L. Bazylevych[1], O. Berezsky[2], M. Zarichnyi[3]

FRÉCHET FUZZY METRIC

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The aim of this note is to introduce a fuzzy counterpart of the Fréchet distance between curves. We consider both monotonic and non-monotonic case.

1. Introduction. The Fréchet distance between curves in a metric space is introduced in [6]. There are various versions of the Fréchet distance, e.g., discrete Fréchet distance (or coupling distance) [4], homotopic Fréchet distance etc. The Fréchet metric and its modifications found numerous applications, e.g., in computer vision, image processing, molecular biology etc.

The aim of this note is to define a fuzzy counterpart of the Fréchet distance between curves in the fuzzy metric spaces. The theory of fuzzy metric spaces is extensively developing, it has many applications in computer vision and image recognition (see, e.g., [11–13]). In this note we deal with the notion of fuzzy metric space in the sense of George and Veeramani [7, 8]. The main reason of this is because the fuzzy metrics in the sense of George and Veeramani induce metrizable topologies.

Some constructions in metric spaces have their counterparts in the theory of fuzzy metric spaces: fuzzy Hausdorff metric [14], fuzzy Prokhorov metric on the set of probability measures on fuzzy metric spaces [15], fuzzy metric on the set of idempotent and max-min measures [2].

We consider both monotonic and non-monotonic cases of the fuzzy Fréchet distance. Finally, we formulate some open problems.

2.1. Preliminaries.

2.1. Fuzzy metric spaces. The unit segment $[0, 1]$ will be denoted by $\mathbb{I}$. A t-norm is a continuous, associatiative, commutative function $*\colon \mathbb{I} \to \mathbb{I}$ which is monotone (in the sense that $x \le x'$ and $y \le y'$ imply $x * y \le x' * y'$) and 1 is a unit for $*$.

Some examples of t-norms are: min (denoted by $\wedge$), $\cdot$ (i.e., multiplication), $(a, b) \mapsto \max\{0, a + b - 1\}$ (the Łukasiewicz t-norm). There are general constructions of t-norms [17].

We recall the definition of the fuzzy metric space in the sense of [7].

Let X be a set, $*\colon [0, 1] \times [0, 1] \to [0, 1]$ be a continuous t-norm, and $\mathbb{R}+ = (0, +\infty)$. A GV-fuzzy metric on X is a pair $((m, *)$, where the mapping $m\colon X \times X \times \mathbb{R}_+ \to (0, 1]$ satisfies the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}_+$:

$(1GV)$ $m(x, y, t) > 0$;
$(2GV)$ $m(x, y, t) = 1$ if and only if $x = y$;
$(3GV)$ $m(x, y, t) = m(y, x, t)$;
$(4GV)$ $m(x, z, t + s) \ge m(x, y, t) * m(y, z, s)$;
$(5GV)$ $m(x, y, -)\colon \mathbb{R}_+ \to [0, 1]$ is continuous.

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If $(m, *)$ is a GV-fuzzy metric on X, then the triple $(X, m, *)$ is called a GV-fuzzy metric space. Given $x \in X$, $e \in (0,1)$, and $t > 0$, one can define

$$B(x, e, t) = \{y \in X \mid M(x, y, t) > 1 - e\}.$$

It is proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $B(x, e, t)$, where $x \in X$, $0 < e < 1$, $t > 0$.

The topological space (X, τ) is said to be fuzzy metrizable if there is a fuzzy metric M on X such that $\tau = \tau_M$. Then, it was proved that a topological space is fuzzy metrizable if and only if it is metrizable.

Given a topological space X, we denote by $\exp X$ the set of all nonempty compact subsets in X. Let $(X, M, *)$ be a fuzzy metric space. Given $x \in X$ and $A, B \in \exp X$, let $M(x, B, t) = \sup_{y \in B} M(x, y, t)$ and $M(A, B, t) = \inf_{x \in A} M(x, B, t)$. Then the Hausdorff fuzzy metric is a function $M_H\colon \exp X \times \exp X \times (0, \infty) \to (0, 1]$ defined by the formula

$$M_H(A, B, t) = \min\{M(A, B, t), M(B, A, t)\}.$$

Then M_H is known to be a fuzzy metric on $\exp X$ (see [14]).

Let (X, d) be a metric space. Then the function $M_d\colon X \times X \times (0, \infty) \to \mathbb{R}$ defined by the formula

$$M_d(x, y, t) = \frac{t}{d(x, y) + t}$$

is known to be a fuzzy metric for $* = \cdot$ (see [16]).

2.2. Fréchet distance. Let X be a topological space. By $\mathcal{C}(X)$ we denote the set of all parametric curves in X. The curves are considered up to parametrization. In the sequel we are interested in the case when X is a (fuzzy) metric space.

Let (X, d) be a metric space. By $\mathbb{I}$ we denote the segment $[0, 1]$. By $\mathcal{H}(\mathbb{I})$ we denote the group of homeomorphisms of $\mathbb{I}$.

Having two continuous parametric curves $\gamma_i\colon \mathbb{I} \to X$, $i = 1, 2$, their Fréchet distance is defined by the formula

$$d_F(\gamma_1, \gamma_2) = \inf_{\alpha} \sup\{d(\gamma_1(\alpha(t)), \gamma_2(\alpha_2(t))) \mid t \in \mathbb{I}\}.$$

It is well known that d_F is a metric on the set $\mathcal{C}(X)$.

One can define the notion of the Gromov-Fréchet distance:

$$d_{GF}(\gamma_1, \gamma_2) = \inf\{d_F(j_1(\gamma_1), j_2(\gamma_2)) \mid j_i\colon \gamma_i(\mathbb{I}) \to Z \text{ are isometric embeddings, } i = 1, 2\}.$$

Given curves in $\mathbb{R}^n$, one defines the isometric Fréchet distance:

$$d_F^{\mathrm{Iso}}(\gamma_1, \gamma_2) = \inf\{d_F(\gamma_1, h(\gamma_2)) \mid h\colon \mathbb{R}^n \to \mathbb{R}^n \text{ is an isometry}\}.$$

Note that

$$d_F^{\mathrm{Iso}} \geq d_{GF}.$$

Definition 1. Let $(X, m, *)$ be a GV-fuzzy metric space, $\gamma_i\colon \mathbb{I} \to X$, $i = 1, 2$, be parametric curves in X. Define

$$M_F(\gamma_1, \gamma_2, t) = \sup_{\alpha \in \mathcal{H}(\mathbb{I})} \inf_{s \in \mathbb{I}} M(\gamma_1(\alpha(s)), \gamma_2(s), t).$$

Clearly, $M_F(\gamma_1,\gamma_2,t)$ is well-defined. It immediately follows from the definition that
$$M_F(\gamma_1,\gamma_2,t)\le M_H(\gamma_1(\mathbb{I}),\gamma_2(\mathbb{I}),t).$$

Theorem 1. *The function M_F is a fuzzy metric on the set $\mathcal{C}(X)$.*

Proof. We first remark that $M_F(\gamma_1,\gamma_2,t)>0$. Indeed, since the function M is continuous and $\gamma_i(\mathbb{I})$, $i=1,2$, is compact, we conclude that, for any $\alpha\in\mathcal{H}(\mathbb{I})$,

$$\inf_{s\in\mathbb{I}} M(\gamma_1(\alpha(s)),\gamma_2(s),t)>0$$

and $M_F(\gamma_1,\gamma_2,t)>0$.

Now, if $M_F(\gamma_1,\gamma_2,t)=1$, then $M_H(\gamma_1(\mathbb{I}),\gamma_2(\mathbb{I}),t)=1$. Therefore, $\gamma_1(\mathbb{I})=\gamma_2(\mathbb{I})$ and, consequently, $\gamma_1=\gamma_2$.

Clearly, $M_F(\gamma_1,\gamma_2,t)=M_F(\gamma_2,\gamma_1,t)$.

Suppose now that $\gamma_1,\gamma_2,\gamma_3\in\mathcal{C}(X)$ and $M_F(\gamma_1,\gamma_2,t)=a$, $M_F(\gamma_2,\gamma_3,s)=b$. Given $r>0$, one can find $\alpha,\beta\in\mathcal{H}(\mathbb{I})$ such that

$$\inf_{p\in\mathbb{I}} M(\gamma_1(\alpha(p)),\gamma_2(p),t)>a-r,\quad \inf_{p\in\mathbb{I}} M(\gamma_2(\beta(p)),\gamma_3(p),s)>b-r.$$

Then

$$\begin{gathered}M_F(\gamma_1,\gamma_3,t+s)\ge\inf_{p\in\mathbb{I}} M(\gamma_1(\alpha\beta(p)),\gamma_3(p),t+s)\ge\\ \ge\inf_{p\in\mathbb{I}}(M(\gamma_1(\alpha(\beta(p)),\gamma_2(\beta(p)),t)*M(\gamma_2(\beta(p),\gamma_3(p),s))\ge\\ \ge\left(\inf_{p\in\mathbb{I}} M(\gamma_1(\alpha(p)),\gamma_2(p),t)\right)*\left(\inf_{p\in\mathbb{I}} M(\gamma_2(\beta(p)),\gamma_3(p),s)\right)\ge(a-r)*(b-r).\end{gathered}$$

Since $r>0$ is arbitrary and $*$ is continuous, we are done.

Given γ_1,γ_2, we are going to prove that the function $M_F(\gamma_1,\gamma_2,-)\colon(0,\infty)\to(0,1]$ is continuous. Let $t_0\in(0,\infty)$, $M_F(\gamma_1,\gamma_2,t_0)=c$. Let $r>0$. By compactness of $\gamma_1(\mathbb{I})\times\gamma_2(\mathbb{I})$ and continuity of M, there exists $\epsilon>0$ such that the following holds: for any $x\in\gamma_1(\mathbb{I})$, $y\in\gamma_2(\mathbb{I})$, any $t\in(t_0-\epsilon,t_0+\epsilon)$, we have
$$|M(x,y,t)-M(x,y,t_0)|<r.$$

Then clearly, for any $t\in(t_0-\epsilon,t_0+\epsilon)$, $\alpha\in\mathcal{H}(\mathbb{I})$ and any $s\in\mathbb{I}$,

$$|M_F(\gamma_1(\alpha(s)),\gamma_2(s),t_0)-M_F(\gamma_1(\alpha(s)),\gamma_2(s),t)|\le r$$

and therefore

$$|\inf_{s\in\mathbb{I}} M_F(\gamma_1(\alpha(s)),\gamma_2(s),t_0)-\inf_{s\in\mathbb{I}} M_F(\gamma_1(\alpha(s)),\gamma_2(s),t)|\le 2r.$$

Now,

$$\begin{gathered}|M_F(\gamma_1,\gamma_2,t)-M_F(\gamma_1,\gamma_2,t_0)|=\\ =|\sup_{\alpha\in\mathcal{H}(\mathbb{I}}\inf_{s\in\mathbb{I}} M_F(\gamma_1(\alpha(s)),\gamma_2(s),t)-\sup_{\alpha\in\mathcal{H}(\mathbb{I}}\inf_{s\in\mathbb{I}} M_F(\gamma_1(\alpha(s)),\gamma_2(s),t_0)|\le 4r.\end{gathered}$$

The continuity is proved.

□

Let (X, d) be a metric space and let $(M_d, \cdot)$ denote the fuzzy metric generated by d.

Proposition 1. $(M_d)_F = M_{d_F}$.

Proof. This easily follows from the condition: given $t \in (0, \infty)$,

$$d(x, y) \le d(x', y') \iff M(x, y, t) \ge M(x', y', t)$$

for any $x, y, x', y' \in X$. $\square$

Remark 1. *A similar result can be proved for the fuzzy metric* $(M'_d, \wedge)$, *where*

$$M'_d(x, y, t) = e^{-\frac{d(x,y)}{t^n}},\ n \in \mathbb{N}.$$

3. Non-monotonic case. The following lemma is proved in [1], we include its proof for the sake of completeness. Let X be a topological space and let $A, B \subset X$ be disjoint closed subsets. A closed set $C \subset X$ is called a separator in X between A and B if there are disjoint open sets $U, V \subset X$ such that $X \setminus C = U \cup V$, with $A \subset U$ and $B \subset V$.

Lemma 1. *Let* $f_i \colon \mathbb{I} \to \mathbb{I}$, $i \in \{1, 2, 3, 4\}$, *be continuous piecewise-linear maps such that* $f_i^{-1}(0) = \{0\}$, $f_i^{-1}(1) = \{1\}$, $i \in \{1, 2, 3, 4\}$. *Then there exist continuous piecewise-linear maps* $g_i \colon \mathbb{I} \to \mathbb{I}$, $i \in \{1, 2, 3\}$, *such that, for every* $t \in \mathbb{I}$, *there exist* $\theta, \tau \in \mathbb{I}$ *such that* $g_1(t) = f_1(\theta)$, $g_2(t) = f_2(\theta) = f_3(\tau)$, $g_3(t) = f_4(\tau)$.

Proof. Since the set $\{(f_1(t), f_2(t) \mid t \in \mathbb{I}\}$ is a connected one-dimensional polyhedron, it contains a set L homeomorphic to a segment such that $(0, 0)$ and $(1, 1)$ are the endpoints of L. The set

$$K = \{(t, f_3(\tau), f_4(\tau)) \mid t, \tau \in \mathbb{I}\} \cap (L \times \mathbb{I})$$

is a separator between the singletons $\{(0, 0, 1)\}$ and $\{(1, 1, 0)\}$. From theorem on separators for cubes (see, e.g., [5, Theorem 1.8.1]) it follows that the points $(0, 0, 0)$ and $(1, 1, 1)$ belong to the same connected component of the set K. Since K is a subpolyhedron of $\mathbb{I}^3$, there exists a continuous piecewise-linear map $g = (g_1, g_2, g_3) \colon \mathbb{I} \to \mathbb{I}^3$ such that $g(0) = (0, 0, 0)$, $g(1) = (1, 1, 1)$ and $g(\mathbb{I}) \subset K$. It is easy to see that the functions g_1, g_2, g_3 are as required. $\square$

By $\mathcal{M}(\mathbb{I})$ we denote the set continuous selfmaps of $\mathbb{I}$ that preserve the endpoints of $\mathbb{I}$.

Let $(X, M, *)$ be a fuzzy metric space. Given $\gamma_1, \gamma_2 \in \mathcal{C}(X)$ and $t \in (0, \infty)$, define

$$M'_F(\gamma_1, \gamma_2, t) = \sup_{\tau_1, \tau_2 \in \mathcal{M}(\mathbb{I})} \inf_{p \in \mathbb{I}} M(\gamma_1(\tau_1(p)), \gamma_2(\tau_2(p)), t).$$

As in the monotonic case,

$$M'_F(\gamma_1, \gamma_2, t) \le M_H(\gamma_1(\mathbb{I}), \gamma_2(\mathbb{I}), t).$$

Theorem 2. *The function* $(M'_F, *)$ *is a fuzzy metric on* $\mathcal{C}(X)$.

Proof. Let us prove property (4) from the definition of the fuzzy metric. Given $\gamma_i \in \mathcal{C}(X)$, $i = 1, 2, 3$, $s, t \in (0, \infty)$, and $\varepsilon > 0$, one can find $\tau_1, \tau_2, \tau_2^*, \tau_3 \in \mathcal{M}(\mathbb{I})$ such that

$$|M'_F(\gamma_1, \gamma_2, s) - \inf_{p \in \mathbb{I}} M(\gamma_1(\tau_1(p)), \gamma_2(\tau_2(p)), s)| < \varepsilon,$$
$$|M'_F(\gamma_2, \gamma_3, t) - \inf_{p \in \mathbb{I}} M(\gamma_2(\tau_2^*(p)), \gamma_3(\tau_3(p)), t)| < \varepsilon.$$

Since the function M is continuous, one may assume that the functions $\tau_1, \tau_2, \tau_2^*, \tau_3$ are piecewise-linear and $\varphi^{-1}(a) = \{a\}$, for every $\varphi \in \{\tau_1, \tau_2, \tau_2^*, \tau_3\}$ and $a \in \{0, 1\}$. Apply Lemma 1 and obtain continuous maps $g_i \in \mathcal{M}(\mathbb{I})$.

Now, suppose that

$$\inf_{p\in\mathbb{I}} M(\gamma_1(g_1(p)), \gamma_3(g_3(p)), t+s) = M(\gamma_1(g_1(p_0)), \gamma_3(g_3(p_0)), t+s).$$

By Lemma 1, there exist $p', p'' \in \mathbb{I}$ such that

$$g_1(p_0) = \tau_1(\theta),$$

Then we obtain

$$M'_F(\gamma_1, \gamma_3, t+s) \geq M(\gamma_1(g_1(p_0)), \gamma_3(g_3(p_0)), t+s) \geq$$
$$\geq M(\gamma_1(\tau_1(\theta)), \gamma_2(g_3(p_0)), t) * M(\gamma_2(), \gamma_3(), s) \geq M'_F(\gamma_1, \gamma_2, t) * M'_F(\gamma_2, \gamma_3, s) - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (4) is proved.

The continuity of $t \mapsto M(\gamma_1, \gamma_2, t)$ can be proved similarly as in the proof of Theorem 1. □

4. Remarks. Some versions of the Fréchet distance are also considered in the literature. In particular, one can define the Fréchet distance between simple closed curves, i.e., embeddings of S^1 into a space X. There is a natural fuzzy counterpart of this notion.

The notion of complete fuzzy metric space is introduced in [9]. It is known that not all fuzzy metric spaces can be completed and the characterization of completable fuzzy metric spaces is given in [10]. The hyperspace $(\exp X, M_H, *)$ is completable if and only if $(X, M, *)$ is completable (see [14, Theorem 5]). This leads to the question whether the analog of this result holds for the Fréchet fuzzy metric.

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[1,3]Ivan Franko National University of Lviv
Lviv, Ukraine
[1] izar@litech.lviv.ua
[3] zarichnyi@yahoo.com

[2] Ternopil National Economic University
Ternopil, Ukraine
ob@tneu.edu.ua

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