UDC 517.555

V. P. Baksa[1], A. I. Bandura[2], T. M. Salo[3]

BOUNDEDNESS OF THE L-INDEX IN A DIRECTION OF THE SUM AND PRODUCT OF SLICE HOLOMORPHIC FUNCTIONS IN THE UNIT BALL

V. P. Baksa, A. I. Bandura, T. M. Salo, *Boundedness of the L-index in a direction of the sum and product of slice holomorphic functions in the unit ball*, Mat. Stud. **57** (2022), 216–224.

Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a fixed direction. We consider slice holomorphic functions of several complex variables in the unit ball, i.e. we study functions which are analytic in intersection of every slice $\{z^0+t\mathbf{b}: t\in\mathbb{C}\}$ with the unit ball $\mathbb{B}^n=\{z\in\mathbb{C}^n:\ |z|:=\sqrt{|z|_1^2+\ldots+|z_n|^2}<1\}$ for any $z^0\in\mathbb{B}^n$. For this class of functions there is considered the concept of boundedness of L-index in the direction $\mathbf{b}$, where $L:\mathbb{B}^n\to\mathbb{R}_+$ is a positive continuous function such that $L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}$ and $\beta>1$ is some constant. There are presented sufficient conditions that the sum of slice holomorphic functions of bounded L-index in direction belong this class. This class of slice holomorphic functions is closed under the operation of multiplication.

1. Introduction. Here we continue our investigations initialized in [1, 2]. There was introduced a concept of L-index boundedness in direction for slice analytic functions of several complex variables and obtained many criteria of L-index boundedness in direction. Here we present some applications of these criteria to deduce sufficient conditions providing that sum, product of slice analytic functions is a function of bounded L-index in direction.

We consider a general problem. At this point, we should point to our article, in which we write about this as Prof. S.Yu. Favorov's problem [4].

Problem 1. *Is it possible to deduce main facts of theory of analytic functions having bounded L-index in the direction $\mathbf{b}\in\mathbb{B}^n\setminus\{0\}$ for functions which are holomorphic on the slices $\{z^0+t\mathbf{b}: t\in\mathbb{C}\}$ and are joint continuous?*

Let us introduce some notations from [1]. Let $\mathbb{R}_+=(0,+\infty)$, $\mathbb{R}_+^*=[0,+\infty)$, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{1}=(1,\ldots,1)$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a given direction, $\mathbb{B}^n=\{z\in\mathbb{C}^n: |z|<1\}$ be a unit ball, $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$ be a unit disc, $L:\ \mathbb{B}^n\to\mathbb{R}_+$ be a continuous function. For a given $z\in\mathbb{B}^n$, we denote $S_z=\{t\in\mathbb{C}: z+t\mathbf{b}\in\mathbb{B}^n\}$. Clearly, $\mathbb{D}=\mathbb{B}^1$. The slice functions on S_z for fixed $z^0\in\mathbb{B}^n$ we will denote as $g_{z^0}(t)=F(z^0+t\mathbf{b})$ and $l_{z^0}(t)=L(z^0+t\mathbf{b})$ for $t\in S_z$. Besides, we denote by $\langle a,c\rangle=\sum\limits_{j=1}^n a_j\overline{c_j}$ the Hermitian inner product in $\mathbb{C}^n$, where a, $c\in\mathbb{C}^n$.

2010 *Mathematics Subject Classification:* 32A10, 32A17, 58C10.
Keywords: bounded index L-index in direction; slice holomorphic function; directional derivative; unit ball.
doi:10.30970/ms.57.2.216-224

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slices $\{z^0+t\mathbf{b}: t\in S_{z^0}\}$ for each $z^0\in\mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for every $p\in\mathbb{N}$, $\partial_{\mathbf{b}}^p F(z)=g_z^{(p)}(0)$, where $g_z(t)=F(z+t\mathbf{b})$ is an analytic function of complex variable $t\in S_z$ for given $z\in\mathbb{B}^n$.

Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction do not imply holomorphy in whole n-dimensional unit ball. There were presented some examples to demonstrate it [1].

A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said [1] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0\in\mathbb{Z}_+$ such that for all $m\in\mathbb{Z}_+$ and each $z\in\mathbb{C}^n$ inequality

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\le\max_{0\le k\le m_0}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} \tag{1}$$

is true. The least such integer number m_0, obeying (1), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F,L,\mathbb{B}^n)$. For $n=1$, $\mathbf{b}=1$, $L(z)=l(z)$, $z\in\mathbb{C}$ inequality (1) defines a function of bounded l-index with the l-index $N(F,l)\equiv N_1(F,l,\mathbb{C})$ [11], and if in addition $l(z)\equiv 1$, then we obtain a definition of index boundedness with index $N(F)\equiv N_1(F,1,\mathbb{C})$ [12, 13]. It is also worth to mention paper [18], which introduces the concept of generalized index. It is quite close to the bounded l-index. Similarly, analytic function $F:\mathbb{B}^n\to\mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if it satisfies (1) for all $z\in\mathbb{B}^n$.

It should be noted that the function L, in addition to the properties of positivity and continuity, must also have the property of some regularity of behavior. Thus, we assume the following additional restrictions on the function L (see also, for example, [3–6, 17]). For $z\in\mathbb{B}^n$ we denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{B}^n}\sup_{t_1,t_2\in S_z}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}: |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L:\mathbb{B}^n\to\mathbb{R}_+$, satisfying for every $\eta\in[0,\beta]$

$$\lambda_{\mathbf{b}}(\eta)<+\infty \tag{2}$$

and for all $z\in\mathbb{B}^n$

$$L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}, \tag{3}$$

where $\beta>1$ is some constant.

2. Auxiliary propositions. In our investigations we need the following propositions obtained in [2]. The next theorems describe local behavior of the slice holomorphic function in the unit ball. They present necessary or sufficient conditions of boundedness of L-index in direction for this class of functions.

Theorem 1 ([2])**.** *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$. A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L-index in a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ if and only if for any r_1 and any r_2 with $0<r_1<r_2\le\beta$, there exists number $P_1=P_1(r_1,r_2)\ge 1$ such that for each $z^0\in\mathbb{B}^n$*

$$\max\left\{|F(z^0+t\mathbf{b})|: |t|=r_2/L(z^0)\right\}\le P_1\max\left\{|F(z^0+t\mathbf{b})|:\ |t|=r_1/L(z_0)\right\}. \tag{4}$$

Also we will use an analog of logarithmic criterion for function from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$. As necessary conditions the criterion was obtained by G. H. Fricke [9,10] for entire functions of one complex variable having bounded index.

Denote

$$G_r(F) := G_r^{\mathbf{b}}(F) := \bigcup_{z\in\mathbb{B}^n:\ F(z)=0} \{z + t\mathbf{b}\colon |t| < r/L(z)\}.$$

By $n_{z^0}(r) = n_{\mathbf{b}}\big(r, z^0, 1/F\big) := \sum_{|a_k^0|\le r} 1$ we denote counting function of zeros a_k^0 for the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t \in \mathbb{C}\colon |t| \le r\}$ for given $z^0 \in \mathbb{B}^n$. If for given $z^0 \in \mathbb{B}^n$ and for all $t \in S_z\colon\ F(z^0+t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z\in\mathbb{B}^n} n_z(r/L(z))$.

Theorem 2 ([2]). *Let $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. If the function F has bounded L-index in the direction $\mathbf{b}$, then*

1) for every $r \in (0, \beta]$ there exists $P = P(r) > 0$ that for each $z \in \mathbb{B}^n\backslash G_r^{\mathbf{b}}(F)$

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \le PL(z); \tag{5}$$

2) for every $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for each $z^0 \in \mathbb{B}^n$ with $F(z^0+t\mathbf{b}) \not\equiv 0$

$$n_{\mathbf{b}}\left(r/L(z^0), z^0, 1/F\right) \le \widetilde{n}(r).$$

Theorem 3 ([2]). *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, $\mathbb{B}^n\backslash G_\beta^{\mathbf{b}}(F) \ne \varnothing$. If the following conditions are satisfied*

1) there exists $r_1 \in (0, \beta/2)$ (either there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1;\infty)$;

2) there exist $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2{\cdot}n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{B}^n\backslash G_{r_2}(F)$ inequality (5) is true;

then the function F has bounded L-index in the direction $\mathbf{b}$.

3. Product of functions of bounded L-index in direction. Now we consider an application of Theorems 2 and 3. The following proposition can be obtained using similar considerations as in the case of analytic in the unit ball functions of bounded L-index in direction [8].

Proposition 1. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a function of bounded L-index in the direction $\mathbf{b}$, $\Phi \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ and $\Psi(z) = F(z)\Phi(z)$. The function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if the function $\Phi(z)$ is of bounded L-index in the direction $\mathbf{b}$.*

Proof. Our proof is similar to proof for analytic in the unit ball functions in [8] but now we use Theorem 2, deduced for functions holomorphic on the slices in the unit ball. Since an analytic function $F(z)$ has bounded L-index in the direction $\mathbf{b}$, by Theorem 2 for every $r \in (0, \beta)$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{B}^n$, satisfying $F(z^0 + t\mathbf{b}) \not\equiv 0$, the estimate $n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r)$ holds. Hence,

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Psi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) + \widetilde{n}(r).$$

Thus, condition 2 of Theorem 2 either holds or does not hold for functions $\Psi(z)$ and $\Phi(z)$ simultaneously. If $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$, then for every $r \in (0,\beta)$ there exist numbers $P_F(r)>0$ and $P_\Phi(r)>0$ such that $\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \le P_f(r)L(z)$, $\left|\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}\right| \le P_\Phi(r)L(z)$ for each $z \in (\mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Phi))$. Since $\mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Psi) \subset (\mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Phi))$ and $\left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le \left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| + \left|\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}\right|$, for all $z \in \mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Psi)$ we have $\left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le (P_F(r)+P_\Phi(r))L(z)$. Hence, by Theorem 3, the function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$.

On the contrary, let $\Psi(z)$ be of bounded L-index in the direction $\mathbf{b}$, $r \in (0,\beta)$. At first, we show that for every $z^0 \in \mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)$ and for every $\widetilde{d}^k = z^0 + d_k^0\mathbf{b}$, where d_k^0 are zeros of function $\Phi(z^0+t\mathbf{b})$, we have $|z^0-\widetilde{d}^k| > \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}$. On the other hand, let there exist $z^0 \in \mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Phi)$ and $\widetilde{d}^k = z^0 + d_k^0\mathbf{b}$ such that $|z^0-\widetilde{d}^k| \le \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}$. Then by the definition of $\lambda_{\mathbf{b}}$, we have the next estimate $L(\widetilde{d}^k) \le \lambda_{\mathbf{b}}(r)L(z^0)$, and hence $|z^0-\widetilde{d}^k| = |\mathbf{b}|\cdot|d_k^0| \le \frac{r|\mathbf{b}|}{2L(\widetilde{d}^k)}$, i.e. $|d_k^0| \le \frac{r}{2L(\widetilde{d}^k)}$, but it contradicts $z^0 \in \mathbb{B}^n\setminus G_r^{\mathbf{b}}(\Phi)$.

We consider $\overline{K}_0 = \left\{z^0+t\mathbf{b} : |t| \le \frac{r}{2L(z^0)\lambda_{\mathbf{b}}(r)}\right\}$. It does not contain zeros of $\Phi(z^0+t\mathbf{b})$, but it may contain zeros $\widetilde{c}^k = z^0 + c_k^0\mathbf{b}$ of the function $\Psi(z^0+t\mathbf{b})$. Since $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, by Theorem 2 the set $\overline{K}_0$ contains at most $\widetilde{n}_1 = \widetilde{n}_1\left(\frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ zeros c_k^0 of the function $\Psi(z^0+t\mathbf{b})$. For all $c_k^0 \in \overline{K}_0$, using the definition of $Q_{\mathbf{b}}(\mathbb{B}^n)$, we obtain the following inequality $L(z^0+c_k^0\mathbf{b}) \ge \frac{1}{\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}L(z^0)$. Thus, every set $m_k^0 = \left\{z^0+t\mathbf{b} : |t-c_k^0| \le \frac{r_1}{L(z^0+c_k^0\mathbf{b})}\right\}$ with $r_1 = \frac{r}{4(\widetilde{n}_1+1)\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)\lambda_{\mathbf{b}}(r)}$ is contained in the set $s_k^0 = \left\{z^0+t\mathbf{b} : |t-c_k^0| \le \frac{r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)}\right\}$. The total sum of diameters of these sets does not exceed

$$\frac{2\widetilde{n}_1 r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)} = \frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}\cdot\frac{\widetilde{n}_1}{(\widetilde{n}_1+1)} < \frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}.$$

Therefore, there exists $r^* \in \left(0, \frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ such that if $|t| = \frac{r^*}{L(z^0)}$, then $z^0+t\mathbf{b} \notin G_{r_1}^{\mathbf{b}}(\Psi)$, and therefore $z^0+t\mathbf{b} \notin G_{r_1}^{\mathbf{b}}(F)$. By Theorem 2, for all these points $z^0+t\mathbf{b}$ we obtain

$$\left|\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}\right| \le \left|\frac{\partial_{\mathbf{b}}\Psi(z^0+t\mathbf{b})}{\Psi(z^0+t\mathbf{b})}\right| + \left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right| \le (P_\Psi^*+P_F^*)L(z^0+t\mathbf{b}), \qquad (6)$$

where P_Ψ^* and P_F^* depend only on r_1, i.e. only of r. Since the function $\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}$ is analytic in $\overline{K}_0$, applying the maximum modulus principle to the function $\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}$ as a function of variable t, we obtain that the modulus of this function at the point $t=0$ does not exceed the maximum modulus of this function on the circle $\left\{t \in \mathbb{C} : |t| = \frac{r^*}{L(z^0)}\right\}$. It means that obtained inequality (6) also holds for z^0 instead $z^0+t\mathbf{b}$.

Thus, for arbitrary $r \in (0,\beta)$ and $z^0 \in \mathbb{B}^n\setminus G_r^{\mathbf{b}}(F)$ we have proved the first condition of Theorem 3. Above we have already shown that the second condition of Theorem 3 is also true. Hence, by the mentioned theorem, the function $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$. □

4. Sum of functions of bounded L-index in direction. Above we wrote that the product of analytic in the unit ball functions of bounded L-index in a direction is a function from the same class ([8]). But the class of analytic functions of bounded index is not closed under the addition. The corresponding example was constructed by W. Pugh (see [15, 17]) in the case of entire function of single variable. A generalization of Pugh's example for entire functions of bounded L-index in direction is proposed in [7].

Let us consider an intersection of the hyperplane $\langle z, \mathbf{b}\rangle = 0$ with the unit ball. The intersection we denote by $A = \{z \in \mathbb{B}^n : \ \langle z, \mathbf{b}\rangle = 0\}$, where $\langle z, \mathbf{b}\rangle := \sum_{j=1}^n z_j b_j$. Obviously that $\bigcup_{z^0\in A}\{z^0 + t\mathbf{b} : |t| \le \frac{1-|z_0|}{|\mathbf{b}|}\} = \mathbb{B}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$, then there exists $t_0 \in S_{z^0}$ such that $F(z^0 + t_0\mathbf{b}) \neq 0$. We denote

$$B(z^0, t) = \left\{t_0 \in S_{z^0} \colon |t_0 - t| < \min\left\{\frac{\beta}{2L(z^0 + t\mathbf{b})}, \frac{1 - |z^0 + \mathbf{b}t|}{2|\mathbf{b}|}\right\}, F(z^0 + t_0\mathbf{b}) \neq 0\right\},$$

$$B(z^0) = \bigcup_{|t|\le(1-|z^0|)/|\mathbf{b}|} B(z^0, t).$$

Theorem 4. *Let $L : \mathbb{B}^n \to \mathbb{R}_+$ be a positive continuous function satisfying (3) with $\beta \ge 3$, the functions F, $G \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ satisfy the following conditions:*
1) $G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ with $N_{\mathbf{b}}(G, L, \mathbb{B}^n) = N < +\infty$;
2) there exists $\alpha \in (0, 1)$ such that for all $z \in \mathbb{B}^n$ and $p \ge N + 1$ ($p \in \mathbb{N}$)

$$\frac{|\partial_{\mathbf{b}}^p G(z)|}{p!L^p(z)} \le \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z)|}{k!L^k(z)} : 0 \le k \le N\right\}; \tag{7}$$

3) for every $z = z^0 + t\mathbf{b} \in \mathbb{B}^n$ with $z^0 \in A$ and some $t_0 \in B(z^0, t)$ with $r = |t - t_0|L(z^0 + t\mathbf{b})$ the inequality

$$\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0 + t\mathbf{b})|}{k!L^k(z^0 + t\mathbf{b})} : \ 0 \le k \le N\right\}; \tag{8}$$

is valid;
4) either $(\exists c > 0)(\forall z^0 \in A)(\forall t \in S_{z^0})$ $(\exists t_0 \in B(z^0, t)$ obeying (8) and if $|t-t_0|L(z^0+t\mathbf{b}) \le 1)$, then

$$\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2}{L(z^0 + t\mathbf{b})}\right\}\Big/|F(z^0 + t_0\mathbf{b})| \le c < +\infty,$$

or for $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ $(\exists c > 0)(\forall z^0 \in A)$ $(\exists t_0 \in B(z^0))$ such that (8) is true and

$$\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2\lambda_{\mathbf{b}}(1)}{L(z^0 + t_0\mathbf{b})}\right\}\Big/|F(z^0 + t_0\mathbf{b})| \le c < +\infty, \tag{9}$$

where $\beta \ge 2\lambda_{\mathbf{b}}(1)$.

Then for every $\varepsilon \in \mathbb{C}$, $|\varepsilon| \le \frac{1-\alpha}{2c}$, the function

$$H(z) = G(z) + \varepsilon F(z) \tag{10}$$

has bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(H, L, \mathbb{B}^n) \le N$.

Proof. We repeat our arguments from [8] where this theorem is proved for functions analytic in the unit ball. We write Cauchy's formula for the slice holomorphic function $F(z^0+t\mathbf{b})$ as analytic function of one complex variable t

$$\frac{\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})}{p!}=\frac{1}{2\pi i}\int_{|t'-t|=\frac{r}{L(z^0+t\mathbf{b})}}\frac{F(z^0+t'\mathbf{b})}{(t'-t)^{p+1}}dt'. \tag{11}$$

For the chosen $r=|t-t_0|L(z^0+t\mathbf{b})$ we deduce $\frac{r}{L(z^0+t\mathbf{b})}=|t'-t|\geq|t'-t_0|-|t-t_0|=|t'-t_0|-\frac{r}{L(z^0+t\mathbf{b})}$. Hence,

$$|t'-t_0|\leq\frac{2r}{L(z^0+t\mathbf{b})}. \tag{12}$$

Equality (11) yields

$$\begin{aligned}\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}&\leq\frac{1}{2\pi L^p(z^0+t\mathbf{b})}\cdot\frac{L^{p+1}(z^0+t\mathbf{b})}{r^{p+1}}\frac{2\pi r}{L(z^0+t\mathbf{b})}\times\\ &\times\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t|=\frac{r}{L(z^0+t\mathbf{b})}\right\}\leq\\ &\leq\frac{1}{r^p}\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}.\end{aligned} \tag{13}$$

If $r=|t-t_0|L(z^0+t\mathbf{b})>1$, then (13) yields

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\leq\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{14}$$

Let $r=|t-t_0|L(z^0+t\mathbf{b})\in(0;1]$. Setting $r=1$ in (11) and (12), we analogously deduce

$$\begin{aligned}&\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\leq\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}=\\ &=\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}}\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\leq\\ &\leq\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\leq\\ &\leq c\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\},\end{aligned} \tag{15}$$

where

$$c=\sup_{z^0\in A,|t|<(1-|z^0|)/|\mathbf{b}|}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\geq 1$$

and $t_0=t_0(z,t)\in B(z^0,t)$ is chosen in (8) and $|t_0-t|\leq 1/L(z^0+t\mathbf{b})$. From $|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}$ one has $|t'|\leq|t_0|+\frac{2}{L(z^0+t\mathbf{b})}\leq|t|+\frac{3}{L(z^0+t\mathbf{b})}$. Therefore, $\beta\geq 3$.

If $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, then $\sup\left\{\frac{L(z^0+t_0\mathbf{b})}{L(z^0+t\mathbf{b})}:\ |t-t_0|\leq\frac{1}{L(z^0+t\mathbf{b})}\right\}\leq\lambda_{\mathbf{b}}(1)$. This means that $L(z^0+t\mathbf{b})\geq\frac{L(z^0+t_0\mathbf{b})}{\lambda_{\mathbf{b}}(1)}$. Using this inequality, we choose in (15)

$$c:=\sup_{z^0\in A}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2\lambda_{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\geq 1$$

with t_0 chosen in (8). Taking into account (14) and (15), one has

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \le c\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\} \tag{16}$$

for all $n\in\mathbb{N}\cup\{0\}$, $r\ge 0$, $z^0\in A$, $t\in S_{z^0}$.

We differentiate (10) p times, $p\ge N+1$, and apply (7), (16) and (8)

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \le \frac{|\partial_{\mathbf{b}}^p G(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} + \frac{|\varepsilon||\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \le \alpha\max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N\right\}+$$
$$+c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le(\alpha+c|\varepsilon|)\max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N\right\}. \tag{17}$$

If $s\le N$, then (16) is valid for $p=s$, but (7) does not hold. Thus, the differentiation of (10) leads to the following estimate

$$\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} \ge \frac{|\partial_{\mathbf{b}}^S G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - \frac{|\varepsilon||\partial_{\mathbf{b}}^s F(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\ge$$
$$\ge\frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}, \tag{18}$$

where $0\le s\le N$. From (8) and (18) we deduce

$$\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}\ge(1-c|\varepsilon|)\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}. \tag{19}$$

If $c|\varepsilon|<1$, then (17) and (19) imply

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \tag{20}$$

for $p\ge N+1$. Assume that $\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\le 1$. Hence, $|\varepsilon|\le\frac{1-\alpha}{2c}$.

Let $N_{\mathbf{b}}(F,L,z^0+t\mathbf{b})$ be the L-index in the direction $\mathbf{b}$ of the function F at the point $z^0+t\mathbf{b}$, i.e. $N_{\mathbf{b}}(F,L,z^0+t\mathbf{b})$ is the smallest number m_0, for which inequality (1) holds with $z=z^0+t\mathbf{b}$.

For $|\varepsilon|\le\frac{1-\alpha}{2c}$ validity of (20) means that for all $z^0\in A$ and every $t\in S_{z^0}$ such that $F(z^0+t\mathbf{b})\ne 0$ the L-index in the direction $\mathbf{b}$ at the point $z^0+t\mathbf{b}$ does not exceed N, i.e., $N_{\mathbf{b}}(F,L,z^0+t\mathbf{b})\le N$.

If for some $z^0\in A$ $\quad F(z^0+t\mathbf{b})\equiv 0$, one has $H(z^0+t\mathbf{b})\equiv G(z^0+t\mathbf{b})$ and $N_{\mathbf{b}}(F,L,z^0+t\mathbf{b})=N_{\mathbf{b}}(G,L,z^0+t\mathbf{b})\le N$. Thus, $H(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H,L,\mathbb{B}^n)\le N$. □

Every slice holomorphic function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ with $N_{\mathbf{b}}(F,L,\mathbb{B}^n)=0$ satisfies inequality (9) (see proof of necessity in [2, Theorem 2]).

If $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, then condition 2) in Theorem 4 always holds. The following theorem is valid.

Theorem 5. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\alpha \in (1/\beta, 1)$ and $F, G \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, which satisfy condition:*
1) *$G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$;*
2) *for every $z = z^0 + t\mathbf{b} \in \mathbb{B}^n$, where $z^0 \in A$, and some $t_0 \in B(z^0, t)$, and $r = |t - t_0| L(z^0 + t\mathbf{b})$*

$$\max\left\{|F(z^0 + t'\mathbf{b})| : |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \leq \max_{0 \leq k \leq N_{\mathbf{b}}(G_\alpha, L_\alpha, \mathbb{B}^n)} \left\{\frac{|\partial_{\mathbf{b}}^k G(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})}\right\};$$

3) $c := \sup\limits_{z^0 \in A} \dfrac{\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0 + t_0\mathbf{b})}\right\}}{|F(z^0 + t_0\mathbf{b})|} < \infty$ *where t_0 is chosen in 2).*

If $|\varepsilon| \leq \frac{1-\alpha}{2c}$, then the function $H(z) = G(z) + \varepsilon F(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L, \mathbb{B}^n) \leq N_{\mathbf{b}}(G_\alpha, L_\alpha, \mathbb{B}^n)$, where $G_\alpha(z) = G(z/\alpha)$, $L_\alpha(z) = L(z/\alpha)$.

Proof. Condition 2) in Theorem 4 always holds for $N = N_b(G_\alpha, L_\alpha)$ instead $N = N_{\mathbf{b}}(G, L)$. Indeed, by Theorem 1, inequality (4) is satisfied for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{t}{\alpha}$ and $\frac{t_0}{\alpha}$ instead z^0, t and t_0 in (4) we obtain

$$\max\left\{|G((z^0 + t\mathbf{b})/\alpha)| : |t - t_0| = \frac{r_2\alpha}{L((z^0 + t_0\mathbf{b})/\alpha)}\right\} \leq$$
$$\leq P_1 \max\left\{|G((z^0 + t\mathbf{b})/\alpha)| : |t - t_0| = \frac{r_1\alpha}{L((z_0 + t_0\mathbf{b})/\alpha)}\right\}. \tag{21}$$

By Theorem 1, inequality (21) means that $G_\alpha = G(z/\alpha)$ has bounded L_α-index in the direction $\mathbf{b}$ and vice versa. Then for $p \geq N_{\mathbf{b}}(G_\alpha, L_\alpha) + 1$ and $\alpha \in (1/\beta, 1)$

$$\frac{|\partial_{\mathbf{b}}^p G_\alpha(z)|}{p! L_\alpha^p(z)} = \frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! \alpha^p L^p(z/\alpha)} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^s G_\alpha(z)|}{s! L_\alpha^s(z)} : \ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} =$$
$$= \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! \alpha^s L^s(z/\alpha)} : \ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

Multiplying by α^p, we deduce

$$\frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! L^p(z/\alpha)} \leq \max\left\{\frac{\alpha^{p-s}|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)} : \ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} \leq$$
$$\leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)} : \ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}. \tag{22}$$

Since z is arbitrary, inequality (22) yields (7). □

Acknowledgments. The research of the first author was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

REFERENCES

1. A. Bandura, M. Martsinkiv, O. Skaskiv, *Slice holomorphic functions in the unit ball having a bounded L-index in direction*, Axioms, **10** (2021), №1, Article ID: 4. https://doi.org/10.3390/axioms10010004

2. A. Bandura, L. Shegda, O. Skaskiv, L. Smolovyk, *Some criteria of boundedness of L-index in a direction for slice holomorphic functions in the unit ball*, Internat. J. Appl. Math., **34** (2021), №4, 775-793. http://doi.org/10.12732/ijam.v34i4.13
3. A. Bandura, T. Salo, O. Skaskiv, *Some criteria of boundedness of L-index in a direction for slice holomorphic functions in the unit ball*, Mat Stud., **57** (2022), №1, 68–78. http://doi.org/10.30970/ms.57.1.68-78
4. A. Bandura, O. Skaskiv, *Slice holomorphic functions in several variables with bounded L-index in direction*, Axioms, **8** (2019), №3, Article ID: 88. http://doi.org/10.3390/axioms8030088
5. A. Bandura, O. Skaskiv, *Functions analytic in the Unit ball having bounded L-index in a direction*, Rocky Mountain J. Math., **49** (2019) №4, 1063–1092. doi: 10.1216/RMJ-2019-49-4-1063
6. A.I. Bandura, O.B. Skaskiv, I.R. Tymkiv, *Composition of entire and analytic functions in the unit ball*, Carpathian Math. Publ., **14** (2022), №1, 95–103. doi:10.15330/cmp.14.1.95-103
7. A.I. Bandura, *Product of two entire functions of bounded L-index in direction is a function with the same class*, Bukovyn. Mat. Zh., **4** (2016), №1-2, 8–12.
8. A.I. Bandura, *Composition, product and sum of analytic functions of bounded L-index in direction in the unit ball*, Mat. Stud., **50** (2018), №2, 115–134. http://doi.org/10.15330/ms.50.2.115-134
9. G.H. Fricke, *Functions of bounded index and their logarithmic derivatives*, Math. Ann., **206** (1973), 215–223.
10. G.H. Fricke, *Entire functions of locally slow growth*, J. Anal. Math., **28** (1975), №1, 101–122.
11. M.M. Sheremeta, *On the l-index boundedness of some composition of functions*, Mat. Stud., **47** (2017), №2, 207–210. http://doi.org/10.15330/ms.47.2.207-210
12. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, in: Entire Functions and Related Parts of Analysis, J. Korevaar (ed.), Proceedings of Symposia in Pure Math., **11**, Am. Math. Soc., Providence (1968), 298–307.
13. J.J. Macdonnell, Some convergence theorems for Dirichlet-type series whose coefficients are entire functions of bounded index, Doctoral dissertation, Catholic University of America, Washington, 1957.
14. F. Nuray, R.F. Patterson, *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*, Mat. Stud., **49** (2018), №1, 67–74. http://doi.org/10.15330/ms.49.1.67-74
15. W.J. Pugh, *Sums of functions of bounded index,* Proc. Amer. Math. Soc., **22** (1969), 319–323. https,//doi.org/10.1090/S0002-9939-1969-0243067-9
16. M.N. Sheremeta, *Entire functions and Dirichlet series of bounded l-index,* Russian Math. (Iz. VUZ), **36** (1992), №9, 76–82.
17. M. Sheremeta, Analytic functions of bounded index, VNTL Publishers, Lviv, 1999.
18. S. Strelitz, *Asymptotic properties of entire transcendental solutions of algebraic differential equations*, Contemp. Math., **25** (1983), 171–214. http://doi.org/10.1090/conm/025/730048

[1,3]Lviv Politechnic National University
Lviv, Ukraine
[1]vitalinabaksa@gmail.com
[3]tetyan.salo@gmail.com

[2]Ivano-Frankivsk National Technical University of Oil and Gas
Ivano-Frankivsk, Ukraine
[2]andriykopanytsia@gmail.com

Received 12.12.2021
Revised 20.05.2022