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THE STRIP OF ANALYTICITY OF REFLECTIONLESS POTENTIALS

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We study the problem of the analytical extension of bounded reflectionless potentials for the Schrödinger operator in the Hilbert space $L_2(\mathbb{R})$ and improve an estimate on the strip of analyticity established by V. A. Marchenko.

1. Introduction.

1.1. The Bargmann potentials. A classical reflectionless potential, or the Bargmann potential [1], for the Schrödinger operator in the Hilbert space $L_2(\mathbb{R})$ is a function given via

$$q(x) = -2\frac{d^2}{dx^2}\log\det\left(\delta_{js} + m_j m_s \frac{e^{-(\kappa_j+\kappa_s)x}}{\kappa_j+\kappa_s}\right)_{1\le j,s\le n}, \qquad x\in\mathbb{R}, \tag{1}$$

where δ_{js} is the Kronecker delta, $\boldsymbol{\kappa} = (\kappa_j)_{j=1}^n$ and $\boldsymbol{m} = (m_j)_{j=1}^n$ are sequences of positive numbers, moreover, the numbers κ_j are different. Denote by $\mathcal{Q}_0$ the set of all classical reflectionless potentials, and denote by $\mathcal{Q}_0(\alpha)$ ($\alpha > 0$) the set of all $q \in \mathcal{Q}_0$ such that the numbers κ_j in (1) satisfy the condition

$$\kappa_j \le \alpha, \qquad j \in \{1,\dots,n\}.$$

1.2. Bounded reflectionless potentials. As shown in [2] by V. A. Marchenko, each potential $q \in \mathcal{Q}_0(\alpha)$ can be extended to an analytic function $q\colon \Omega_\alpha \to \mathbb{C}$ into the strip

$$\Omega_\alpha = \{z\in\mathbb{C}\colon |\operatorname{Im} z| < \alpha^{-1}\}$$

that satisfies therein the estimate

$$|q(x+iy)| \le \frac{2\alpha^2}{(1-\alpha|y|)^2}, \qquad x\in\mathbb{R}, \quad y\in(-\alpha^{-1},\alpha^{-1}). \tag{2}$$

Denote by $\mathcal{Q}(\alpha)$ the closure of the set $\mathcal{Q}_0(\alpha)$ in the topology of uniform convergence on the compact sets in $\mathbb{R}$. Using the estimate (2), we can prove that the set $\mathcal{Q}(\alpha)$ is sequentially compact in the topology of uniform convergence on the compact sets in $\mathbb{R}$, moreover, each potential $q \in \mathcal{Q}(\alpha)$ can be extended to an analytic function in Ω_α that satisfies the estimate (2).

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The set

$$\mathcal{Q} := \bigcup_{\alpha>0} \mathcal{Q}(\alpha)$$

is called the set of the bounded reflectionless potentials.

The function (see [3])

$$q(x) = \frac{-2\alpha^2}{\operatorname{ch}^2[\alpha(x-\xi)]}, \qquad x \in \mathbb{R}, \qquad \xi \in \mathbb{R}, \tag{3}$$

is an example of the reflectionless potential from the class $\mathcal{Q}_0(\alpha)$. But, as is easy to see, the function q is analytic in the strip $|\operatorname{Im} z| < \pi/2\alpha$, which is wider than the strip Ω_α defined in [2]. This is the reason why the authors decided to study the possibility of improving the estimate (2).

The main result of this paper is the following theorem.

Theorem. *Each function $q \in \mathcal{Q}(\alpha)$ can be analytically extended into the strip*

$$\Pi_\alpha := \{z \in \mathbb{C} \colon |\operatorname{Im} z| < \pi/2\alpha\}$$

and

$$|q(x+iy)| \le \frac{2\alpha^2}{\cos^2 \alpha y}, \qquad x \in \mathbb{R}, \quad |y| < \frac{\pi}{2\alpha}. \tag{4}$$

Moreover, the estimate (4) *is reached in the class $\mathcal{Q}(\alpha)$ on the potentials of the form* (3).

Since the set $\mathcal{Q}(\alpha)$ is the closure of $\mathcal{Q}_0(\alpha)$ in the topology of uniform convergence on the compact sets in $\mathbb{R}$, it suffices to prove Theorem for the potentials from the class $\mathcal{Q}_0(\alpha)$. The main tool in proving Theorem is the formula for the potentials $q \in \mathcal{Q}_0$ obtained in the recent article [4].

2. The proof of Theorem. Here and further, we denote by H and H_1 the Hilbert spaces over the complex plane $\mathbb{C}$, and we denote by $\mathcal{B}(H, H_1)$ ($\mathcal{B}(H)$) the Banach space (the Banach algebra) of all linear everywhere defined continuous operators $A : H \to H_1$ ($A : H \to H$).

First, we prove three auxiliary lemmas.

Lemma 1. *Let K and Γ be positive operators in $\mathcal{B}(H)$, and $y \in \mathbb{R}$, $|y| < \frac{\pi}{2\|K\|}$. Then the operator $(e^{2iyK} + \Gamma)$ is invertible in the algebra $\mathcal{B}(H)$.*

Proof. Obviously, we can only consider the case $y \ge 0$. Let us put

$$\delta := \tfrac{1}{2}(\pi - 2\|yK\|), \qquad \gamma := \sin\delta.$$

Then

$$\delta I \le 2yK + \delta I \le (\pi - \delta)I,$$

thus

$$\sin(2yK + \delta I) \ge (\sin\delta)I = \gamma I. \tag{5}$$

Note that the invertibility of the operator $(e^{2iyK} + \Gamma)$ is equivalent to the invertibility of the operator

$$W := e^{i\delta}(e^{2iyK} + \Gamma).$$

Taking into account (5) and the inequality $\Gamma \geq 0$, we get that the imaginary part $\operatorname{Im} W$ of W satisfies the bound

$$\operatorname{Im} W = \operatorname{Im}(-W^*) = \sin(2yK + \delta I) + (\sin\delta)\Gamma \geq \sin(2yK + \delta I) \geq \gamma I.$$

Hence, the operators W and W^* are bounded below, moreover,

$$\|Wf\| \geq \gamma\|f\|, \quad \|W^*f\| \geq \gamma\|f\|, \qquad f \in H.$$

Therefore, we conclude that W is invertible in the algebra $\mathcal{B}(H)$. □

Lemma 2. *Let the conditions of Lemma 1 hold, $R \in (H, H_1)$, and*

$$K\Gamma + \Gamma K = R^*R. \tag{6}$$

Then for the operator $M := (e^{2iyK} + \Gamma)^{-1}R^*$ the inequality

$$\|M^*KM\| \leq \frac{\|K\|^2}{2\cos^2(y\|K\|)} \tag{7}$$

holds.

Proof. Without loss of generality, we may assume that $y \neq 0$. Let $\lambda \in \mathbb{R}$ and $\lambda y > 0$. We introduce the notation

$$K_\lambda := (K - i\lambda I)^{-1}, \qquad B := RK_\lambda M.$$

Taking into account (6), we obtain that

$$(K + i\lambda I)(e^{-2iyK} + \Gamma) + (e^{2iyK} + \Gamma)(K - i\lambda I) = R^*R + 2Kh(K), \tag{8}$$

where

$$h(t) := \cos 2yt + \lambda\frac{\sin 2yt}{t}, \qquad t > 0.$$

Multiplying the equality (8) on the left by the operator $M^*K_{-\lambda}$ and on the right by the operator $K_\lambda M$, we get

$$B + B^* = B^*B + 2M^*g(K)h(K)M, \tag{9}$$

where

$$g(t) := \frac{t}{t^2 + \lambda^2}, \qquad t \geq 0.$$

Since the functions $f_1(t) = \frac{\sin t}{t}$ and $f_2(t) = \cos t$ strictly decrease on the interval $[0, \pi]$, the function h strictly decreases on the interval $[0, \frac{\pi}{2|y|}]$ too. Let us put

$$\beta := y\|K\|, \qquad \lambda := \|K\| \operatorname{tg}\beta, \qquad \gamma = \frac{\cos^2\beta}{\|K\|^2}.$$

Then

$$h(t) \geq h(\|K\|) = \cos 2\beta + \operatorname{tg}\beta \sin 2\beta = 1, \qquad t \in [0, \|K\|].$$

It is easy to see that for all $t \in [0, \|K\|]$

$$g(t) \geq \frac{t}{\|K\|^2 + \lambda^2} = \gamma t.$$

Thus

$$g(t)h(t) \geq \gamma t, \qquad t \in [0, \|K\|],$$

and, hence

$$g(K)h(K) \geq \gamma K.$$

From the above, using (9), we obtain that

$$2\gamma M^* K M \leq 2M^* g(K)h(K)M = B + B^* - B^*B = I - (I - B^*)(I - B) \leq I$$

and, as a result, we have

$$\|M^* K M\| \leq \frac{1}{2\gamma} = \frac{\|K\|^2}{2\cos^2(y\|K\|)}.$$

□

Lemma 3. *Let K, Γ be positive operators in $\mathcal{B}(H)$, $R \in \mathcal{B}(H, H_1)$, and let the equality (6) holds. Then for an arbitrary $z \in \Pi_\alpha$, where $\alpha = \|K\|$, the operator $(e^{2zK} + \Gamma)$ is invertible in the algebra $\mathcal{B}(H)$ and*

$$\|e^{zK} K^{1/2} (e^{2zK} + \Gamma)^{-1} R^*\| \leq \frac{\|K\|}{\sqrt{2}\cos(y\|K\|)}.$$

Proof. Let $z = x + iy$, where $x, y \in \mathbb{R}$, and $|y| < \frac{\pi}{2\alpha}$. Let us consider the operator

$$L := e^{zK} K^{1/2} (e^{2zK} + \Gamma)^{-1} R^*.$$

We set

$$\Gamma_1 := e^{-xK} \Gamma e^{-xK}, \qquad R_1 := R e^{-xK}.$$

Then the operator L can be written as

$$L = e^{iyK} K^{1/2} (e^{2iyK} + \Gamma_1)^{-1} R_1^*.$$

Note that the operator Γ_1 is positive and the equalities

$$K\Gamma_1 + \Gamma_1 K = R_1^* R_1, \quad L^* L = R_1((e^{-2iyK} + \Gamma_1)^{-1} K (e^{2iyK} + \Gamma_1)^{-1} R_1^*$$

hold. Therefore, in view of Lemma 2, the inequality

$$\|L^* L\| \leq \frac{\|K\|^2}{2\cos^2(y\|K\|)}$$

holds. This inequality implies that

$$\|L\| \leq \frac{\|K\|}{\sqrt{2}\cos(y\|K\|)}.$$

□

Proof of Theorem . Let $\boldsymbol{\kappa} := (\kappa_j)_{j=1}^n$ and $\boldsymbol{m} := (m_j)_{j=1}^n$ be sequences from the formula (1). We introduce an $n \times n$ matrix G with entries

$$G_{k,l} := \frac{\kappa_k \kappa_l}{\kappa_k + \kappa_l},$$

and two diagonal matrices

$$K = \mathrm{diag}\{\kappa_1, \dots, \kappa_n\}, \quad A = \mathrm{diag}\{a_1, \dots, a_n\}$$

with $a_j := \kappa_j / m_j$. It was shown in [4] that $G > 0$ and that the potential q defined by formula (1) can be presented as

$$q(x) = 2\frac{d}{dx}\langle (A^2 e^{2xK} + G)^{-1}\boldsymbol{\kappa}, \boldsymbol{\kappa}\rangle_{\mathbb{C}^n}, \qquad x \in \mathbb{R}, \tag{10}$$

where $\langle \cdot\, , \,\cdot \rangle_{\mathbb{C}^n}$ is the standard scalar product in $\mathbb{C}^n$. Note that the fact that the potential q belongs to the class $\mathcal{Q}_0(\alpha)$ means that $\|K\| \le \alpha$.

The formula (10) can be rewritten as

$$q(x) = 2\frac{d}{dx} R^*(e^{2xK} + \Gamma)^{-1} R, \quad x \in \mathbb{R}, \tag{11}$$

where the operator $R : \mathbb{C}^n \to \mathbb{C}$ acts by the formula

$$Rc = \langle c, \boldsymbol{m}\rangle_{\mathbb{C}^n}, \quad c \in \mathbb{C}^n,$$

and

$$\Gamma = A^{-1} G A^{-1}.$$

It is easy to see that the (k, l)-entry of the matrix Γ is equal to

$$\Gamma_{k,l} := \frac{m_k m_l}{\kappa_k + \kappa_l}.$$

Calculating the derivative in (11), we get that

$$q(x) = -4R(e^{2xK} + \Gamma)^{-1} e^{2xK} K (e^{2xK} + \Gamma)^{-1} R^*, \quad x \in \mathbb{R}.$$

Lemma 3 implies that the matrix-valued function

$$F(z) := (e^{2zK} + \Gamma)^{-1}$$

does not have any singular points in the strip Π_α. Hence, F is analytic in Π_α. Thus the formula

$$q(z) = -4R(e^{2zK} + \Gamma)^{-1} e^{2zK} K (e^{2zK} + \Gamma)^{-1} R^*, \quad z \in \Pi_\alpha, \tag{12}$$

defines the analytic extension of the potential q as claimed.

The equality (12) implies that

$$|q(z)| \le 4\|L(\bar{z})^*\| \|L(z)\|, \quad z \in \Pi_\alpha,$$

where

$$L(z) = e^{zK} K^{1/2} (e^{2zK} + \Gamma)^{-1} R^*.$$

But, in view of Lemma 3,

$$\|e^{zK}K^{1/2}(e^{2zK}+\Gamma)^{-1}R^*\| \le \frac{\|K\|}{\sqrt{2}\cos(y\|K\|)}, \quad z \in \Pi_\alpha.$$

Thus

$$|q(z)| \le 4\|L(\bar{z})\|\|L(z)\| \le \frac{2\|K\|^2}{\cos^2(y\|K\|)}, \quad z \in \Pi_\alpha.$$

The proof of Theorem is complete. □

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