\langle a\rangle \times \langle b\rangle \times \langle x\rangle) \rtimes \langle c\rangle$, $|a| = |b| = 3, |x| = \infty$, $|c| = 2$, $[a, c] = [b, c] = 1$, $c^{-1}xc = x^{-1}$. In this group

$$N_G(C_{\bar{p}}) \subseteq N_G(\langle ac\rangle) \cap N_G(\langle ax\rangle) = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle) \cap (\langle a\rangle \times \langle b\rangle \times \langle x\rangle) = \langle a\rangle \times \langle b\rangle = Z(G).$$

So, the norm of cyclic subgroups of non-prime order coincides with the center of the group

$$N_G(C_{\bar{p}}) = \langle a\rangle \times \langle b\rangle = Z(G)$$

and the norm is the elementary Abelian group of order 9.

The following result determines sufficient conditions for the norm $N_G(C_{\bar{p}})$ be central.

Lemma 4. *If the center $Z(G)$ of a non-periodic group G contains elements of infinite order, then the norm $N_G(C_{\bar{p}})$ is Abelian and coincides with the group center $N_G(C_{\bar{p}}) = Z(G)$.*

Proof. Let $Z(G)$ contain elements of infinite order. Since $Z(G) \subseteq N_G(C_{\bar{p}})$, the norm $N_G(C_{\bar{p}})$ is a non-periodic Abelian group by Proposition 1. Let us show that every element from $N_G(C_{\bar{p}})$ is permutable with all elements of infinite order from G.

Let $x \in N_G(C_{\bar{p}})$, $y \in G$, $|y| = \infty$ and $[x, y] \neq 1$. Since $N_G(C_{\bar{p}})$ is a non-periodic Abelian group, it is generated by elements of infinite order. Thus, we can regard that $|x| = \infty$.

By the infinity of the subgroup $\langle y\rangle$, we get that it is $N_G(C_{\bar{p}})$-admissible, so $x^{-1}yx = y^{-1}$ and $\langle x\rangle \cap \langle y\rangle = E$. Taking into account that $[x^2, y] = 1$ and the subgroup $\langle x^2y\rangle$ is x-admissible, we get $x^{-1}x^2yx = x^{-2}y^{-1} = x^2y^{-1}$. But in this case one has $x^4 = 1$. This contradicts its choice. So, $[x, y] = 1$ for any element $x \in N_G(C_{\bar{p}})$ and $y \in G$, $|y| = \infty$.

Let $y \in G$, $|y| < \infty$. Suppose that $[x, y] \neq 1$. Let us take an element $z \in Z(G)$, $|z| = \infty$. Then $|yz| = \infty$ and the subgroup $\langle yz\rangle$ is $N_G(C_{\bar{p}})$-admissible. If $x^{-1}yzx = yz$, then $x^{-1}yx = y$ by the equalities $x^{-1}yzx = x^{-1}yxz = yz$, which contradicts the assumption. Thus, $x^{-1}yzx = (yz)^{-1}$. On the other hand,

$$x^{-1}yzx = x^{-1}yxx^{-1}zx = x^{-1}yxz = y^{-1}z^{-1},$$

that is $x^{-1}yx = y^{-1}z^{-2}$, which contradicts the above.

Thus, $[\langle y\rangle, N_G(C_{\bar{p}})] = E$ for every element $y \in G$, so, $N_G(C_{\bar{p}}) = Z(G)$. The lemma is proved. □

Corollary 4. *An arbitrary non-periodic central-by-finite group G has the Abelian norm $N_G(C_{\bar{p}})$ and $N_G(C_{\bar{p}}) = Z(G)$.*

Lemma 5. *If the center $Z(G)$ of a non-periodic group G contains elements of composite order, then its norm $N_G(C_{\bar{p}})$ is Abelian.*

Proof. Let us suppose that the norm $N_G(C_{\bar{p}})$ is non-Abelian. Then it is non-periodic and almost Dedekind by Lemma 4. Since the center of such a group does not contain elements of non-prime order by Corollary 1, the assumption is false and the norm $N_G(C_{\bar{p}})$ is Abelian. The lemma is proved. □

Combining the results of Lemma 4 and Lemma 5, we get the following statement.

Corollary 5. *If the center $Z(G)$ of a non-periodic group G contains non-identity elements of non-prime order, then the norm $N_G(C_{\bar{p}})$ of such a group is Abelian.*

Let us note that the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order in a non-periodic group G is quite closely related to the norm $N_G(C_\infty)$ of infinite cyclic subgroups, which is an intersection of the normalizers of all infinite cyclic subgroups of this group (see [12]). It is explained by the fact that the class of non-periodic groups, in which all infinite cyclic subgroups are normal, contains the class of non-periodic groups with a normal system of cyclic subgroups of non-prime order. Therefore, the norm of $N_G(C_{\bar{p}})$ of a non-periodic group is contained in the norm $N_G(C_\infty)$ of infinite cyclic subgroups of a group $N_G(C_{\bar{p}}) \subseteq N_G(C_\infty)$. Clearly, in torsion free groups these norms coincide $N_G(C_{\bar{p}}) = N_G(C_\infty)$.

This allows the usage of some results for groups with restrictions on the norm of infinite cyclic subgroups [9] for the characterization of groups with restrictions on the norm of cyclic subgroups of non-prime order of a group.

3. Non-periodic groups with the non-Abelian norm of cyclic subgroups of non-prime order. Let us consider the impact of the properties of the norm of non-prime order cyclic subgroups on the properties of a group. In this section we will consider non-periodic groups in which the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is non-Abelian. Also, the relations between the given norm $N_G(C_{\bar{p}})$ and the norm $N_G(C_\infty)$ of infinite cyclic subgroups of a group will be studied.

By Proposition 1 in torsion free groups the norm $N_G(C_{\bar{p}})$ is Abelian. Let us show that it is the central subgroup of a group (so it coincides with the norm of infinite cyclic subgroups and Baer norm).

Theorem 2. *If G is a torsion free group, then its norm $N_G(C_{\bar{p}})$ coincides with the center of a group, with the norm $N(G)$ of a group, and with the norm $N_G(C_\infty)$ of infinite cyclic subgroups $N_G(C_{\bar{p}}) = N_G(C_\infty) = N(G) = Z(G)$.*

Proof. Suppose that $N_G(C_{\bar{p}}) \neq Z(G)$. Then there exist such elements $x \in N_G(C_{\bar{p}})$ and $y \in G$, that $[x, y] \neq 1$. By the definition of a subgroup $N_G(C_{\bar{p}})$ we get $x^{-1}yx = y^{-1}$. Therefore, $\langle x\rangle \cap \langle y\rangle = E$ and since $[x^2, y] = 1$, $\langle x^2y\rangle$ is x-invariant subgroup. So

$$x^{-1}x^2yx = (x^2y)^{-1} = y^{-1}x^{-2} = x^{-2}y^{-1} = x^2y^{-1}.$$

But in this case $x^4 = 1$, which contradicts the condition. Thus, $N_G(C_{\bar{p}}) = Z(G)$. The equalities $N_G(C_{\bar{p}}) = N_G(C_\infty)$ and $N(G) = N_G(C_{\bar{p}})$ in torsion free groups are evident. The theorem is proved. □

Corollary 6. *A torsion free group G, which is a finite extension of the norm $N_G(C_{\bar{p}})$, is Abelian.*

Proof. Let $[G : N_G(C_{\bar{p}})] < \infty$. Then by Theorem 2, $N_G(C_{\bar{p}}) = Z(G)$, so $[G : Z(G)] < \infty$. By Theorem 1.4 [5], in this case $|G'| < \infty$. Since G is torsion free group, it can be only when $G' = E$. Thus, a group G is Abelian. □

Let us consider the mixed non-periodic groups. The following examples claim that the norm $N_G(C_{\bar{p}})$, which is a proper subgroup of a mixed non-periodic group G, can either coincide with the norm $N_G(C_\infty)$ or not coincide.

Example 2. $G = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle$, $|a| = |b| = \infty$, $|c| = 3$, $c^{-1}ac = b$, $c^{-1}bc = a^{-1}b^{-1}$. In this group, one has $Z(G) = E$ and all infinite cyclic subgroups are contained in the group $\langle a, b\rangle$. Moreover, the set of cyclic subgroups of non-prime order coincides with the set of infinite cyclic subgroups. So, $N_G(C_{\bar{p}}) = N_G(C_\infty)$. Since $\langle c\rangle \nsubseteq N_G(\langle a\rangle)$, $\langle c\rangle \nsubseteq N_G(C_{\bar{p}})$ and $N_G(C_{\bar{p}}) = N_G(C_\infty) = \langle a\rangle \times \langle b\rangle$ is the non-central Abelian group, which is generated by all elements of infinite order.

Example 3. $G = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle$, $|a| = |b| = \infty$, $|c| = 6$, $c^{-1}ac = ab$, $c^{-1}bc = a^{-1}$. In this case, one has $Z(G) = E$ and all infinite cyclic subgroups are contained in the subgroup $\langle a, b \rangle$. So, $N_G(C_\infty) = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c^3 \rangle$. But

$$N_G(C_{\bar{p}}) \subseteq N_G(\langle c \rangle) \cap N_G(\langle ac \rangle) = \langle c \rangle \cap \langle ac \rangle = Z(G) = E.$$

Thus, $N_G(C_{\bar{p}}) \neq N_G(C_\infty)$.

By Proposition 1, Lemma 2 and above examples, we come to the following result.

Theorem 3. *If G is a mixed non-periodic group, then its norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is either Abelian (torsion or non-periodic) or non-periodic non-Abelian.*

Further, we will consider the mixed non-periodic groups with the non-Abelian norm $N_G(C_{\bar{p}})$. By Theorem 3, the norm $N_G(C_{\bar{p}})$ is non-periodic almost Dedekind. So, by Proposition 1, $N_G(C_{\bar{p}}) = C \rtimes \langle b \rangle$, where C is non-periodic Abelian, $|b| = 2$, $b^{-1}cb = c^{-1}$ for any element $c \in C$.

By Theorem 2 and Proposition 1, we get the following statement.

Corollary 7. *A non-periodic group G, which is a finite extension of the norm $N_G(C_{\bar{p}})$, is almost Abelian.*

Let D be the subgroup generated by all elements of infinite order of a group G.

Lemma 6. *If a non-periodic group G has the non-Abelian norm $N_G(C_{\bar{p}})$, then the subgroup D is Abelian and contains all elements of infinite or composite order of the group.*

Proof. Let the norm $N_G(C_{\bar{p}})$ be non-Abelian, that is $N_G(C_{\bar{p}}) = C \rtimes \langle b \rangle$, where C is a non-periodic Abelian group, $|b| = 2$, $b^{-1}cb = c^{-1}$ for any element $c \in C$.

Let us prove that the subgroup $C \subset N_G(C_{\bar{p}})$ is contained in the center $Z(D)$ of the subgroup D. We take such arbitrary elements $c \in C$ and $a \in D$, that $[c, a] \neq 1$. Without loss of generality we conclude $|a| = |c| = \infty$.

Since $\langle a \rangle \lhd G_1 = \langle a \rangle N_G(C_{\bar{p}})$, $c^{-1}ac = a^{-1}$ and $\langle a \rangle \cap \langle c \rangle = E$. Thus, $[c^2, a] = 1$, $|c^2a| = \infty$ and $\langle c^2a \rangle$ is c-invariant subgroup. But then $c^{-1}(c^2a)c = c^{-2}a^{-1} = c^2a^{-1}$ and $c^4 = 1$. This contradicts the choice of the element c. So, $C \subseteq Z(D)$.

Let us show that the subgroup D contains all elements of composite order of a group G. Let $y \in G$ be an arbitrary element of composite order. Then $\langle y \rangle \lhd G_2 = \langle y \rangle N_G(C_{\bar{p}})$ and $[G_2 : C_{G_2}(y)] < \infty$. So, there exist such an element $c \in C$, $|c| = \infty$, that $[c, y] = 1$. Since $|cy| = \infty$, $cy \in D$, so $y \in D$, which is desired conclusion.

Now we will study how the element b acts on elements of the subgroup D. For any element $a \in D$, $|a| = \infty$ we have $\langle a \rangle \lhd \langle a \rangle N_G(C_{\bar{p}})$. So, if $[a, b] = 1$, then $|ba| = \infty$ and $ba \in D$. By the proved above $[c, ab] = 1$ for any element $c \in C$, $|c| = \infty$, which is impossible. Thus, $b^{-1}ab = a^{-1}$, where $a \in D$ is an arbitrary element of infinite order.

Let $|a| < \infty$, where $a \in D$. Let us take such an element $c \in C$, that $|c| = \infty$. Then $|ca| = \infty$ and $b^{-1}(ca)b = (ca)^{-1} = c^{-1}a^{-1} = c^{-1}b^{-1}ab$, and in this case $b^{-1}ab = a^{-1}$.

Let us denote by x and y such arbitrary elements of the group D, that $[x, y] \neq 1$. Then

$$b^{-1}(xy)b = (xy)^{-1} = y^{-1}x^{-1} = b^{-1}xbb^{-1}yb = x^{-1}y^{-1},$$

so, $[x, y] = 1$, which contradicts their choice. Thus, the subgroup D is Abelian. The lemma is proved. □

From the proof of Lemma 6 we get the following result characterizing the properties of non-periodic groups with the non-Abelian norm $N_G(C_{\bar{p}})$.

Theorem 4. *A non-periodic group G has the non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order if and only if all elements of infinite order of the group generate a normal Abelian subgroup D, which contains all elements of non-prime order of a group G, and there exist the element b of order 2, $b^{-1}ab = a^{-1}$ for any element $a \in D$. Moreover,*

$$N_G(C_{\bar{p}}) = D \rtimes \langle b \rangle .$$

Corollary 8. *If the norm $N_G(C_{\bar{p}})$ of a non-periodic group G is non-Abelian, then the quotient group $G/N_G(C_{\bar{p}})$ is torsion and does not contain elements of infinite and composite order.*

Lemma 7. *If a non-periodic group G with the non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order contains a normal infinite cyclic subgroup, then $N_G(C_{\bar{p}}) = G$.*

Proof. By the condition of the lemma and Theorem 4, $N_G(C_{\bar{p}}) = D \rtimes \langle b \rangle$, where D is a non-periodic Abelian group, $|b| = 2$ and $b^{-1}ab = a^{-1}$ for any element $a \in D$.

Let $\langle x \rangle \lhd G$, $|x| = \infty$. Then by Theorem 4 $x \in D$, $b^{-1}xb = x^{-1}$, $[G : C_G(\langle x \rangle)] = 2$ and so, $G = C_G(\langle x \rangle) \rtimes \langle b \rangle$.

Let y be an arbitrary non-identity element from $C_G(\langle x \rangle)$. If y is of non-prime order, then $y \in D$ by Lemma 6. Let $|y| = p$, where p is prime. Since $[x, y] = 1$, $|xy| = \infty$. Then $xy \in D$ and $y \in D$. Thus, $C_G(\langle x \rangle) = D$ and $N_G(C_{\bar{p}}) = G$, which is the desired conclusion. □

Theorem 5. *A non-periodic group G has the non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order if and only if G is non-Abelian and every cyclic subgroup of non-prime order of a group G is normal in it, and $G = N_G(C_{\bar{p}})$.*

Proof. The sufficiency is evident, so we will prove only necessity.

Let the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order of non-periodic group G be non-Abelian. Then $N_G(C_{\bar{p}}) = D \rtimes \langle b \rangle$, and by Theorem 4 the subgroup D contains all elements of non-prime order of a group G, $|b| = 2$ and $b^{-1}ab = a^{-1}$ for any element $a \in D$.

Suppose that $N_G(C_{\bar{p}}) \neq G$ and x is an arbitrary element of a group G, which is not contained in the norm $N_G(C_{\bar{p}})$. Then $|x| = p$, where p is prime.

Let $p \neq 2$. In the quotient group $\overline{G} = G/D$ we have $|\overline{b}| = 2, \overline{b} \in Z(\overline{G})$. So, the element $\overline{xb}$ is of order $2p$. Thus, its preimage xb is also of non-prime order. By Theorem 4, one has $xb \in N_G(C_{\bar{p}})$ and $x \in N_G(C_{\bar{p}})$. This contradicts its choice.

Let $p = 2$ and $|x| = 2$. Then $[\overline{x}, \overline{b}] = 1$ and $|\overline{xb}| = 2$. One has $|xb| = 2$, because in other case the element xb is of non-prime order and is contained in D. Therefore, $x \in N_G(C_{\bar{p}})$, which contradicts its choice.

Let us show, that for an arbitrary element $a \in D$, $|a| = \infty$ we have $[x, a] \neq 1$. And in another case $|xa| = \infty$ and by Lemma 6, $xa \in D$. So, $x \in D$, which contradicts its choice. Therefore, by the conditions $D \lhd G$ and $[D, \langle x \rangle] \subseteq D$, we can regard that $x^{-1}ax = ac$, $c \in D, c \neq 1$.

Since $|x| = 2$, $[x^2, a] = 1$. Thus, $a = x^{-2}ax^2 = acx^{-1}cx$ and $x^{-1}cx = c^{-1}$. If $|c| > 2$, then by $[xb, c] = 1$ we conclude that the element xbc is of non-prime order. By Lemma 6, $xbc \in D$, so $x \in N_G(C_{\bar{p}})$, which is impossible. Thus, $|c| = 2$. But in this case $[a^2, x] = 1$, $|a^2x| = \infty$ and $a^2x \in D$, $x \in D$. Thus, the assumption is false and $G = N_G(C_{\bar{p}})$.

The theorem is proved. □

Corollary 9. *If the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order of a non-periodic group G is a proper subgroup of a group, then it is Abelian.*

Corollary 10. *A non-periodic group G with the non-Abelian norm $N_G(C_{\bar{p}})$ is soluble of degree 2.*

Corollary 11. *If a non-periodic group G contains a non-normal cyclic subgroup of composite or infinite order, then its norm $N_G(C_{\bar{p}})$ is Abelian.*

Corollary 12. *If the norm $N_G(C_{\bar{p}})$ of a non-periodic group G is non-Abelian, then every cyclic subgroup of non-prime order is normal in G.*

In other words, from the non-Abelianity of the norm $N_G(C_{\bar{p}})$ we get the normality of all subgroups of non-prime (in particular, infinite) in a group. So, we get the following statement.

Corollary 13. *If the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order of a non-periodic group G is non-Abelian, then all infinite cyclic subgroups are normal in G and*

$$G = N_G(C_{\bar{p}}) = N_G(C_\infty).$$

Let us note that the condition of the non-Abelianity of the norm $N_G(C_{\bar{p}})$ in Corollary 13 is substantial. Example 3 confirms the existence of groups with unit norm $N_G(C_{\bar{p}})$, in which the condition $N_G(C_{\bar{p}}) = N_G(C_\infty)$ (and the norm of infinite cyclic subgroups is non-Abelian) is violated. On the other hand, with some additional restrictions on the norm $N_G(C_\infty)$ these norms can coincide.

Corollary 14. *Let G be a non-periodic group with the non-Abelian norm $N_G(C_\infty)$ of infinite cyclic subgroups. The norms $N_G(C_\infty)$ and $N_G(C_{\bar{p}})$ coincide if and only if $N_G(C_\infty)$ contains all elements of composite order of a group G and does not contain non-normal cyclic subgroups of order 4. In this case,*

$$N_G(C_{\bar{p}}) = N_G(C_\infty) = G.$$

The proof of Corollary 14 follows from Theorem 5 and the main theorem [9], which characterizes mixed non-periodic groups with the non-Abelian norm $N_G(C_\infty)$.

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