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ON THE TRACE OF PERMUTING TRI-DERIVATIONS ON RINGS

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In the paper we examined the some effects of derivation, trace of permuting tri-derivation and endomorphism on each other in prime and semiprime ring. Let R be a $2,3$-torsion free prime ring and $F\colon R\times R\times R\to R$ be a permuting tri-derivation with trace f, $d\colon R\to R$ be a derivation. In particular, the following assertions have been proved:

- if $[d(r),r]=f(r)$ for all $r\in R$, then R is commutative or $d=0$ (Theorem 1);
- if $g\colon R\to R$ is an endomorphism such that $F(d(r),r,r)=g(r)$ for all $r\in R$, then $F=0$ or $d=0$ (Theorem 2);
- if $F(d(r),r,r)=f(r)$ for all $r\in R$, then (i) $F=0$ or $d=0$, (ii) $d(r)\circ f(r)=0$ for all $r\in R$ (Theorem 3).

On the other hand, if there exist permuting tri-derivations $F_1,F_2\colon R\times R\times R\to R$ such that $F_1(f_2(r),r,r)=f_1(r)$ for all $r\in R$, where f_1 and f_2 are traces of F_1 and F_2, respectively, then (i) $F_1=0$ or $F_2=0$, (ii) $f_1(r)\circ f_2(r)=0$ for all $r\in R$ (Theorem 4).

1. Introduction and preliminaries. Throughout this paper, R will denote an associative ring. Recall that a ring R is prime if for any $r,s\in R$, $rRs=0$ implies $r=0$ or $s=0$ and R is semiprime if $rRr=0$ implies $r=0$. An element $r\in R$ is called n-torsion free, where $n>1$ is an integer, if $nr=0$ implies $r=0$. A ring R is called n-torsion free ring if every element in R is n-torsion free. For any $r,s\in R$, the symbol $[r,s]$ will denote the commutator $rs-sr$ and the symbol $r\circ s$ will stand for the anti-commutator $rs+sr$. We will use the commutator identities: $[rs,t]=[r,s]t+r[s,t]$ and $[r,st]=[r,s]t+s[r,t]$. An additive map $d\colon R\to R$ is called a derivation if $d(rs)=d(r)s+rd(s)$ holds for all $r,s\in R$. The commutativity of prime rings with derivations was introduced by E. C. Posner in [6]. In [5], M. A. Öztürk achieved some results by introducing the idea of permuting tri-derivations in rings. A mapping $F(.,.,.)\colon R\times R\times R\to R$ is called permuting if the equation $F(r_1,r_2,r_3)=F(r_{\pi(1)},r_{\pi(2)},r_{\pi(3)})$ holds for all $r_1,r_2,r_3\in R$ and every permutation $\pi(1),\pi(2),\pi(3)$. A map $f\colon R\to R$ defined by $f(r)=F(r,r,r)$ is called the trace of F, where $F(.,.,.)\colon R\times R\times R\to R$ is a permuting map. Clearly, if $F(.,.,.)\colon R\times R\times R\to R$ is permuting tri-additive (i.e., additive in all three arguments), then the trace of F satisfies the relation $f(r+s)=f(r)+f(s)+3F(r,r,s)+3F(r,s,s)$ for all $r,s\in R$. A permuting tri-additive map $F(.,.,.)\colon R\times R\times R\to R$ is called a permuting tri-derivation if $F(rw,s,t)=F(r,s,t)w+rF(w,s,t)$ for all $r,s,t,w\in R$. The trace of F is an odd function. In [4], [1], [7] and [8], the authors investigated some properties of permuting tri-derivations in prime and semiprime rings.

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In the present paper we study and investigate some results involving the traces of permuting tri-derivations, derivations and endomorphisms in the prime and semiprime rings. Our results extend some results contained in [2].

Lemma 1 ([6])**.** *Let R be a prime ring and d be a derivation of R such that $[d(r), r] = 0$ for all $r \in R$. Then R is commutative or $d = 0$.*

Lemma 2 ([3])**.** *Let R be a 2-torsion free prime ring. If $r, s \in R$ such that $rxs + sxr = 0$ for all $x \in R$, then $r = 0$ or $s = 0$.*

Lemma 3 ([5])**.** *Let R be a $2,3$-torsion free ring, F be a permuting tri-additive mapping of R and f be the trace of F. If $f(r) = 0$ for all $r \in R$, then $F = 0$.*

Lemma 4 ([5])**.** *Let R be a $2,3$-torsion free semiprime ring. Suppose that $F(.,.,.)\colon R\times R\times R \to R$ is a permuting tri-derivation with trace f. If $F(f(r), r, r) = 0$ for all $r \in R$, then $F = 0$.*

2. Results.

Theorem 1. *Let R be a $2,3$-torsion free prime ring. Suppose that $F\colon R \times R \times R \to R$ is a permuting tri-derivation with trace f. Let $d\colon R \to R$ be a derivation. If $[d(r), r] = f(r)$ for all $r \in R$, then R is commutative or $d = 0$.*

Proof. We have

$$[d(r), r] = f(r) \tag{1}$$

for all $r \in R$. Substituting $r + s$ for r in (1) and using (1), we get

$$[d(r), s] + [d(s), r] = 3F(r, r, s) + 3F(r, s, s) \tag{2}$$

for all $r, s \in R$. Substituting $r = -r$ in (2), comparing with (2) and using the fact that R is a $2,3$-torsion free ring, we find that

$$F(r, r, s) = 0 \tag{3}$$

for all $r, s \in R$. Replacing r by $r + w$ in (3), we have

$$0 = F(r, r, s) + 2F(r, w, s) + F(w, w, s) \tag{4}$$

for all $r, s, w \in R$. Combining (3) and (4), we get $2F(r, s, w) = 0$ for all $r, s, w \in R$. Since R is the 2-torsion free ring, we have $F(r, s, w) = 0$ for all $r, s, w \in R$. This implies that $F = 0$ and so $f = 0$. In view of Lemma 1, the theorem is proved. □

Theorem 2. *Let R be a $2,3$-torsion free prime ring. Suppose that $F\colon R \times R \times R \to R$ is a permuting tri-derivation with trace f and $d\colon R \to R$ is a derivation. If $g\colon R \to R$ is an endomorphism such that $F(d(r), r, r) = g(r)$ for all $r \in R$, then $F = 0$ or $d = 0$.*

Proof. Let g be an endomorphism satisfying

$$F(d(r), r, r) = g(r) \tag{5}$$

for all $r \in R$. Linearizing (5) and using (5), we have

$$0 = 2F(d(r), r, s) + F(d(r), s, s) + F(d(s), r, r) + 2F(d(s), r, s) \tag{6}$$

for all $r, s \in R$. Putting $-s$ instead of s in (6) and subtracting the result from equation (6), we find that

$$0 = 2F(d(r), r, s) + F(d(s), r, r) \tag{7}$$

for all $r, s \in R$, since R is 2-torsion free. Replacing s by sr in (7), we get

$$0 = 3sF(d(r), r, r) + d(s)f(r) + F(s, r, r)d(r) \tag{8}$$

for all $r, s \in R$. Replacing s by rs in (8), we obtain

$$0 = 3rsF(d(r), r, r) + d(r)sf(r) + rd(s)f(r) + f(r)sd(r) + rF(s, r, r)d(r) \tag{9}$$

for all $r, s \in R$. In view of (8) and (9), we get $0 = d(r)sf(r) + f(r)sd(r)$ for all $r, s \in R$. Application of Lemma 2 gives that $d(r) = 0$ or $f(r) = 0$ for all $r \in R$. Then by Lemma 3, the theorem is proved. □

Theorem 3. *Let R be a 2,3-torsion free prime ring, $F\colon R \times R \times R \to R$ be a permuting tri-derivation with trace f. If $d\colon R \to R$ is a derivation such that $F(d(r), r, r) = f(r)$ for all $r \in R$, then*
(i) $F = 0$ or $d = 0$,
(ii) $d(r) \circ f(r) = 0$ for all $r \in R$.

Proof. Given that

$$F(d(r), r, r) = f(r) \tag{10}$$

for all $r \in R$. Replacing r by $r + s$ in (10) and using (10) in the last equation, we get

$$2F(d(r), r, s) + F(d(r), s, s) + F(d(s), r, r) + 2F(d(s), r, s) = 3F(r, r, s) + 3F(r, s, s) \tag{11}$$

for all $r, s \in R$. Replacing s by $-s$ in (11) and subtracting the new equation from (11), we have

$$2F(d(r), r, s) + F(d(s), r, r) = 3F(r, r, s) \tag{12}$$

for all $r, s \in R$, since R is 2-torsion free. Taking sr for s in (12), we obtain

$$2F(d(r), r, s)r + 2sF(d(r), r, r) + F(d(s), r, r)r + d(s)f(r)+ \\ +F(s, r, r)d(r) + sF(d(r), r, r) = 3F(r, r, s)r + 3sf(r)$$

for all $r, s \in R$. Using equations (10) and (12), we conclude that

$$0 = d(s)f(r) + F(s, r, r)d(r) \tag{13}$$

for all $r, s \in R$. Replacing s by rs in (13) and using (13), we find that $0 = d(r)sf(r) + +f(r)sd(r)$ for all $r, s \in R$. Lemma 2 implies that $d(r) = 0$ for all $r \in R$ or $f(r) = 0$ for all $r \in R$. From Lemma 3, we have $F = 0$. Replacing s by r in (13), we have $d(r)f(r) + f(r)d(r) = 0$ for all $r \in R$. This means that $d(r) \circ f(r) = 0$ for all $r \in R$. □

Theorem 4. *Let R be a 2,3-torsion free prime ring. Suppose that there exist permuting tri-derivations $F_1, F_2\colon R \times R \times R \to R$ such that $F_1(f_2(r), r, r) = f_1(r)$ for all $r \in R$, where f_1 and f_2 are traces of F_1 and F_2, respectively. Then:*

(i) $F_1 = 0$ or $F_2 = 0$, (ii) $f_1(r) \circ f_2(r) = 0$ for all $r \in R$.

Proof. Assume that F_1 and F_2 are permuting tri-derivations satisfying

$$F_1(f_2(r), r, r) = f_1(r) \tag{14}$$

for all $r \in R$. The linearization of (14) gives

$$\begin{aligned}2F_1(f_2(r), r, s) + F_1(f_2(r), s, s) + F_1(f_2(s), r, r) + 2F_1(f_2(s), r, s)+\\ +3F_1(F_2(r, r, s), r, r) + 6F_1(F_2(r, r, s), r, s) + 3F_1(F_2(r, r, s), s, s) + 3F_1(F_2(r, s, s), r, r)+\\ +6F_1(F_2(r, s, s), r, s) + 3F_1(F_2(r, s, s), s, s) = 3F_1(r, r, s) + 3F_1(r, s, s)\end{aligned} \tag{15}$$

for all $r, s \in R$. Replacing s by $-s$ in (15) and comparing with (15), we obtain

$$\begin{aligned}2F_1(f_2(r), r, s) + F_1(f_2(s), r, r) + 3F_1(F_2(r, r, s), r, r)+\\ +3F_1(F_2(r, r, s), s, s) + 6F_1(F_2(r, s, s), r, s) = 3F_1(r, r, s)\end{aligned} \tag{16}$$

for all $r, s \in R$, since R is 2-torsion free. Replacing r by $r + s$ in (16) and applying (14) and (16), we get

$$\begin{aligned}0 = 12F_1(f_2(s), s, s) + 9F_1(F_2(r, r, s), s, s) + 18F_1(F_2(r, s, s), r, s)+\\ +18F_1(F_2(r, s, s), s, s) + 3F_1(f_2(s), r, r) + 12F_1(f_2(s), r, s)\end{aligned} \tag{17}$$

for all $r, s \in R$. Taking $s = -s$ in (17), comparing with (17) and using the fact that R is $2, 3$-torsion free ring, we get

$$0 = 3F_1(F_2(r, s, s), s, s) + 2F_1(f_2(s), r, s) \tag{18}$$

for all $r, s \in R$. Now substituting rs for r in (18) and using (18), we obtain

$$0 = F_2(r, s, s)f_1(s) + F_1(r, s, s)f_2(s) \tag{19}$$

for all $r, s \in R$, because R is 3-torsion free. Replacing r by sr in (19), we find that $f_1(s)rf_2(s) + f_2(s)rf_1(s) = 0$ for all $r, s \in R$. Lemma 2 implies that $f_1(s) = 0$ or $f_2(s) = 0$ for all $s \in R$. From Lemma 3, we have $F_1 = 0$ or $F_2 = 0$. Replacing s by r in (19), we get $f_2(r)f_1(r) + f_1(r)f_2(r) = 0$ for all $r \in R$. Hence, $f_1(r) \circ f_2(r) = 0$ for all $r \in R$. □

Theorem 5. *Let R be a $2, 3, 5$-torsion free semiprime ring. Suppose that there exist permuting tri-derivations $F_1, F_2\colon R \times R \times R \to R$ such that $F_1(f_1(r), r, r) = f_2(r)$ for all $r \in R$, where f_1 and f_2 are traces of F_1 and F_2, respectively. Then $F_1 = 0$.*

Proof. Assume that

$$F_1(f_1(r), r, r) = f_2(r) \tag{20}$$

for all $r \in R$. Linearization of (20) yields

$$\begin{aligned}F_1(f_1(r), r, r) + 2F_1(f_1(r), r, s) + F_1(f_1(r), s, s) + F_1(f_1(s), r, r) + 2F_1(f_1(s), r, s)+\\ +F_1(f_1(s), s, s) + 3F_1(F_1(r, r, s), r, r) + 6F_1(F_1(r, r, s), r, s) + 3F_1(F_1(r, r, s), s, s)+\\ +3F_1(F_1(r, s, s), r, r) + 6F_1(F_1(r, s, s), r, s) + 3F_1(F_1(r, s, s), s, s) =\\ = f_2(r) + f_2(s) + 3F_2(r, r, s) + 3F_2(r, s, s)\end{aligned} \tag{21}$$

for all $r, s \in R$. Putting $s = -s$ in (21) and comparing with the obtained equation, we get

$$2F_1(f_1(r), r, s) + F_1(f_1(s), r, r) + 3F_1(F_1(r, r, s), r, r)+ \\ +3F_1(F_1(r, r, s), s, s) + 6F_1(F_1(r, s, s), r, s) = 3F_2(r, r, s) \tag{22}$$

for all $r, s \in R$, since R is 2-torsion free. Substituting $r+s$ for r in (22), using (20) and (22), we get $0 = 12F_1(f_1(s), s, s)+9F_1(F_1(r, r, s), s, s)+18F_1(F_1(r, s, s), r, s)+18F_1(F_1(r, s, s), s, s)+ + 3F_1(f_1(s), r, r) + 12F_1(f_1(s), r, s)$ for all $r, s \in R$. Since R is 3-torsion free, we have

$$0 = 4F_1(f_1(s), s, s) + 3F_1(F_1(r, r, s), s, s) + 6F_1(F_1(r, s, s), r, s)+ \\ +6F_1(F_1(r, s, s), s, s) + F_1(f_1(s), r, r) + 4F_1(f_1(s), r, s) \tag{23}$$

for all $r, s \in R$. Replacing r by $r + s$ in (23) and using (23) we get $0 = 20F_1(f_1(s), s, s) + 12F_1(F_1(r, s, s), s, s) + 8F_1(f_1(s), r, s)$ for all $r, s \in R$. The last expression implies that $40F_1(f_1(s), s, s) = 0$ for all $s \in R$. Since R is $2, 5$-torsion free ring, we have $F_1(f_1(s), s, s) = 0$ for all $s \in R$. In view of Lemma 4, the last equation implies that $F_1 = 0$. □

Corollary 1. *Let R be a $2, 3, 5$-torsion free semiprime ring. Let $F\colon R \times R \times R \to R$ be a permuting tri-derivation with trace f. If $F(f(r), r, r) = f(r)$ for all $r \in R$, then $F = 0$.*

Theorem 6. *Let R be a $2, 3, 5$-torsion free semiprime ring. Suppose that $F\colon R\times R\times R \to R$ is a permuting tri-derivation with trace f. If $g\colon R \to R$ is an endomorphism such that $F(f(r), r, r) = g(r)$ for all $r \in R$, then $F = 0$.*

Proof. We have

$$F(f(r), r, r) = g(r) \tag{24}$$

for all $r \in R$. Linearizing of (24) and using (24), we obtain

$$0 = 2F(f(r), r, s) + F(f(r), s, s) + F(f(s), r, r) + 2F(f(s), r, s)+ \\ +3F(F(r, r, s), r, r) + 6F(F(r, r, s), r, s) + 3F(F(r, r, s), s, s)+ \\ +3F(F(r, s, s), r, r) + 6F(F(r, s, s), r, s) + 3F(F(r, s, s), s, s) \tag{25}$$

for all $r, s \in R$. Taking $s = -s$ in (25) and subtracting the result from (25), we obtain

$$0 = 2F(f(r), r, s) + F(f(s), r, r) + 3F(F(r, r, s), r, r)+ \\ +3F(F(r, r, s), s, s) + 6F(F(r, s, s), r, s) \tag{26}$$

for all $r, s \in R$, since R is 2-torsion free. Replacing r by $r + s$ in (26) and using (26), we get

$$0 = 15F(f(s), s, s) + 12F(F(r, r, s), r, s) + 9F(F(r, r, s), s, s) + 24F(F(r, s, s), s, s)+ \\ +18F(F(r, s, s), r, s) + 16F(f(s), r, s) + 2F(f(r), s, s) + 6F(F(r, s, s), r, r) + 3F(f(s), r, r) \tag{27}$$

for all $r, s \in R$. Putting $-s$ instead of s in (27) and comparing with (27), since f is odd and R is $2, 3$-torsion free, we get

$$0 = 5F(f(s), s, s) + 3F(F(r, r, s), s, s) + 6F(F(r, s, s), r, s) + F(f(s), r, r) \tag{28}$$

for all $r, s \in R$. Putting $r = r + s$ in (28) and using (28), we get

$$0 = 12F(F(r, s, s), s, s) + 8F(f(s), r, s) + 10F(f(s), s, s) \tag{29}$$

for all $r, s \in R$. Replacing r by s in (29), it is obvious to verify that $F(f(s), s, s) = 0$, because R is $2, 3, 5$-torsion free. It follows from Lemma 4 that we have $F = 0$. This completes the proof. □

Theorem 7. *Let R be a $2,3$-torsion free prime ring. Suppose that $F\colon R\times R\times R\to R$ is a permuting tri-derivation with trace f. Let $d\colon R\to R$ be a derivation. If $d([f(r),r])=0$ for all $r\in R$, then $[f(r),r]d(r)=0$ for all $r\in R$.*

Proof. Assume that

$$d([f(r),r])=0 \tag{30}$$

for all $r\in R$. In the given condition, replacement of r by $r+s$ leads to

$$\begin{gathered}0=d([f(r),s])+d([f(s),r])+3d([F(r,r,s),r])+\\+3d([F(r,r,s),s])+3d([F(r,s,s),r])+3d([F(r,s,s),s])\end{gathered} \tag{31}$$

for all $r,s\in R$. Putting $-s$ instead of s in (31) and subtracting the new equation from (31), we get

$$d([f(r),s])+d([f(s),r])+3d([F(r,r,s),r])+3d([F(r,s,s),s])=0 \tag{32}$$

for all $r,s\in R$, since f is odd and R is 2-torsion free. Replacing r by $r+s$ in (32), using (30) and (32), we get

$$6d([F(r,r,s),s])+9d([F(r,s,s),s])+6d([F(r,s,s),r])+3d([f(s),r])=0 \tag{33}$$

for all $r,s\in R$. Taking $s=-s$ in (33), subtracting the resulting equation from (33), we get $0=18d([F(r,s,s),s])+6d([f(s),r])$ for all $r,s\in R$. Since R is $2,3$-torsion free ring, we conclude that $0=3d([F(r,s,s),s])+d([f(s),r])$ for all $r,s\in R$. Replacing r by sr in this equality and using it, we get $0=4[f(s),s]d(r)+3d(f(s))[r,s]+3f(s)d([r,s])=0$ for all $r,s\in R$. Again we replace s with r in the last equality. Since R is a 2-torsion free prime ring, we get $[f(r),r]d(r)=0$ for all $r\in R$. □

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