УДК 517.982

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ON LINEAR SECTIONS OF ORTHOGONALLY ADDITIVE OPERATORS

A. Gumenchuk, I. Krasikova, M. Popov. *On linear sections of orthogonally additive operators*, Mat. Stud. **58** (2022), 94–102.

Our first result asserts that, for linear regular operators acting from a Riesz space with the principal projection property to a Banach lattice with an order continuous norm, the C-compactness is equivalent to the AM-compactness. Next we prove that, under mild assumptions, every linear section of a C-compact orthogonally additive operator is AM-compact, and every linear section of a narrow orthogonally additive operator is narrow.

1. Introduction. Orthogonally additive operators on Riesz spaces naturally generalize linear operators, and in recent years a number of results on linear operators were generalized to the orthogonally additive ones by different authors, see e.g. [1, 3, 5, 10, 11, 12, 13, 16, 19, 21] and the bibliography therein. Necessary background for the theory of orthogonally additive operators was prepared by J. M. Mazón and S. Segura de León in [10, 11] (1990), and since then a number of mathematicians actively study different problems on orthogonally additive operators on Riesz spaces.

Our attention was drawn to linear sections of orthogonally additive operators introduced and studied in a recent paper by the third named author [21]. A linear section S of an orthogonally additive operator $T\colon E \to F$ by a given level $\mathbf{L}$ of the domain Riesz space E is defined to be a linear operator $S\colon E \to F$ which equals T on $\mathbf{L}$ (for precise definitions see below). One given orthogonally additive operator may have a large variety of linear sections by different levels [21]. We are interested in the question of what compact-like properties of T does any S inherit.

In passing, we obtain a result for regular linear operators. The notion of C-compactness for orthogonally additive (in particular, linear) operators was introduced by J. M. Mazón and S. Segura de León in [11] (1990) as a weak version of the well known AM-compactness. To be more precise, recall that x is called a *fragment* (*component* in the terminology of [2]) of y (x, y are elements of a Riesz space E), if $x \perp (y - x)$. The set of all fragments of an element $e \in E$ is denoted by $\mathfrak{F}_e$. An orthogonally additive (in particular, a linear) operator $T\colon E \to F$, where E is a Riesz space and F a Banach space is said to be:

- *AM-compact* if $T([x, y])$ is a precompact subset of F for all $x, y \in E$;
- *C-compact* if $T(\mathfrak{F}_e)$ is a precompact subset of F for every $e \in E$.

2010 *Mathematics Subject Classification:* 47B38, 47B65, 47H30.
Keywords: orthogonally additive operator; narrow operator; C-compact operator; AM-compact operator.
doi:10.30970/ms.58.1.94-102

Since $\mathfrak{F}_e \subseteq [-|e|, |e|]$ for all $e \in E$, every AM-compact operator is C-compact. One of our results asserts that every C-compact regular linear operator acting from a Riesz space with the principal projection property to a Banach lattice with an order continuous norm is AM-compact. A partial case of this theorem, where the domain space is a Dedekind σ-complete Banach lattice, can be derived from a known result of [11].

In standard terminology and notation we follow Aliprantis-Burkinshaw textbook [2]. Given elements $x, x_1, \ldots, x_m$ of a Riesz space E, the notation $x = \bigsqcup_{i=1}^m x_i$ means that $x = \sum_{i=1}^m x_i$ and $x_i \perp x_j$ as $i \neq j$. The *lateral order* $\sqsubseteq$ on E is defined by setting $x \sqsubseteq y$ ($x, y \in E$) if and only if x is a *fragment* of y (see [12] for a detailed study of the lateral order). Given any sets A, B, C with $C \subseteq A$ and a function $f \colon A \to B$, by $f|_C$ we denote the restriction of f to C.

1.1. Lateral bands, consistent sets and levels of a Riesz space. Let E be a Riesz space. Since every subset $A \subseteq E$ is laterally bounded from below by zero, A is said to be *laterally bounded* provided A is laterally bounded from above, that is, there exists $e \in E$ such that $A \subseteq \mathfrak{F}_e$. An infimum and supremum of a subset $A \subseteq E$ with respect to the lateral order (in case of existence) are denoted using the symbols in bold $\bigcap A$ and $\bigcup A$. The following statement characterizes lateral infima a suprema in terms of the given order on E.

Proposition 1 ([21]). *Let E be a Riesz space and $e \in E$. Then the following assertions hold:*
1. *The set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $\cup$ and $\cap$. Moreover, $x \cup y = (x^+ \vee y^+) - (x^- \vee y^-)$ and $x \cap y = (x^+ \wedge y^+) - (x^- \wedge y^-)$ for all $x, y \in \mathfrak{F}_e$.*
2. *Assume $e \geq 0$. Then the following hold:*
   - (a) *The lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\leq$.*
   - (b) *Let a nonempty subset A of $\mathfrak{F}_e$ have a lateral supremum $a = \bigcup A$ (respectively, a lateral infimum $a = \bigcap A$). Then:*
     i. If $y = \sup A$ (respectively, $y = \inf A$) exists in E then $y = a$.
     ii. If, moreover, E has the principal projection property then $\sup A$ *(respectively,* $\inf A$*) exists in E and by* (i) *equals a.*

A subset $G \subseteq E$ is said to be *laterally solid* provided that $\mathfrak{F}_x \subseteq G$ for all $x \in G$. A laterally solid subset $I \subseteq E$ is called a *lateral ideal* of E if for every $x, y \in I$ with $x \perp y$ one has $x + y \in I$. A lateral ideal B of E is called a *lateral band* of E if for every laterally bounded subset A of B the existence of $\bigcup A$ implies $\bigcup A \in B$. Obviously, the intersection of any nonempty family of lateral ideals (or lateral bands) is a lateral ideal (respectively, a lateral band). The *lateral ideal* (or *lateral band*) *generated by* a nonempty subset A of E is defined to be the intersection of all lateral ideals (respectively, lateral bands) of E including A. For every $e \in E$ the set $\mathfrak{F}_e$ is simultaneously the lateral ideal and lateral band generated by the singleton $\{e\}$, and is called the *principal lateral ideal* and *principal lateral band* of E.

A subset G of a Riesz space E is said to be *consistent* if every two-point subset $\{x, y\}$ of G is laterally bounded (equivalently, every finite subset of G is laterally bounded [12, Proposition 5.2]). The lateral band $\mathcal{B}(G)$ in a Riesz space E generated by a consistent set G is consistent [12, Theorem 6.10]. A consistent lateral band of E is called a *level* of E. A level which is not included in another level is called a *maximal level.* A level $\mathbf{L}$ in E is called a *principal level* provided $\mathbf{L} = \mathfrak{F}_e$ for some $e \in E$. Obviously, a principal level $\mathfrak{F}_e$ in a Riesz space E is a maximal level if and only if e is a weak order unit of E.

Example 1 (Example 2.2 of [21]). Let (Ω, Σ, μ) be a finite atomless measure space, $0 \le p \le \infty$ and $E = L_p(\mu)$. Fix any $z \in L_0(\mu)$ and set $\mathbf{L}_z = \{x \in E : x \sqsubseteq z\}$. Then

1. $\mathbf{L}_z$ is a level in E;
2. $\mathbf{L}_z$ is a maximal level in E if and only if $\operatorname{supp} z = \Omega$;
3. $\mathbf{L}_z$ is a principal level $\mathbf{L}_z = \mathfrak{F}_z$ if and only if $z \in E$.

If $\mathbf{L}'$ and $\mathbf{L}''$ are *orthogonal levels* (that is, $e' \perp e''$ for all $e' \in \mathbf{L}'$ and $e'' \in \mathbf{L}''$) then the *direct sum* defined by setting $\mathbf{L}' \oplus \mathbf{L}'' = \{x + y : x \in \mathbf{L}', y \in \mathbf{L}''\}$ is a level as well. A level $\mathbf{L}$ in E is said to be *positive* (respectively, *negative*) provided $\mathbf{L} \subset E^+$ (respectively, $x \le 0$ for each $x \in \mathbf{L}$). The relation $\mathbf{L} \ge 0$ (respectively, $\mathbf{L} \le 0$) means that the level $\mathbf{L}$ is positive (respectively, negative). Every level $\mathbf{L}$ in a Riesz space E admits a unique decomposition into a direct sum of levels $\mathbf{L} = \mathbf{L}^+ \oplus \mathbf{L}^-$, where $\mathbf{L}^+ \ge 0$ and $\mathbf{L}^- \le 0$. In particular, for any principal level $\mathbf{L} = \mathfrak{F}_e$ one has $\mathfrak{F}_e^+ = \mathfrak{F}_{e^+}$ and $\mathfrak{F}_e^- = \mathfrak{F}_{-e^-}$ [21, Proposition 2.5].

1.2. Orthogonally additive operators. Let E be a Riesz space and F a real vector space. A function $T \colon E \to F$ is called an *orthogonally additive operator* if $T(x + y) = T(x) + T(y)$ holds for all $x, y \in E$ with $x \perp y$. Obviously, $T(0) = 0$ for an orthogonally additive operator T. The set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Let E, F be Riesz spaces. An orthogonally additive operator $T \colon E \to F$ is said to be:

- *positive* if $Tx \ge 0$ holds in F for all $x \in E$;
- *order bounded* if T sends order bounded subsets of E to order bounded subsets of F.

An order bounded orthogonally additive operator $T \colon E \to F$ is called an *abstract Uryson operator.*

Observe that the only linear operator which is positive in the sense of orthogonally additive operators is zero. A positive orthogonally additive operator need not be order bounded. Indeed, every function $T \colon \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an orthogonally additive operator and obviously, not all such functions are order bounded. The set of all abstract Uryson operators from E to F is denoted by $\mathcal{U}(E, F)$.

Consider the following order on $\mathcal{U}(E, F)$: $S \le T$ whenever $T - S \ge 0$. Then $\mathcal{U}(E, F)$ becomes an ordered vector space.

Theorem 1 (Theorem 3.2 of [10]). *Let E and F be Riesz spaces with F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete Riesz space. Moreover, for each $S, T \in \mathcal{U}(E, F)$ and $x \in E$ the following conditions hold:*

1. $(T \vee S)(x) = \sup\{T(y) + S(z) : x = y \sqcup z\}$; 2. $(T \wedge S)(x) = \inf\{T(y) + S(z) : x = y \sqcup z\}$;
3. $T^+(x) = \sup\{Ty : y \sqsubseteq x\}$; 4. $T^-(x) = -\inf\{Ty : y \sqsubseteq x\}$; 5. $|T(x)| \le |T|(x)$.

1.3. Different types of order convergence and order continuity. We use the term *order convergence* of nets in a Riesz space F in the sense of *strong order convergence* (see [7]), because it is equivalent to the weak order convergence if either F is Dedekind complete or the net is order (or laterally) increasing, which are the cases below.

A net $(x_\alpha)_{\alpha \in A}$ in a Riesz space E is

- *order convergent* to a limit $x \in E$ if there is a net $(y_\alpha)_{\alpha \in A}$ in E such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for some $\alpha_0 \in A$ and all $\alpha \ge \alpha_0$; in this case we write $x_\alpha \xrightarrow{o} x$;

- *horizontally convergent* (*laterally* to a limit $x \in E$or *up-laterally* in other terminology) provided $x_\alpha \sqsubseteq x_\beta$ for all $\alpha < \beta$ and $x = \bigsqcup_{\alpha \in A} x_\alpha$ (the latter condition is equivalent to $x_\alpha \xrightarrow{o} x$ due to the lateral increase); in this case we write $x_\alpha \xrightarrow{h} x$.

If E is a Riesz space and $w \in E$ then by E_w we denote the principal ideal of E generated by w.

Let E, F be Riesz spaces and $D \subseteq E$. A function $f\colon D \to F$ is said to be

- *vertically order σ-continuous* on D if D is an ideal of E and for every $w \in D^+$, every $x \in E_w$ and every increasing sequence $(x_n)_{n=1}^\infty$ in E_w such that $0 \le x - x_n \le \frac{1}{n} w$ one has $f(x_n) \xrightarrow{o} f(x)$;
- *horizontally order continuous* (up-laterally-to-order continuous in terminology of [12], and disjointly continuous in terminology of [10]) on D if D is a lateral ideal of E and for every $e \in D$ and every net (e_α) in $\mathfrak{F}_e$ the condition $e_\alpha \xrightarrow{h} e$ implies $f(e_\alpha) \xrightarrow{o} f(e)$;
- *order continuous* on D if D is an ideal of E and f sends order convergent nets in D to order convergent nets in F.

Similarly we define the *horizontal σ-order continuity* and *σ-order continuity* by replacing nets with sequences. Obviously, the order continuity implies the rest of continuities but not converse (see [4] for details).

The case of linear operators is of special interest. Notice that every regular linear operator is vertically order σ-continuous.

Proposition 2 ([21], Proposition 4.6). *Let E, F be Riesz spaces with F Archimedean. Then every regular linear operator $T\colon E \to F$ is vertically order σ-continuous on E.*

The second special property of linear operators says that the horizontal continuity is equivalent to the order continuity.

Proposition 3 (Proposition 3.9 of [10]). *Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space and $S\colon E \to F$ a regular linear operator. Then the following assertions hold:*

1. *if S is horizontally order continuous then S is order continuous;*
2. *if S is horizontally order σ-continuous then S is order σ-continuous.*

The following example (appeared in [9, Example 4.2] in a different context) shows that the vertical order σ-continuity for a positive linear functional (which holds anyway by Proposition 2) does not imply its horizontal order continuity (and hence its order continuity).

Example 2. There exists a positive linear bounded functional $f \in L_\infty^*$ which is not horizontally order continuous.

Proof. Denote by $\mathcal{B}$ the Boolean algebra of the Borel subsets of $[0,1]$ equals up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$. Then the linear bounded functional $f\colon L_\infty \to \mathbb{R}$ defined by

$$f(x) = \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A x\, d\mu, \quad x \in L_\infty$$

is as desired. The fact that f is not horizontally order continuous was proved in [9, Example 4.2]. Repeat this simple argument. Choose a nested sequence (A_n) in $\mathcal{U}$ with $\mu(A_n) \to 0$. Then the sequence of characteristic functions $x_n := \mathbf{1}_{[0,1] \setminus A_n}$ has the lateral supremum $x := \mathbf{1}_{[0,1]}$, however $f(x_n) = 0$ for all $n \in \mathbb{N}$ and $f(x) = 1$. □

1.4. Linear sections of an orthogonally additive operator. Let E be a Riesz space and $\mathbf{L}$ a level in E. Denote by $E_{\mathbf{L}}$ the minimal ideal of E including $\mathbf{L}$. Let F be a linear space and $T\colon E \to F$ an orthogonally additive operator. A linear operator $S\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ is called a *linear section of T by $\mathbf{L}$* if $S|_{\mathbf{L}} = T|_{\mathbf{L}}$ and $S|_{\mathbf{L}^d} = 0$.

The following results guarantee the existence (and in some cases the uniqueness) of linear sections which inherit some properties of a given orthogonally additive operator.

Theorem 2 (Theorem 4.7 of [21]). *Let E, F be Riesz spaces. Assume E has the principal projection property, F is Dedekind complete and $T \in \mathcal{U}(E,F)$. Then for every level $\mathbf{L}$ of E there is a unique regular linear section $S = \Psi_{\mathbf{L}}(T)\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by $\mathbf{L}$. Moreover, if $\mathbf{L} \geq 0$ then $S^+ = (\Psi_{\mathbf{L}}(T))^+ = \Psi_{\mathbf{L}}(T^+)$. In particular, if T is positive as an orthogonally additive operator and $\mathbf{L} \geq 0$ then S is positive as a linear operator.*

Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space, $T \in \mathcal{U}(E,F)$ and $\mathbf{L}$ a level of E. The regular linear section $S = \Psi_{\mathbf{L}}(T)\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by $\mathbf{L}$, the existence and uniqueness of which Theorem 2 asserts, is called the *canonical linear section* of T by $\mathbf{L}$.

Theorem 3 (Theorem 4.12 of [21]). *Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space, $T \in \mathcal{U}(E,F)$ and $\mathbf{L}$ a positive level of E. If T is horizontally order continuous (horizontally order σ-continuous) on $\mathbf{L}$ then the canonical linear section $S = \Psi_{\mathbf{L}}(T)$ of T by $\mathbf{L}$ is order continuous (order σ-continuous) on its domain.*

2. The C-compactness implies the AM-compactness for linear operators. To the best of our knowledge, the notion of C-compactness was introduced in [11]. Obviously, every AM-compact orthogonally additive operator is C-compact. Several interesting results on the C-compactness for orthogonally additive operators were obtained in [11] and [16].

Combines some convexity technical tools with Freudenthal's spectral theorem we obtain the following lemma, which is the main technical tool for two of our main results.

We say that a positive vector e of a Riesz space E is a *strong order unit* of E provided E equals the ideal of E generated by e, that is, for every $x \in E$ there exists $\lambda > 0$ such that $|x| \leq \lambda e$.

The following lemma is the main technical tool for two main results of the section.

Lemma 1. *Let E be a Riesz space with the principal projection property and a strong order unit, and F a Banach lattice with an order σ-continuous norm. A linear regular operator $T\colon E \to F$ is AM-compact if and only if there exists a strong order unit $e \in E^+$ of E such that the set $T(\mathfrak{F}_e)$ is precompact.*

To prove Lemma 1, we need the following known lemmas.

Lemma 2 (Lemma 2.3 of [8]). *Let E be a Riesz space E, $e \in E^+$ and $x \in E$ an e-step function with $|x| \leq e$. Then there exist $n \in \mathbb{N}$, $\lambda_j \in [0,1]$ and $y_j \in E$, $j = 1, \dots, n$ such that $|y_j| = e$, $\sum_{j=1}^n \lambda_j = 1$ and $x = \sum_{j=1}^n \lambda_j y_j$.*

Lemma 3 (Section 11.2.1, Theorem 3 of [6]). *The convex hull of any precompact subset of a Banach space is precompact.*

Proof of Lemma 1. The "only if" part is obvious, so we prove the "if" part. It is enough to prove the implication for positive operators. Let $T \geq 0$, $e \in E^+$ be a weak order unit of E

with $T(\mathfrak{F}_e)$ precompact. First we show that the set $T\big([-e,e]\big)$ is precompact in F. By the precompactness of $T(\mathfrak{F}_e)$, the following set is precompact in F:

$$F_1 := T(\mathfrak{F}_e) - T(\mathfrak{F}_e) = \{Tx - Ty :\, x, y \in \mathfrak{F}_e\}.$$

Set $E_1 := \big\{w \in E :\, |w| = e\big\}$. Being a subset of F_1, the set $T(E_1)$ is precompact as well. Denote by S the set of all e-step functions in E order bounded by e, that is,

$$S := \Big\{\bigsqcup_{k=1}^{m} a_k e_k :\, m \in \mathbb{N},\, a_k \in [-1,1],\, e = \bigsqcup_{k=1}^{m} e_k\Big\}.$$

By Lemma 2, $S \subseteq \operatorname{conv} E_1$. Hence, $T(S) \subseteq T(\operatorname{conv} E_1) = \operatorname{conv} T(E_1)$. By Lemma 3, $\operatorname{conv} T(E_1)$ is precompact and thus, so is $T(S)$. So, to prove that $T\big([-e,e]\big)$ is precompact, it is enough to show that $T\big([-e,e]\big) \subseteq \overline{T(S)}$. Fix any $y \in T\big([-e,e]\big)$ and prove that $y \in \overline{T(S)}$. Say, $y = Tx$, where $x \in E$ with $|x| \le e$. Since $y = T(x^+) - T(x^-)$, it is enough to prove that $T(x^+) \in \overline{T(S)}$ and $T(x^-) \in \overline{T(S)}$. In other words, it is enough to consider the case where $x > 0$. So, let $0 < x \le e$. Using Freudenthal's spectral theorem [2, Theorem 2.8], choose a sequence (u_n) of e-step functions such that $0 \le u_n \uparrow x$ and $x - u_k \le \frac{1}{k} e$ for all $k \in \mathbb{N}$. Observe that $u_n \in S$ for all $n \in \mathbb{N}$. By the vertical σ-continuity of T, the sequence (Tu_n) order tends to Tx, and by the order σ-continuity of norm in F, one has $\|Tu_n - y\| \to 0$. Hence, $y \in \overline{T(S)}$, and the precompactness of $T\big([-e,e]\big)$ is proved.

Given any $w \in E^+$, we choose $\lambda > 0$ so that $w \le \lambda e$. Then $T\big([-w,w]\big) \subseteq \lambda T\big([-e,e]\big)$, and the precompactness of $T\big([-w,w]\big)$ follows from that of $T\big([-e,e]\big)$. By the arbitrariness of $w \in E^+$, T is AM-compact. □

Now we are ready to prove the main results of the section.

Theorem 4. *Let E be a Riesz space with the principal projection property and F a Banach lattice with an order σ-continuous norm. Then every C-compact linear regular operator $T\colon E \to F$ is AM-compact.*

Proof. Fix any $w \in E^+$ and prove that $T\big([-w,w]\big)$ is precompact. Let E_w be the ideal of E generated by w and $T_1 := T|_{E_w}$ be the restriction of T to E_w. By the C-compactness of T, the set $T_1(\mathfrak{F}_w) = T(\mathfrak{F}_w)$ is precompact. Taking into account that w is a strong order unit of E_w, we obtain by Lemma 1 that T_1 is AM-compact, and hence, $T\big([-w,w]\big) = T_1\big([-w,w]\big)$ is precompact. □

Remark 1. Theorem 4 can be obtained from [11, Theorem 3.4] for the case where E is a σ-Dedekind complete Banach lattice. However, our method of proof is completely different.

Theorem 5. *Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space and $T \in \mathcal{U}(E,F)$ a C-compact operator. Then the canonical linear section $S = \Psi_{\mathbf{L}}(T)\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by an arbitrary level $\mathbf{L}$ of E is AM-compact.*

For the proof, we need one more known lemma.

Lemma 4 (Lemma 4.8 of [21]). *Let $\mathbf{L}$ be a level of a Riesz space E. Then the ideal $E_{\mathbf{L}}$ of E generated by A equals $E_{\mathbf{L}} = \bigcup_{e \in \mathbf{L}} E_e$, where E_e is the principal ideal generated by e.*

Proof of Theorem 5. Fix any $w \in E_{\mathbf{L}} \oplus \mathbf{L}^d$, say $w = w_1 \sqcup w_2$, where $w_1 \in E_{\mathbf{L}}$ and $w_2 \in \mathbf{L}^d$. Prove that $S\big([-w, w]\big)$ is precompact. Since $S|_{\mathbf{L}^d} = 0$, we obtain that $S\big([-w, w]\big) = S\big([-w_1, w_1]\big)$. Using Lemma 4, choose $e \in \mathbf{L}$ so that $w_1 \in E_e$. By the C-compactness of T one has $S(\mathfrak{F}_e) = T(\mathfrak{F}_e)$ is precompact. Taking into account that e is a strong order unit of E_e, we obtain by Lemma 1 that $S_1 := S|_{E_e}$ is AM-compact, and hence, $S\big([-w_1, w_1]\big) = S_1\big([-w_1, w_1]\big)$ is precompact. $\square$

3. Narrowness of linear sections. Narrow operators generalize compact operators defined on function spaces and Riesz spaces. Formally narrow linear operators defined on Köthe F-spaces were introduced in [20] and [14], but actually they were studied by different authors before the name of narrow operators appeared (see [22] for detailed information). Then the notion was generalized to linear operators on Riesz spaces in [9], and for orthogonally additive operators on Riesz spaces in [18]. Remark also that narrow operators were defined on a more wide class of lattice normed spaces in [15] for linear operators and in [17] for orthogonally additive operators. Under natural assumptions, AM-compact (even C-compact) operators are narrow (see [9] and [18] for linear operators and [16] for orthogonally additive operators). Exceptional Example 2 gives a nonnarrow AM-compact linear bounded functional on the Banach Köthe space L_∞, the norm of which is not absolutely continuous.

Let E be a Riesz space and F a Banach space (or more general, an F-space). An orthogonally additive operator $T\colon E \to F$ is said to be *narrow at a point* $w \in E$ provided for every $\varepsilon > 0$ there is a decomposition $w = w' \sqcup w''$ such that $\|T(w') - T(w'')\| < \varepsilon$. The operator T is called *strictly narrow at* w if there is a decomposition $w = w' \sqcup w''$ such that $T(w') = T(w'')$. T is called *narrow* (*strictly narrow*) if it is so at every point $w \in E$.

Not less interesting is the following version of narrowness. Let E, F be Riesz spaces. An orthogonally additive operator $T\colon E \to F$ is said to be *order narrow at a point* $w \in E$ provided there is a net of decompositions $w = w'_\alpha \sqcup w''_\alpha$ such that $\big(T(w'_\alpha) - T(w''_\alpha)\big) \xrightarrow{\text{o}} 0$. The operator T is called *order narrow* if it is so at every point $w \in E$.

Observe that, if a linear operator T is narrow (in any sense) at each $x \in E^+$ then T is narrow. However, an orthogonally additive operator, which is narrow at each positive element, need not be narrow, as the following example shows: $T(x) = x^-$ for all $x \in E$.

Theorem 6. *Let E be a Riesz space with the principal projection property, F a Dedekind complete Banach lattice and $T \in \mathcal{U}(E, F)$ a narrow operator. Then the canonical linear section $S = \Psi_{\mathbf{L}}(T)\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by an arbitrary level $\mathbf{L}$ of E is narrow.*

For the proof, we need the following lemma.

Lemma 5. *Let E be a Riesz space with the principal projection property and a strong order unit e, F a Banach lattice. If a regular linear operator $S\colon E \to F$ is narrow at all fragments of e then S is narrow.*

Proof. Let x be any e-step function, say, $x = \bigsqcup_{k=1}^m a_k e_k$, where $a_k \in \mathbb{R} \setminus \{0\}$ and $\bigsqcup_{k=1}^m e_k \sqsubseteq e$. To show that S is narrow at x, given any $\varepsilon > 0$, for every $k \in \{1, \dots, m\}$ we choose a decomposition $e_k = e'_k \sqcup e''_k$ such that $\|S(e'_k - e''_k)\| < \varepsilon / m|a_k|$. Now define a decomposition $x = x' \sqcup x''$ by setting $x' := \bigsqcup_{k=1}^m a_k e'_k$ and $x'' := \bigsqcup_{k=1}^m a_k e''_k$. Then
$$\|S(x' - x'')\| = \Big\| \sum_{k=1}^m a_k S(e'_k - e''_k) \Big\| \le \sum_{k=1}^m |a_k| \|S(e'_k - e''_k)\| < \varepsilon.$$
So S is narrow at all e-step functions.

Now let x be an arbitrary element of E^+ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ to satisfy

$$\big\||S|\,e\big\| < \frac{n\varepsilon}{4}. \tag{1}$$

Using Freudenthal's spectral theorem [2, Theorem 2.8] and the assumption that e is a strong order unit of E, choose an e-step function x_0 so that

$$0 \le x - x_0 \le \frac{1}{n} e. \tag{2}$$

By the above, S is narrow at x_0. Choose a decomposition

$$x_0 = y \sqcup z \tag{3}$$

such that $\|S(y-z)\| < \varepsilon/4$. Let P_u denote the order projection of E onto the band generated by an element $u \in E$ (see [2, Theorem 1.47]). By (1) of [2, Theorem 1.48], (3) implies $P_y x \sqcup P_z x = P_{x_0} x \sqsubseteq x$. Then $x = x' \sqcup x''$, where $x' = P_y x$ and $x'' = P_z x + x - P_{x_0} x$. We show that $\|S(x' - x'')\| < \varepsilon$. By (3), $P_y x_0 = y$ and $P_z x_0 = z$. Hence

$$|x' - y| = |P_y x - P_y x_0| \le x - x_0 \overset{by\,(2)}{\le} \frac{1}{n} e;$$
$$|z - x''| = |P_z x_0 - P_z x - x + P_{x_0} x| \le (x - x_0) + (x - P_{x_0} x) \le$$
$$\le \frac{1}{n} e + x - P_{x_0} x_0 = \frac{1}{n} e + x - x_0 \le \frac{2}{n} e.$$

Thus, we finally obtain

$$\|S(x' - x'')\| \le \big\||S||x' - y|\big\| + \|S(y-z)\| + \big\||S||z - x''|\big\| \le$$
$$\le \Big\||S|\Big(\frac{1}{n} e\Big)\Big\| + \frac{\varepsilon}{4} + \Big\||S|\Big(\frac{2}{n} e\Big)\Big\| \overset{by\,(1)}{<} \varepsilon.$$

□

Proof of Theorem 6. Fix any $w \in E_{\mathbf{L}} \oplus \mathbf{L}^d$, say $w = w_1 \sqcup w_2$, where $w_1 \in E_{\mathbf{L}}$ and $w_2 \in \mathbf{L}^d$. Using Lemma 4, choose $e \in \mathbf{L}$ so that $w_1 \in E_e$. Since $Sx = T(x)$ for all $x \in \mathfrak{F}_e$, by the narrowness of T we deduce that S is narrow at all fragments of e. Since e is a strong order unit of E_e, by Lemma 5 the restriction $S|_{E_{\mathbf{L}}}$ of S to $E_{\mathbf{L}}$ is narrow. Taking into account that $S|_{\mathbf{L}^d} = 0$, one can easily show that S is narrow. □

We do not know if an analogue of Theorem 6 is true for order narrow operators.

Problem 1. *Let E be a Riesz space with the principal projection property, F a Dedekind complete Riesz space and $T \in \mathcal{U}(E, F)$ an order narrow operator. Is the canonical linear section $S = \Psi_{\mathbf{L}}(T)\colon E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by an arbitrary level $\mathbf{L}$ of E order narrow?*

Acknowledgements. The authors are grateful to Marat Pliev for valuable remarks.

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Received 03.06.2022
Revised 27.08.2022