UDC 517.9

Yu. M. Sybil

ON THE BOUNDARY INTEGRAL EQUATION METHOD OF SOLVING BOUNDARY VALUE PROBLEMS FOR THE TWO DIMENSIONAL LAPLACE EQUATION

Yu. M. Sybil. *On the boundary integral equation method of solving boundary value problems for the two dimensional Laplace equation*, Mat. Stud. **62** (2024), 60–76.

We consider approach based on the integral representation of solutions in domain which consists of bounded and unbounded parts that gives us opportunity to reduce different transmission type problems to connected with them equivalent boundary equations of the first and the second kind. We suppose also that solutions of some of these boundary problems are unbounded at infinity. Interior and exterior Dirichlet and Neumann boundary value problems for Laplace equation are restrictions of the solutions os more general this transmission problems. Interior Neumann and exterior Dirichlet boundary value problems we also can solve using integral equation of the second kind that have not unique solution. Corresponding modified equations are constructed in this case and solutions of obtained equations are unique. We also show correctness of all obtained boundary equations of the second type given on closed Lipschitz curve in some Hilbert spaces without compactness of corresponding integral operators.

1. Introduction. The main problem that differs two-dimensional boundary value problems for Laplace equation in unbounded domains from the three-dimensional case is the fact that we have to take to attention solutions which don't tend to zero but even are unbounded at infinity. Different types of the boundary value problems for the two dimensional Laplace equation in smooth domains were posed, investigated and solved by many authors and now are well known classic [5–7,10,11,13,14]. Most of them used the theory of Fredholm operators based on the compactness of corresponding integral operators.

We made attempt to consider boundary integral equation method for solving two dimensional interior, exterior and transmission type Dirichlet and Neumann boundary value problems for Laplace equation in Lipschitz domains based on common approach of integral representation of solutions with help of Green formulas. We also show correctness of obtained boundary integral equations of the second type without using of compactness corresponding integral operators.

As initial fact we use continuity and surjectivity of trace operators in certain Hilbert spaces connected with Lipschitz boundary [1,3,4], continuity of potentials of the simple and double layers and their boundary values in appropriate spaces [3,9] and positive definiteness of boundary value of the potential on simple layer and boundary value of normal derivative of the potential of double layer [8].

2020 *Mathematics Subject Classification:* 31A10, 35J05, 47G20.
Keywords: Laplace equation; Dirichlet and Neumann boundary value problems; potentials of simple and double layer; integral equations of the first and second kind.
doi:10.30970/ms.62.1.60-76

We also essentially use equivalence of the considered boundary value problems and obtained boundary equations that based on the integral representation of solutions.

As it was mentioned above some considered boundary equations have not unique solution. For instance, boundary integral equations for interior Neumann and exterior Dirichlet boundary value problems. We can reduce these equations to some modified ones that have unique solution [7, 10, 12].

2. Integral representation of harmonic function in two dimensional domain with closed Lipschitz boundary. Let $\Omega_+ \subset \mathbb{R}^2$ be a bounded Lipschitz domain. This means that its boundary Σ locally is the graph of Lipschitz function [1, 3]. Let us note that curve Σ can be piecewise smooth and have corners. If $x, y \in \mathbb{R}^2$ then $x = (x_1, x_2)$, $y = (y_1, y_2)$, $|x-y|^2 = \sum_{i=1}^{2}(x_i - y_i)^2$. Suppose that $0 \in \Omega_+$.

Let Σ_R and K_R are circle and disk

$$\Sigma_R = \{x \in \mathbb{R}^2\colon |x| = R\}, \quad K_R = \{x \in \mathbb{R}^2\colon |x| < R\}, \quad R > 0.$$

We denote by $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$ the closure of Ω_+ and $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega}_+$, $\Omega' = \Omega_+ \cup \Omega_-$. Since Σ is Lipschitz almost everywhere we can define outward pointing into Ω_- unit vector of the normal $\vec{n}_x$, $x \in \Sigma$.

As usually we consider in $\Omega_\pm$ Sobolev spaces $H^1(\Omega_\pm)$ of real functions with inner product and norm

$$(u,v)_{H^1(\Omega_\pm)} = \int_{\Omega_\pm} \{(\nabla u(x), \nabla v(x)) + u(x)v(x)\}\, dx, \quad \|u\|^2_{H^1(\Omega_\pm)} = \int_{\Omega_\pm} \{|\nabla u(x)|^2 + u^2(x)\}\, dx,$$

where

$$\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}\right), \quad (\nabla u(x), \nabla v(x)) = \sum_{i=1}^{2} \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}.$$

We introduce the following functional spaces

$$H^1_{\mathrm{loc}}(\overline{\Omega_-}) = \{u(x), x \in \Omega_- |\ \varphi u \in H^1(\Omega_-), \varphi \in C_0^\infty(\mathbb{R}^2)\},$$
$$H^1_{\mathrm{comp}}(\Omega_-) = \{u \in H^1(\Omega_-) | \varphi u = u \text{ for some } \varphi \in C_0^\infty(\mathbb{R}^2)\},$$
$$H^1(\Omega') = \{u(x), x \in \Omega' |\ r_{\Omega_+} u \in H^1(\Omega_+), r_{\Omega_-} u \in H^1_{\mathrm{loc}}(\overline{\Omega_-})\},$$

where $C_0^\infty(\mathbb{R}^2)$ is a linear spaces of infinitely differentiable functions with compact support in $\mathbb{R}^2$ and $r_{\Omega_\pm} u$ is a restriction of function u on $\Omega_\pm$.

Also we consider space $H^1(\Omega_+, L)$ with norm and inner product

$$(u,v)_{H^1(\Omega_+,L)} = (u,v)_{H^1(\Omega_+)} + (Lu, Lv)_{L_2(\Omega_+)}, \quad \|u\|^2_{H^1(\Omega_+,L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

where $L = -\Delta$ is Laplace operator.

In domain Ω_- we consider space $H^1(\Omega_-, L) = \{u \in H^1_{\mathrm{loc}}(\overline{\Omega}_-) |\ Lu \in L_2(\Omega_-)\}$ and in domain Ω' space $H^1(\Omega', L) = \{u(x), x \in \Omega' |\ r_{\Omega_\pm} u \in H^1(\Omega_\pm, L)\}$.

We use the trace space $H^{1/2}(\Sigma)$ and dual space $H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'$. We have dense inclusion $H^{1/2}(\Sigma) \subset L_2(\Sigma) \subset H^{-1/2}(\Sigma)$ if we consider $L_2(\Sigma)$ as a pivot space [5, 8]. We denote as $\langle \cdot, \cdot \rangle$ the relation of duality between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$.

The impotent role in future will play the next proposition [3, 4].

Proposition 1. *Trace map $\gamma_0^+\colon H^1(\Omega_+) \to H^{1/2}(\Sigma)$ is continuous, surjective and has a continuous right inverse $(\gamma_0^+)^{-1}\colon H^{1/2}(\Sigma) \to H^1(\Omega_+)$.*

Trace map $\gamma_0^-\colon H^1_{loc}(\overline{\Omega}_-) \to H^{1/2}(\Sigma)$ is continuous, surjective and has a continuous right inverse $(\gamma_0^-)^{-1}\colon H^{1/2}(\Sigma) \to H^1_{loc}(\overline{\Omega}_-)$.

Let us note that moreover $(\gamma_0^-)^{-1}\colon H^{1/2}(\Sigma) \to H^1_{\mathrm{comp}}(\Omega_-)$.

Using extension by continuity of the operator of normal derivative $\frac{\partial}{\partial n_x}$ from space $C^1(\overline{\Omega}_+)$ to space $H^1(\Omega_+, L)$ we get continuous trace map $\gamma_1^+\colon H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ [3]

$$\langle \gamma_1^+ u, w\rangle = -\int_{\Omega_+} v(x)Lu(x)dx + \int_{\Omega_+} (\nabla u(x), \nabla v(x))dx,$$

where $u \in H^1(\Omega_+, L)$, $w \in H^{1/2}(\Sigma)$ and $v = (\gamma_0^+)^{-1}w$.

Analogously we have continuous trace map $\gamma_1^-\colon H^1(\Omega_-, L) \to H^{-1/2}(\Sigma)$

$$\langle \gamma_1^- u, w\rangle = \int_{\Omega_-} v(x)Lu(x)dx - \int_{\Omega_-} (\nabla u(x), \nabla v(x))dx,$$

where $u \in H^1(\Omega_-, L)$, $w \in H^{1/2}(\Sigma)$ and $v = (\gamma_0^-)^{-1}w$.

Besides the operators $\gamma_0^\pm$ i $\gamma_1^\pm$ we consider operators $[\gamma_0] = \gamma_0^+ - \gamma_0^-\colon H^1(\Omega') \to H^{1/2}(\Sigma)$ and $[\gamma_1] = \gamma_1^+ - \gamma_1^-\colon H^1(\Omega', L) \to H^{-1/2}(\Sigma)$.

For $u \in H^1(\Omega_+, L)$ and $v \in H^1(\Omega_+)$ we have the first Green formula [3]

$$\int_{\Omega_+} (\nabla u(x), \nabla v(x))dx = \int_{\Omega_+} v(x)Lu(x)dx + \langle \gamma_1^+ u, \gamma_0^+ v\rangle, \tag{1}$$

and for $u, v \in H^1(\Omega_+, L)$ – the second Green formula

$$\int_{\Omega_+} (v(x)Lu(x) - u(x)Lv(x))\, dx = \langle \gamma_1^+ v, \gamma_0^+ u\rangle - \langle \gamma_1^+ u, \gamma_0^+ v\rangle. \tag{2}$$

In this paper we consider boundary value problems for functions $u(x) \in H^1(\Omega_+)$ and $u(x) \in H^1_{\mathrm{loc}}(\overline{\Omega}_-)$ which satisfy in distributional sense Laplace equation

$$Lu(x) = -\Delta u(x) = 0. \tag{3}$$

These functions belong to $C^\infty(\Omega_\pm)$ after changing their values on set of zero measure [13].

Since the domain Ω_- is unbound we also consider the functions $u \in H^1_{\mathrm{loc}}(\overline{\Omega}_-)$ which satisfy Laplace equation (3) and the next condition at infinity

$$\lim_{|x|\to\infty} \left(u(x) - \frac{\alpha}{2\pi} \ln \frac{1}{|x|} \right) = c_*, \tag{4}$$

where $c_* \in \mathbb{R}$ depends on u and $\alpha \in \mathbb{R}$ is a some constant which does not depend on u.

This condition means that $u(x) = w(x) + \frac{\alpha}{2\pi} \ln \frac{1}{|x|}$, where function $w(x)$ satisfies Laplace equation (3) and the following condition at infinity

$$\lim_{|x|\to\infty} w(x) = c_*, \quad |\nabla w(x)| = O\Big(\frac{1}{|x|^2}\Big), \quad |x| \to \infty. \tag{5}$$

Let us denote fundamental solution of Laplace operator L as $Q(x, y)$

$$Q(x, y) = \frac{1}{2\pi} \ln \frac{d}{|x - y|}, \quad d > 0, \quad x \neq y, \quad L_x Q(x, y) = \delta(|x - y|).$$

In the following for the sake of brevity we use notation $\langle [\gamma_1]u, Q(x, \cdot)\rangle$ for $\langle [\gamma_1]u(\cdot), Q(x, \cdot)\rangle$ and so on.

Theorem 1. *For function $u \in H^1(\Omega', L)$ which satisfies Laplace equation (3) in Ω' and condition at infinity (4) it holds the next integral representation*

$$u(x) = \langle [\gamma_1]u, Q(x,\cdot)\rangle - \langle \gamma_1^+ Q(x,\cdot), [\gamma_0]u\rangle + c_\infty, \quad x \in \Omega', \tag{6}$$

where $[\gamma_i]u = \gamma_i^+ u - \gamma_i^- u$, $i = 0, 1$, $c_\infty = c_\star - \frac{\alpha}{2\pi}\ln d$.

Proof. Using second Green formula (2) for $v(y) = Q(x,y)$, $x \in \Omega_+$, we can get the following integral representation of function u in Ω_+ [3]

$$u(x) = \langle \gamma_1^+ u, Q(x,\cdot)\rangle - \langle \gamma_1^+ Q(x,\cdot), \gamma_0^+ u\rangle. \tag{7}$$

If $x \in \Omega_-$ then from (2) we obtain

$$0 = \langle \gamma_1^+ u, Q(x,\cdot)\rangle - \langle \gamma_1^+ Q(x,\cdot), \gamma_0^+ u\rangle. \tag{8}$$

Let us denote $B = K_R \setminus \overline{\Omega}_+$. For the function $u(x)$ in B it holds next integral representation

$$u(x) = -\langle \gamma_1^- u, Q(x,\cdot)\rangle + \langle \gamma_1^- Q(x,\cdot), \gamma_0^- u\rangle + \int_{\Sigma_R} \left\{ \frac{\partial u(y)}{\partial n_y} Q(x,y) - \frac{\partial Q(x,y)}{\partial n_y} u(y) \right\} ds_y.$$

If function u satisfies condition at infinity (4) then we may present $u(x)$ in Ω_- as $u(x) = w_0(x) + w(x) + c_\infty$, where $w_0(x) = \frac{\alpha}{2\pi}\ln\frac{d}{|x|}$, $c_\infty = c_\star - \frac{\alpha}{2\pi}\ln d$, $w(x)$ satisfies Laplace equation (3) in Ω_- and conditions at infinity (5) with $c_\star = 0$.

Let us denote $w_1(y) = \frac{1}{2\pi}\ln\frac{|y|}{|x-y|}$ for the fixed $x \in B$. Then $Q(x,y) = w_1(y) + \frac{1}{2\pi}\ln\frac{d}{|y|}$ and we have

$$\frac{\partial w_0(y)}{\partial n_y} Q(x,y) - \frac{\partial Q(x,y)}{\partial n_y} w_0(y) = \frac{\partial w_0(y)}{\partial n_y} w_1(y) - \frac{\partial w_1(y)}{\partial n_y} w_0(y).$$

It is easy to verify that function $w_1(y)$ satisfies Laplace equation (3) in domain $\Omega_- \setminus \overline{K}_R$ and conditions at infinity (5) with $c_\infty = 0$. Then we have

$$\left| \int_{\Sigma_R} \left(\frac{\partial w_0(y)}{\partial n_y} w_1(y) - \frac{\partial w_1(y)}{\partial n_y} w_0(y) \right) ds_y \right| \le \int_{\Sigma_R} \left(\frac{\alpha}{2\pi}\frac{1}{|y|}|w_1(y)| + c_1 \frac{1}{|y|^2}\frac{\alpha}{2\pi}\ln\frac{d}{|y|} \right) ds_y =$$
$$= \frac{\alpha}{2\pi R}\int_{\Sigma_R} |w_1(y)| ds_y + c_1 \frac{1}{R^2}\frac{\alpha}{2\pi}\ln\frac{d}{R} 2\pi R \to 0, \qquad R \to \infty,$$

where $c_1 > 0$ is some constant.

Since the function $w(x)$ satisfies conditions at infinity (5) with $c_\star = 0$ we have [13]

$$\lim_{R\to\infty} \int_{\Sigma_R} \left\{ \frac{\partial w(y)}{\partial n_y} Q(x,y) - \frac{\partial Q(x,y)}{\partial n_y} w(y) \right\} ds_y = 0.$$

If $x \in B$ then [13]

$$-\int_{\Sigma_R} \frac{\partial Q(x,y)}{\partial n_y} ds_y = 1.$$

Thus if function u satisfies condition at infinity (4) we obtain

$$\lim_{R\to\infty} \int_{\Sigma_R} \left\{ \frac{\partial u(y)}{\partial n_y} Q(x,y) - \frac{\partial Q(x,y)}{\partial n_y} u(y) \right\} ds_y = c_\infty.$$

As a consequence for $x \in \Omega_-$ we can get

$$u(x) = -\langle \gamma_1^- u, Q(x,\cdot)\rangle + \langle \gamma_1^- Q(x,\cdot), \gamma_0^- u\rangle + c_\infty. \tag{9}$$

Let $x \in \Omega_+$. In domain B using (2) we have

$$0 = -\langle \gamma_1^- u, Q(x,\cdot)\rangle + \langle \gamma_1^- Q(x,\cdot), \gamma_0^- u\rangle + \int_{\Sigma_R} \left\{ \frac{\partial u(y)}{\partial n_y} Q(x,y) - \frac{\partial Q(x,y)}{\partial n_y} u(y) \right\} ds_y.$$

When $R \to \infty$ we obtain

$$0 = -\langle \gamma_1^- u, Q(x,\cdot)\rangle + \langle \gamma_1^- Q(x,\cdot), \gamma_0^- u\rangle + c_\infty. \tag{10}$$

By using formulas (7), (8), (9) and (10) for function $u \in H^1(\Omega')$ which satisfies equation (3) and condition at infinity (4) we obtain representation $u(x) = \langle [\gamma_1]u, Q(x,\cdot)\rangle - \langle \gamma_1^+ Q(x,\cdot), [\gamma_0]u\rangle + c_\infty,\ x \in \Omega'$. □

We denote

$$V\tau(x) = \int_\Sigma Q(x,y)\tau(y) ds_y, \quad W\mu(x) = \int_\Sigma \frac{\partial Q(x,y)}{\partial n_y} \mu(y) ds_y,$$

where $\tau \in L_1(\Sigma)$, $\mu \in H^{1/2}(\Sigma)$.

Let us note that $V\tau(x)$ satisfies condition at infinity (4) with $\alpha = \langle \tau, 1\rangle$ and $c_* = 0$.

For potentials of simple $V\tau$ and double layer $W\mu$ it holds the jump relations which can be written in the next form [3].

Proposition 2. *Let $\tau \in H^{-1/2}(\Sigma)$ and $\mu \in H^{1/2}(\Sigma)$. Then:*
1. $[\gamma_0]V\tau = 0, \quad [\gamma_1]V\tau = \tau.$
2. $[\gamma_0]W\mu = -\mu, \quad [\gamma_1]W\mu = 0.$

If we introduce the operators $N\tau = \frac{1}{2}(\gamma_1^+ V\tau + \gamma_1^- V\tau)$, $M\mu = \frac{1}{2}(\gamma_0^+ W\mu + \gamma_0^- W\mu)$, we can rewrite jump relations as

$$\gamma_1^\pm V\tau = \pm\frac{1}{2}\tau + N\tau, \quad \gamma_0^\pm W\mu = \mp\frac{1}{2}\mu + M\mu, \tag{11}$$

If $\tau \in L_2(\Sigma)$ and $\mu \in H^{1/2}(\Sigma)$ then for $x \in \Sigma$

$$N\tau(x) = \int_\Sigma \frac{\partial Q(x,y)}{\partial n_x} \tau(y) ds_y, \quad M\mu(x) = \int_\Sigma \frac{\partial Q(x,y)}{\partial n_y} \mu(y) ds_y.$$

Let us denote: $K_d = \gamma_0^\pm V$, $H = -\gamma_1^\pm W$, $B^\pm = \gamma_1^\pm V$, $C^\pm = \mp\gamma_0^\pm W$.

From [3] we can get the following assertion.

Proposition 3. *The operators*

$$V\colon H^{-1/2}(\Sigma) \to H^1_{loc}(\mathbb{R}^2), \quad W\colon H^{1/2}(\Sigma) \to H^1(\Omega_+), \quad K_d\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma),$$
$$H\colon H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma), \quad B^\pm\colon H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma), \quad C^\pm\colon H^{1/2}(\Sigma) \to H^{1/2}(\Sigma),$$

are continuous.

3. Dirichlet boundary value problems. We consider the interior D_+ and the exterior D_- Dirichlet boundary value problems in Ω_+ and Ω_- respectively from the point of view if we look for solutions of these problems as potential of the simple layer $V\tau(x)$, $x \in \Omega_+$ or $x \in \Omega_-$.

Problem D_+: find function $u \in H^1(\Omega_+)$ which satisfies Laplace equation (3) in Ω_+ and Dirichlet boundary condition $\gamma_0^+ u = g_+ \in H^{1/2}(\Sigma)$.
Problem D_-: find function $u \in H^1_{\text{loc}}(\overline{\Omega_-})$ which satisfies Laplace equation (3) in Ω_-, Dirichlet boundary condition $\gamma_0^- u = g_- \in H^{1/2}(\Sigma)$ and condition at infinity (4) where $\alpha \in \mathbb{R}$ is given.

It's well known that problem D_+ has unique solution for arbitrary $g_+ \in H^{1/2}(\Sigma)$ ([7,8]).

Concerning of uniqueness of solution of the problem D_- we have the following assertion.

Lemma 1. *If in the conditions at infinity (4) $\alpha = 0$ then problem D_- with boundary conditions $\gamma_0^- u = 0$ has only trivial solution.*

Proof. From the first Green formula (1) in the domain B introduced in theorem 1 we obtain
$$\int_B |\nabla u(x)|^2 dx = \int_{\Sigma_R} \frac{\partial u(y)}{\partial n_y} u(y) ds_y \leq \left\{2\pi R \cdot \frac{c_1}{R^2} \cdot c_*\right\} \to 0 \quad \text{as} \quad R \to \infty,$$
where $c_1 > 0$ is some constant. Therefor $\int_{\Omega_-} |\nabla u(x)|^2 dx = 0$ and $u(x) = c_*$, $x \in \Omega_-$. So far as $\gamma_0^- u = 0$ then $c_* = 0$. We get $u(x) = 0$ in Ω_-. □

Now we consider the homogenous equation
$$K_d \tau(x) = \int_\Sigma \ln \frac{d}{|x-y|} \tau(y) ds_y = 0. \tag{12}$$
and show that for some $d = d_0$ it has not trivial solution.

Lemma 2. *For a given curve Σ there exists unique constant $d = d_0$ and unique up to multiplication by a constant solution τ_0 of equation (12) with condition $\int_\Sigma \tau_0(y) ds_y \neq 0$.*

Proof. From [8] it follows that for some $d > 0$ equation $K_d \tau_1 = 1$ has unique solution. Let us consider $\alpha_1 = \int_\Sigma \tau_1(y) ds_y$. For function $u = V\tau_1$ in Ω' we have $\tau_1 = [\gamma_1]u$ and $\gamma_0^+ u = \gamma_0^- u = 1$. Since $\gamma_0^+ u = 1$ we obtain $u(x) \equiv 1$, $x \in \Omega_+$. Let function $v_1(x) \equiv 1$, $x \in \mathbb{R}^2$. For the function $v(x) = u(x) - v_1(x)$, $x \in \Omega'$, we get $\tau_1 = [\gamma_1]v$ and $\gamma_0^\pm v = 0$. If $\alpha_1 = 0$ from Lemma 1 it implies that $v = 0$ or $u(x) \equiv 1$, $x \in \mathbb{R}^2$. As a result we have $\tau_1 = 0$ that is impossible. Thus $\alpha_1 \neq 0$.

Then
$$\frac{1}{2\pi} \int_\Sigma \ln \frac{d_0}{|x-y|} \tau_1(y) ds_y = 1 + \frac{\alpha_1}{2\pi} \ln \frac{d_0}{d} = 0$$
and we get $d_0 = de^{-\frac{2\pi}{\alpha_1}}$, where $\alpha_1 = \int_\Sigma \tau_1(y) ds_y$.

We show that d_0 does not depend on the choice of d. Let we have $d_1 > 0$, $d_2 > 0$ and $K_{d_1}\tau_{11} = 1$, $K_{d_2}\tau_{12} = 1$, $\alpha_{11} = \int_\Sigma \tau_{11}(y) ds_y$, $\alpha_{12} = \int_\Sigma \tau_{12}(y) ds_y$.

Then $K_{d_2}\tau_{11} = 1 + \frac{\alpha_{11}}{2\pi} \ln \frac{d_2}{d_1} = \beta$ and $\tau_{12} = \tau_{11}/\beta$, $\alpha_{12} = \alpha_{11}/\beta$.

We obtain $d_{01} = d_1 e^{-\frac{2\pi}{\alpha_{11}}}$, $d_{02} = d_2 e^{-\frac{2\pi}{\alpha_{12}}}$ and $\ln \frac{d_{01}}{d_{02}} = \ln \frac{d_1}{d_2} + 2\pi(\frac{1}{\alpha_{12}} - \frac{1}{\alpha_{11}}) = 0$.

Thus $d_{01} = d_{02}$.

Let us suppose that for $d = d_0$ there exist two linear independent solutions τ_1 and τ_2 of equation (12), i.e. $\tau_1 \neq c\tau_2$ where c is constant. Then $\tau^* = \tau_1 + c\tau_2$ is a solution of (12) for arbitrary constant c. If $\alpha_i = \int_\Sigma \tau_i(y) ds_y$ then for $c = -\alpha_1/\alpha_2$ we obtain condition $\int_\Sigma \tau^*(y) ds_y = 0$. Let $v(x) = V\tau^*(x)$, $x \in \Omega_-$. Since function $v(x)$ is a solution of the problem D_- with boundary condition $\gamma_0^- v(x) = 0$ and conditions at infinity (5) with $c_\star = 0$ we have $v(x) = 0$, $x \in \Omega_-$, or $\tau^* = 0$. Thus $\tau_1 = c\tau_2$. □

Corollary 1. *There exists unique constant d_0 that operator $K_{d_0}\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is not injective, that is, there exists $\tau \neq 0$ such that $K_{d_0}\tau = 0$.*

If τ_1 is a solution of equation $K_d\tau_1 = 1$, where $d \neq d_0$, then $\tau_0 = c\tau_1$ is solution of the equation (12) and c is an arbitrary constant.

As a example for finding of d_0 let us consider the case when Σ is a circle of radius $r = a$. Then equation (12) in polar system can be rewritten as

$$\int_0^{2\pi} \ln \frac{d_0}{a[2 - 2\cos(\varphi - \varphi_0]^{1/2}} \tau(\varphi) d\varphi = 0, \quad 0 \leq \varphi_0 \leq 2\pi.$$

Since geometry of region and boundary function are symmetric it is obvious that solution $\tau(\varphi)$ does not depend on the angle φ. Thus

$$\int_0^{2\pi} \ln \frac{d_0^2}{2a^2(1 - \cos\varphi)} d\varphi = 0 \qquad \text{and} \qquad \int_0^{2\pi} \ln(1 - \cos\varphi) d\varphi = 2\pi \ln \frac{d_0^2}{2a^2}.$$

By using equality $\int_0^{\pi} \ln(1 \pm \cos\varphi) d\varphi = -\pi \ln 2$ we obtain $\ln \frac{d_0}{a} = 0$ or $d_0 = a$.

The next question is how to find constant d_0 for arbitrary contour Σ. Let $d \neq d_0$, $K_d\tau_1 = 1$, $\alpha_1 = \int_\Sigma \tau_1(y) ds_y$. Then

$$\frac{1}{2\pi} \int_\Sigma \ln \frac{d_0}{|x - y|} \tau_1(y) ds_y = 1 + \frac{\alpha_1}{2\pi} \ln \frac{d_0}{d} = 0$$

and we get $d_0 = de^{-\frac{2\pi}{\alpha_1}}$.

Let us note that $d_0 = de^{-\frac{2\pi}{\alpha_1}}$ for a closed curve Σ is called logarithmic capacity and denoted as cap_Σ. Here $K_d\tau_1 = 1$, $d \neq d_0$ and $\alpha_1 = \int_\Sigma \tau_1(y) ds_y$. Thus if Σ is a circle of radius $r = a$ then $\mathrm{cap}_\Sigma = a$ that is well known result [8] .

Let us consider the function

$$u_0(x) = \frac{1}{2\pi} \int_\Sigma \ln \frac{d_0}{|x - y|} \tau_1(y) ds_y, \quad x \in \Omega', \tag{13}$$

where τ_1 is a solution of integral equation $K_d\tau_1 = 1$, $d \neq d_0$. As a consequence we have that $u_0(x) = 0$, $x \in \Omega_+$, and $u_0(x) \neq 0$, $x \in \Omega_-$, $\gamma_0^- u = 0$.

Now we look at the question of positive definiteness of operator K_d which is important for numerical solution of equation $K_d\tau = g$, $g \in H^{1/2}(\Sigma)$. The operator K_d is positive definite if there exists a constant $c > 0$ that for all $\tau \in H^{-1/2}(\Sigma)$ we have

$$\langle \tau, K_d\tau \rangle \geq c\|\tau\|^2_{H^{-1/2}(\Sigma)}. \tag{14}$$

It's easy to see that inequality (14) may have place not for all $\tau \in H^{-1/2}(\Sigma)$. For instance let Σ be a circle of radius $a \neq d$. Then equation $K_d\tau_1 = 1$ has solution $\tau_1 = \left(\ln \frac{d}{a}\right)^{-1}$ and in this case inequality $\langle \tau, K_d\tau \rangle > 0$ fulfils only when $d > a = d_0$.

Let d_0 be constant considered in Lemma 2. Then we have the following assertion [8].

Proposition 4. *1. Operator $K_d\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is positive definite if and only if $d > d_0$.*

2. Operator $K_d\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is isomorphism (continuous bijection) if and only if $d \neq d_0$.

Corollary 2. *Equation $K_d\tau = g$ with condition $d \neq d_0$ has unique solution $\tau \in H^{-1/2}(\Sigma)$ for arbitrary $g \in H^{1/2}(\Sigma)$ and there exists bounded operator $K_d^{-1}\colon H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$.*

As we noted above problem D_+ has unique solution for arbitrary $g_+ \in H^{1/2}(\Sigma)$. Let us consider function $u(x) = V\tau(x)$, $x \in \Omega_+$, $d \neq d_0$, where τ is a unique solution of equation $K_d\tau = g_+$. Then $u(x)$ satisfies equation (3) in Ω_+ and boundary condition $\gamma_0^+ u(x) = g_+(x)$. As a consequence we can get the following corollary.

Corollary 3. *We can present solution of the problem D_+ in the form $u(x) = V\tau(x)$, $d \neq d_0$, where $\tau \in H^{-1/2}(\Sigma)$ is unique solution of equation $K_d\tau = g_+$.*

Lemma 3. *Let $\tau, \sigma \in H^{-1/2}(\Sigma)$. Then $\langle \tau, K_d\sigma\rangle = \langle \sigma, K_d\tau\rangle$.*

Proof. Let $u(x) = V\tau(x)$ and $v(x) = V\sigma(x)$. From the second Green formula (2) in Ω_+ we obtain $\langle \gamma_1^+ u, \gamma_0^+ v\rangle - \langle \gamma_1^+ v, \gamma_0^+ u\rangle = 0$.

If we apply the second Green formula in the domain B introduced in Theorem 1 we can get

$$\langle \gamma_1^- v, \gamma_0^- u\rangle - \langle \gamma_1^- u, \gamma_0^- v\rangle = \int_{\Sigma_R} \left(\frac{\partial v(x)}{\partial n_y} u(x) - \frac{\partial u(x)}{\partial n_y} v(x) \right) ds_y$$

Since $\gamma_0^+ u = \gamma_0^- u = K_d\tau$, $\gamma_0^+ v = \gamma_0^- v = K_d\sigma$, $\gamma_1^+ u - \gamma_1^- u = \tau$ and $\gamma_1^+ v - \gamma_1^- v = \sigma$ we have

$$\langle \tau, K_d\sigma\rangle - \langle \sigma, K_d\tau\rangle = \int_{\Sigma_R} \left(\frac{\partial v(x)}{\partial n_y} u(x) - \frac{\partial u(x)}{\partial n_y} v(x) \right) ds_y$$

We can present functions $u(x)$ and $v(x)$ in the form $u(x) = \alpha(x) + u_0(x)$, $v(x) = \beta(x) + v_0(x)$, $\alpha(x) = \frac{1}{2\pi}\langle \tau, 1\rangle \ln \frac{d}{|x|}$, $\beta(x) = \frac{1}{2\pi}\langle \sigma, 1\rangle \ln \frac{d}{|x|}$, where functions $u_0(x)$ and $v_0(x)$ satisfy Laplace equation (3) in Ω_- and conditions at infinity (5) with $c_\infty = 0$.

It's easy verify that

$$\int_{\Sigma_R} \left(\frac{\partial \alpha(x)}{\partial n_y} \beta(x) - \frac{\partial \beta(x)}{\partial n_y} \alpha(x) \right) ds_y = 0.$$

If we consider functions $u_0(x)$ and $v_0(x)$ then

$$\left| \int_{\Sigma_R} \left(\frac{\partial v_0(x)}{\partial n_y} u_0(x) - \frac{\partial u_0(x)}{\partial n_y} v_0(x) \right) ds_y \right| \leq$$

$$\leq \int_{\Sigma_R} \left(\left| \frac{\partial v_0(x)}{\partial n_y} \right| |u_0(x)| + \left| \frac{\partial u_0(x)}{\partial n_y} \right| |v_0(x)| \right) ds_y \leq 2\pi R \Big(c_1 \frac{1}{R^2} + c_2 \frac{1}{R^2} \Big) \to 0 \text{ as } R \to \infty,$$

where $c_1 > 0$ and $c_2 > 0$ – some constants.

Thus

$$\lim_{R\to\infty} \int_{\Sigma_R} \left(\frac{\partial v(x)}{\partial n_y} u(x) - \frac{\partial u(x)}{\partial n_y} v(x) \right) ds_y = 0$$

and $\langle \tau, K_d\sigma\rangle - \langle \sigma, K_d\tau\rangle = 0$. □

Now we consider boundary value problem D_- relatively of behavior it's solution at infinity. We have two different occasions: when this solution is bounded at infinity ($\alpha = 0$) or unbounded ($\alpha \neq 0$).

Theorem 2. *The problem D_- has unique solution for arbitrary $g_- \in H^{1/2}(\Sigma)$. We can present this solution in the form $u = V\tau + c_\infty$, $d \neq d_0$, where $c_\infty = (\langle \tau_1, g_-\rangle - \alpha)/\langle \tau_1, 1\rangle$, τ is unique solution of equation $K_d\tau = g_- - c_\infty$ and τ_1 is solution of equation $K_d\tau_1 = 1$.*

Proof. Let τ is unique solution of equation $K_d\tau = g_- - c_\infty$ for arbitrary $g_- \in H^{1/2}(\Sigma)$ where $c_\infty = (\langle \tau_1, g_-\rangle - \alpha)/\langle \tau_1, 1\rangle$. Let us consider function $u(x) = V\tau(x) + c_\infty$, $x \in \Omega_-$. Then we have $\gamma_0^- u(x) = K_d\tau + c_\infty = g_-$. By using lemma 3 we obtain

$\langle \tau, 1\rangle = \langle \tau, K_d\tau_1\rangle = \langle \tau_1, K_d\tau\rangle = \langle \tau_1, g - c_\infty\rangle = \langle \tau_1, g\rangle - c_\infty\langle \tau_1, 1\rangle = \langle \tau_1, g\rangle - \langle \tau_1, g\rangle + \alpha = \alpha$.

Thus function $u(x) = V\tau(x)+c_\infty$ satisfies conditions at infinity (4) with $c_* = \frac{\alpha}{2\pi}\ln d+c_\infty$. As a result we get that function $u(x) = V\tau(x) + c_\infty$, $x \in \Omega_-$, is a solution of the problem D_- with boundary condition $\gamma_0^- u(x) = g_-(x)$ and $\langle \tau, 1\rangle = \alpha$.

Now we show that for given $g_-(x)$ this function $u(x)$ is unique solution of the problem D_-. We suppose that there exists another function $v(x)$ which is a solution of the problem D_-. Then function $w(x) = u(x) - v(x)$ satisfies boundary condition $\gamma_0^- w(x) = 0$ and conditions at infinity (5). From Lemma 1 it follows that $w(x) = 0$ or $v(x) = u(x)$, $x \in \Omega_-$.

Let us show that for $g_-(x) = 0$ problem D_- has solution $u_0(x) = \frac{\alpha}{\langle \tau_1,1\rangle}V\tau_1(x) - \frac{\alpha}{\langle \tau_1,1\rangle}$. We have $\gamma_0^- u_0(x) = \frac{\alpha}{\langle \tau_1,1\rangle}K_d\tau_1(x) - \frac{\alpha}{\langle \tau_1,1\rangle} = 0$. Then

$$u_0(x) = \frac{\alpha}{\langle \tau_1, 1\rangle}\frac{1}{2\pi}\int_\Sigma \ln\frac{|x|}{|x-y|}\tau_1(y)ds_y + \frac{\alpha}{2\pi}\ln\frac{1}{|x|} + c_*,$$

where $c_* = \frac{\alpha}{2\pi}\ln d - \frac{\alpha}{\langle \tau_1,1\rangle}$. Thus $u_0(x)$ satisfies conditions at infinity (4). □

Corollary 4. *The problem D_- with condition at infinity (5) has unique solution for arbitrary $g_- \in H^{1/2}(\Sigma)$ and $\lim_{|x|\to\infty} u(x) = \langle \tau_1, g_-\rangle/\langle \tau_1, 1\rangle$.*

It would be useful to note that we can consider conditions at infinity (4) as special type of boundary conditions given on some infinitely remote closed curve.

4. Neumann boundary value problems. Now we consider interior N_+ and exterior N_- Neumann boundary value problems in Ω_+ and Ω_- respectively.

Problem N_+: find function $u \in H^1(\Omega_+)$ which satisfies Laplace equation (3) in Ω_+ and Neumann boundary condition

$$\gamma_1^+ u = f_+ \in H^{-1/2}(\Sigma). \tag{15}$$

Problem N_-: find function $u \in H^1_{\mathrm{loc}}(\overline{\Omega_-})$ which satisfies Laplace equation (3) in Ω_-, Neumann boundary condition

$$\gamma_1^- u = f_- \in H^{-1/2}(\Sigma) \tag{16}$$

and condition at infinity (4) where $\alpha = -\langle f_-, 1\rangle$ and $c_* = 0$.

Theorem 3. *Problem N_+ with homogeneous boundary conditions $\gamma_1^+ u = 0$ has solution $u_0(x) = c$, $x \in \Omega_+$, where c is an arbitrary constant. For $u_0(x)$ we have the following integral representation*

$$u_0(x) = cW\mu_0(x), \quad x \in \Omega_+, \tag{17}$$

where $\mu_0(x) = 1, x \in \Sigma$.

Problem N_- with homogeneous boundary conditions $\gamma_1^- u = 0$ and $\alpha = 0$ in condition at infinity (4) with $c_* = 0$ has only trivial solution.

Proof. The first Green formula (1) in Ω_+ gives us: $\int_{\Omega_+} |\nabla u_0(x)|^2 dx = 0$. Hence $u_0(x) = c$, $x \in \Omega_+$, where c is an arbitrary constant.

Let us consider domain Ω_-. For the domain B (see theorem 1) we have

$$\int_B |\nabla u(x)|^2 dx = \int_{\Sigma_R} \frac{\partial u(y)}{\partial n_y} u(y) ds_y \to 0 \quad \text{as} \quad R \to \infty.$$

Thus $\int_{\Omega_-} |\nabla u(x)|^2 dx = 0$ and $u(x) = c_*$ or $u(x) = 0$, $x \in \Omega_-$.

Let us consider function $u_0(x)$, $x \in \Omega'$ such that $u_0(x) = 1$, $x \in \Omega_+$ and $u_0(x) = 0$, $x \in \Omega_-$. From the integral representation (6) we have $\mu_0 = [\gamma_0]u_0 = 1$ and we get (17). □

Lemma 4. *Problem N_- with homogeneous boundary conditions $\gamma_1^- u = 0$ doesn't have solution if in condition at infinity (4) $\alpha \neq 0$ and $c_* = 0$.*

Proof. If $u(x)$ is such a solution of this problem then from integral representation (6) we have $u(x) = -W\mu(x) + V\gamma_1^+ u(x)$, $x \in \Omega'$, where $\mu = \gamma_0^+ u - \gamma_0^- u$. Since $\langle \gamma_1^+ u, 1\rangle = 0$ we obtain $\lim_{|x|\to\infty} u(x) = 0$ and $\alpha = 0$. □

From Theorem 3 it follows that equation $-\gamma_1^+ W\mu_0 = H\mu_0 = 0$ has solution $\mu_0(x) = 1$, $x \in \Sigma$. Thus if $u_0(x) = -W\mu_0(x)$ then $u_0(x) = 1$, $x \in \Omega_+$ and $u_0(x) = 0$, $x \in \Omega_-$.

Let us introduce spaces $\mathcal{Y} = \{\mu \in H^{1/2}(\Sigma)\colon (\mu, 1)_{L_2(\Sigma)} = 0\}$ and $\mathcal{Z} = \{f \in H^{-1/2}(\Sigma)\colon \langle f, 1\rangle = 0\}$.

We use the following proposition [8].

Proposition 5. *Operator $H\colon \mathcal{Y} \to H^{-1/2}(\Sigma)$ is positive defined, i.e. there exists constant $c > 0$ that for all $\mu \in H^{1/2}(\Sigma)$ which satisfy condition $(\mu, 1)_{L_2(\Sigma)} = 0$ there holds*

$$\langle H\mu, \mu\rangle \geq c\|\mu\|^2_{H^{1/2}(\Sigma)}. \tag{18}$$

As a consequence of Proposition 5 we can get the following assertion.

Theorem 4. *Operator $H\colon \mathcal{Y} \to \mathcal{Z}$ is an isomorphism.*

Proof. Let $\mu \in \mathcal{Y}$ and $u(x) = -W\mu(x)$, $x \in \Omega_+$. Then $\langle H\mu, 1\rangle = \langle \gamma_1^+ u, 1\rangle = 0$ or $H\mu \in \mathcal{Z}$. The space $\mathcal{Y}' = \mathcal{Z}$. Thus continuous operator $H\colon \mathcal{Y} \to \mathcal{Y}'$ is positive defined and therefor bijective ([2], theorem 2.1.16). □

Corollary 5. *Equation $H\mu = f$ has unique solution $\mu \in H^{1/2}(\Sigma)$, $(\mu, 1)_{L_2(\Sigma)} = 0$, for every functional $f \in H^{-1/2}(\Sigma)$ which satisfies $\langle f, 1\rangle = 0$ and $\|\mu\|_{H^{1/2}(\Sigma)} \leq c\|f\|_{H^{-1/2}(\Sigma)}$, where $c > 0$ is some constant.*

As a result for the boundary value problems N_+ and N_- we have the following propositions.

Theorem 5. *Problem N_+ has a solution for functional $f_+ \in H^{-1/2}(\Sigma)$ which satisfies condition $\langle f_+, 1\rangle = 0$. We can represent this solution in the form $u(x) = -W\mu(x) + c$, $x \in \Omega_+$, where μ is unique solution of equation $H\mu = f_+$ with $(\mu, 1)_{L_2(\Sigma)} = 0$ and c is an arbitrary constant.*

The proof is obvious if we take to attention Corollary 5.

Theorem 6. *Problem N_- has unique solution for arbitrary functional $f_- \in H^{-1/2}(\Sigma)$. We can represent this solution in the form $u(x) = -W\mu(x) - Vf_-(x)$, $x \in \Omega_-$, where $\mu(x)$ is a unique solution of equation*

$$H\mu = \frac{1}{2}f_- + Nf_- \tag{19}$$

with $(\mu, 1)_{L_2(\Sigma)} = 0$.

Proof. At first we show that equation (19) has unique solution $\mu(x)$ with $(\mu, 1)_{L_2(\Sigma)} = 0$ for arbitrary $f_- \in H^{-1/2}(\Sigma)$. If we apply the first Green formula (1) for the functions $z(x) = Vf_-(x)$ and $v(x) = 1$, $x \in \Omega_+$, we get $\langle \gamma_1^+ z, 1\rangle = 0$ or $\langle \frac{1}{2}f_- + Nf_-, 1\rangle = 0$. Thus from Corollary 5 we obtain that equation (19) has unique solution $\mu(x)$ with condition $(\mu, 1)_{L_2(\Sigma)} = 0$ for arbitrary $f_- \in H^{-1/2}(\Sigma)$.

Let us consider function $u(x) = -W\mu(x) - Vf_-(x)$, $x \in \Omega_-$, where $\mu(x)$ is a unique solution of equation (19) with condition $(\mu, 1)_{L_2(\Sigma)} = 0$. Then $\gamma_1^- u = H\mu + \frac{1}{2}f_- - Nf_- = f_-$ and $u(x)$ is solution of the problem N_-.

Now we show that for given f_- this function $u(x)$ is unique solution of the problem N_-. We suppose that there exists another function $v(x)$ which is a solution of the problem N_-. From Theorem 1 we have that $v(x) = -W\sigma(x) - V\tau(x)$ for some $\sigma \in H^{1/2}(\Sigma)$ and $\tau \in H^{-1/2}(\Sigma)$. Since $v(x)$ is a solution of the problem N_- then $\langle \tau, 1\rangle = \langle f_-, 1\rangle$. For the function $w(x) = u(x) - v(x) = -W\sigma_0(x) - V(f_- - \tau)(x)$, where $\sigma_0(x) = \mu(x) - \sigma(x)$, we obtain $\langle f_- - \tau, 1\rangle = \langle f_-, 1\rangle - \langle \tau, 1\rangle = \langle f_-, 1\rangle - \langle f_-, 1\rangle = 0$. Thus function $w(x)$ satisfies Laplace equation in Ω_-, is bounded at infinity and $\gamma_1^- w(x) = 0$. From Theorem 3 it follows that $w(x) = 0$ or $v(x) = u(x)$, $x \in \Omega_-$. □

If the solution $u(x)$ of the problem N_- is bounded at infinity, i.e. $\alpha = 0$ or $\langle f_-, 1\rangle = 0$ then we may look for this solution as $u(x) = -W\mu(x)$ where $\mu(x)$ is solution of equation $H\mu = f_-$ with condition $(\mu, 1)_{L_2(\Sigma)} = 0$.

5. Dirichlet boundary value problems of transmission type. In order to present solutions of interior D_+ and exterior D_- Dirichlet problems via potential of double layer we consider the following boundary value problems (problems DT_+ and DT_-).

Problem DT_+: find function $u \in H^1(\Omega', L)$ which satisfies Laplace equation (3) in Ω', Dirichlet boundary condition

$$\gamma_0^+ u = g_+ \in H^{1/2}(\Sigma), \tag{20}$$

boundary condition of transmission type on Σ

$$\gamma_1^+ u = \gamma_1^- u \tag{21}$$

and condition at infinity

$$\lim_{|x|\to\infty} u(x) = 0. \tag{22}$$

As we denoted above $C^+ = -\gamma_0^+ W = \frac{1}{2}I - M$.

Theorem 7. *The problem* DT_+ *is equivalent to the integral equation of the second kind*

$$C^+\mu(x) \equiv \frac{1}{2}\mu(x) - \int_\Sigma \frac{\partial Q(x,y)}{\partial n_y}\mu(y)ds_y = g_+(x), \quad x \in \Sigma, \tag{23}$$

$\mu(y) = g_+(y) - \gamma_0^- u(y)$, *i.e. the solution* u *of the problem* DT_+ *has the form*

$$u(x) = -W\mu(x), \quad x \in \Omega', \tag{24}$$

where $\mu \in H^{1/2}(\Sigma)$ *is solution of integral equation (23). And vice versa if* μ *is a solution of equation (23) then function* u *given by (24) is a solution of the problem* DT_+.

Proof. Integral representation (24) follows from (6). By using boundary conditions and jump relations (11) we get integral equation (23).

Let now function u is given by expression (24) where $\mu \in H^{1/2}(\Sigma)$ is a solution of (23). Then function u satisfies $Lu = 0$ in Ω' and belongs to $H^1(\Omega', L)$. So far as μ is a solution of equation (23) then using the jump relations (11) we are convinced of fulfilment of the boundary conditions (20) and (21). If $x \to \infty$ we can get conditions at infinity (22). □

Theorem 8. *Problem DT_+ has unique solution for arbitrary $g_+ \in H^{1/2}(\Sigma)$.*

Proof. Let us show that problem DT_+ with boundary condition $\gamma_0^+ u = 0$ has only trivial solution. From the first Green formula (1) in Ω_+ we get $\int_{\Omega_+} |\nabla u|^2 dx = 0$. Thus $u(x) \equiv \text{const}$, $x \in \Omega_+$ and $u = 0$ in Ω_+.

Using conditions at infinity (22) in Ω_- we have

$$\int_{\Omega_-} |\nabla u(x)|^2 dx = -\int_{\Sigma} \gamma_1^- u(y) \gamma_0^- u(y) ds_y.$$

Since $u = 0$ в Ω_+ then $\gamma_1^- u = \gamma_1^+ u = 0$ and $u(x) \equiv \text{const}$, $x \in \Omega_-$. Hence $u = 0$ in Ω_-.

For arbitrary function $g_+ \in H^{1/2}(\Sigma)$ there exists function u which is a solution of the problem D_+. Then $\gamma_1^+ u = f \in H^{-1/2}(\Sigma)$ and $< f, 1 >= 0$. From Theorem 6 it follows that in Ω_- there exists unique function $u(x)$ which satisfies in Ω_- Laplace equation, boundary condition $\gamma_1^- u = f$ and condition at infinity (22). It means that there exists unique function $u \in H^1(\Omega', L)$ which is a solution of problem DT_+. □

Theorem 9. *Operator $C^+ \colon H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is an isomorphism, i.e. equation (23) has an unique solution $\mu \in H^{1/2}(\Sigma)$ for arbitrary $g_+ \in H^{1/2}(\Sigma)$ and*

$$\|\mu\|_{H^{1/2}(\Sigma)} \le c \|g_+\|_{H^{1/2}(\Sigma)}, \quad c > 0.$$

Proof. Let σ_0 be a solution of equation $C^+ \sigma_0 = 0$. Then $u_0(x) = -W\sigma_0(x)$ is a solution of the problem DT_+ with condition $\gamma_0^+ u_0 = 0$ and $\sigma_0 = -\gamma_0^- u_0$. Thus $u(x) = 0$, $x \in \Omega_+$, $\gamma_1^- u_0 = \gamma_1^+ u_0 = 0$ and from theorem 6 we obtain $u_0 = 0$ in Ω_- or $\sigma_0 = 0$.

Let us show that operator C^+ is surjective. For arbitrary $g_+ \in H^{1/2}(\Sigma)$ there exists function u which is unique solution of the problem DT_+. From Theorem 7 it follows that $u(x)$ has unique representation $u(x) = -W\mu(x)$ where μ is a solution of equation $C^+ \mu = g_+$. So far as operator $C^+ \colon H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is continuous we can get the continuity of the inverse operator $(C^+)^{-1}$. □

So far as restriction of the function $u(x)$ which is a solution of the problem DT_+ on domain Ω_+ is a solution of the problem D_+ we have the following corollary.

Corollary 6. *We can present solution of the problem D_+ in the form (24) where $\mu \in H^{1/2}(\Sigma)$ is unique solution of the equation (23).*

Problem D_- when in condition at infinity (4) $\alpha = 0$ is connected with the following problem.

Problem DT_-: find function $u \in H^1(\Omega', L)$ which satisfies Laplace equation (3) in Ω', Dirichlet boundary condition in Ω_-

$$\gamma_0^- u = g_- \in H^{1/2}(\Sigma), \tag{25}$$

boundary condition of transmission type (21) on Σ and condition at infinity (5).

Theorem 10. *The problem DT_- is equivalent to the integral equation of the second kind*

$$C^- \mu(x) = \gamma_0^- W\mu(x) \equiv \frac{1}{2}\mu(x) + \int_{\Sigma} \frac{\partial Q(x,y)}{\partial n_y} \mu(y) ds_y = -g_-(x) + c_*, \quad x \in \Sigma \tag{26}$$

$\mu(y) = \gamma_0^+ u(y) - g_-(y)$, i.e. solution u of the problem DT_- has representation

$$u(x) = -W\mu(x) + c_*, \quad x \in \Omega', \tag{27}$$

where $\mu(y)$ is solution of integral equation (26). And vice versa function $u(x)$ given by (27) where $\mu(y)$ is solution of equation (26) is solution of the problem DT_-.

The proof is similar to the proof of Theorem 7.

Theorem 11. *There exists function $u(x)$ which is a solution of the problem DT_- with boundary condition $\gamma_0^- u = 0$. We have $u = 0$ in Ω_-, $u = const$ in Ω_+ and $c_* = 0$.*

Proof. If $\gamma_0^- u = 0$ and $u(x)$ satisfies condition at infinity (5) then $u(x) = 0$ in Ω_- and $c_* = 0$ (Theorem 1). Consequently $\gamma_1^+ u = \gamma_1^- u = 0$. Thus $u = \text{const}$ in Ω_+. □

From Theorems 10 and 11 it follows existence of the function $u_0(x) = -W\mu_0(x)$, $x \in \Omega'$, where $\mu_0 = \gamma_0^+ u_0 = \text{const}$ is a solution of homogenous integral equation of the second order

$$C^-\mu_0(x) = \frac{1}{2}\mu_0(x) + \int_\Sigma \frac{\partial Q(x,y)}{\partial n_y}\mu_0(y)ds_y = 0, \quad x \in \Sigma.$$

Function $u_0(x)$ is a solution of problem DT_- with boundary condition $\gamma_0^- u_0 = 0$ and $c_* = 0$.

Let us consider function $u_0(x) = -W\mu_0(x)$ where $\mu_0(y) = 1$, $y \in \Sigma$. Then $u_0(x) = 0$ in Ω_- and $u_0(x) = 1$ in Ω_+. From the jump relations (11) we obtain $M\mu_0(y) = \frac{1}{2}(\gamma_0^+ W\mu_0(y) + \gamma_0^- W\mu_0(y)) = \frac{1}{2}$, $y \in \Sigma$.

Let us note that $W\mu_0$ where $\mu_0 = 1$ is well known Gauss integral.

Theorem 12. *Problem DT_- has not unique solution for arbitrary function $g_- \in H^{1/2}(\Sigma)$. Constant $c_* = \langle \tau_1, g_-\rangle / \langle \tau_1, 1\rangle$, where τ_1 is a solution of equation $K_d\tau_1 = 1$, $d \neq d_0$.*

Proof. Let $g_- \in H^{1/2}(\Sigma)$. From Theorem 2 we get existence of the function $u(x)$ which is a unique solution of the problem D_- which satisfies condition at infinity (5), $\gamma_0^- u = g_-$ and $c_* = \langle \tau_1, g_-\rangle / \langle \tau_1, 1\rangle$. If we apply the first Green formula in the domain B introduced in Theorem 1 for functions $u(x)$ and $v(x) = 1$, $x \in \Omega_-$, we have

$$\langle \gamma_1^- u, 1\rangle = \int_{\Sigma_R} \frac{\partial u(y)}{\partial n_y} ds_y \to 0, \qquad R \to \infty,$$

since $u(x)$ satisfies condition at infinity (5). Let us consider functional $f = \gamma_1^- u \in H^{-1/2}(\Sigma)$, $< f, 1 >= 0$. From Theorem 5 it follows that in Ω_+ there exists unique up to a constant function $u(x)$ which satisfies in Ω_+ Laplace equation and boundary condition $\gamma_1^+ u = f$. Hence there exists (not unique) function $u \in H^1(\Omega')$ which is a solution of problem DT_-. □

From the above assertions it follows that solution of the problem DT_- has the next integral representation $u(x) = -W\mu(x) + c_*$, $x \in \Omega'$.

Here μ is a solution of equation $C^-\mu = -g_- + c_*$ and $c_* = \langle \tau_1, g_-\rangle / \langle \tau_1, 1\rangle$, where τ_1 is a solution of equation $K_d\tau_1 = 1$ and $\mu_0 = 1$.

Let us denote $\mathcal{X} = \{g \in H^{1/2}(\Sigma) \colon \langle \tau_1, g\rangle = 0\}$ and as we have denoted above $\mathcal{Y} = \{\mu \in H^{1/2}(\Sigma) \colon (\mu, 1)_{L_2(\Sigma)} = 0\}$.

Theorem 13. *Operator $C^- \colon \mathcal{Y} \to \mathcal{X}$ is an isomorphism.*

Proof. From Theorems 10 and 12 it follows that equation $C^-\mu = g_- - c_*$, where $c_* = \langle \tau_1, g_-\rangle / \langle \tau_1, 1\rangle$, has solution for arbitrary $g_- \in H^{1/2}(\Sigma)$. If $g_- \in \mathcal{X}$, i.e. $\langle \tau_1, g_-\rangle = 0$, then equation $C^-\mu = g_-$ has solution. Thus operator $C^- \colon H^{1/2}(\Sigma) \to \mathcal{X}$ is surjective.

Let $(\mu, 1)_{L_2(\Sigma)} = 0$ and $C^-\mu = 0$. Then function $u = -W\mu + c_*$ is a solution of the problem DT_- with $g_-(x) = \beta$, $x \in \Sigma$ and $c_* = \beta$, $\beta = \text{const}$. From Theorem 2 it follows that $u(x) = \beta$, $x \in \Omega_-$. Since $\gamma_1^+ u = \gamma_1^- u = 0$ then $u(x) = a = \text{const}$, $x \in \Omega_+$. So far as $\mu = \gamma_0^+ u - \beta = a - \beta$ and $(\mu, 1)_{L_2(\Sigma)} = (a - \beta, 1)_{L_2(\Sigma)} = 0$ we have $a = \beta$ and $\mu = 0$. Thus operator $C^- \colon \mathcal{Y} \to \mathcal{X}$ is injective. Continuity of the operator C^- completes the proof of theorem. □

Now we consider boundary value problem D_- when we represent its solution as a double layer potential. In this case solution u of the problem D_- satisfies the condition at infinity (5). Using idea from [10] we have the following theorem.

Theorem 14. *We can present solution of the problem D_- with condition at infinity (5) in the form*

$$u(x) = -\int_\Sigma \frac{\partial Q(x,y)}{\partial n_y}\mu(y)ds_y + \int_\Sigma \mu(y)ds_y, \tag{28}$$

where function μ is unique solution of the following integral equation for arbitrary $g \in H^{1/2}(\Sigma)$

$$C_1^- \mu(x) \equiv \frac{1}{2}\mu(x) + \int_\Sigma \left(\frac{\partial Q(x,y)}{\partial n_y} - 1\right)\mu(y)ds_y = -g_-(x), \quad x \in \Sigma. \tag{29}$$

Proof. If u is solution of the problem DT_- then $u = -W\mu + c_*$ where $\mu = \sigma + c_1\mu_0$ is solution of equation $C^-\mu = -g_- + c_*$, σ is unique solution of equation $C^-\sigma = -g_- + c_*$, $(\sigma, 1)_{L_2(\Sigma)} = 0$, $c_* = \langle\tau_1, g_-\rangle/\langle\tau_1, 1\rangle$, $C^-\mu_0 = 0$, $\mu_0(x) \equiv 1, x \in \Sigma$, τ_1 is a solution of equation $K_d\tau_1 = 1$, $d \neq d_0$ and c_1 is an arbitrary constant. For arbitrary $g_- \in H^{1/2}(\Sigma)$ we have $\langle\tau_1, -g_- + c_*\rangle = 0$.

Let $u(x)$ is a solution of the problem DT_-. Then the solution of the problem D_- is a restriction of the function $u(x)$ on Ω_- and has representation $u = -W\mu + c_*$. Then $C_1^-\mu = C^-\mu - (\mu, 1)_{L_2(\Sigma)} = -g_- + c_* - (\mu, 1)_{L_2(\Sigma)}$. If we take $c_1 = c_*/|\Sigma|$ where $|\Sigma| = (\mu_0, 1)_{L_2(\Sigma)}$ then $c_* = (\mu, 1)_{L_2(\Sigma)}$ and solution u of the problem D_- has form $u = -W\mu + (\mu, 1)_{L_2(\Sigma)}$ where $\mu = \sigma + \frac{\langle\tau_1, g_-\rangle}{\langle\tau_1, 1\rangle|\Sigma|}\mu_0$ is unique solution of the equation (29) for arbitrary $g_- \in H^{1/2}(\Sigma)$. □

If we take to attention the continuity of the operator $C_1^- \colon H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)$ as a consequence we have the following theorem.

Theorem 15. *Operator $C_1^- \colon H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)$ is an isomorphism. Equation (29) has unique solution $\mu \in H^{1/2}(\Sigma)$ for arbitrary function $g_- \in H^{1/2}(\Sigma)$ and*

$$\|\mu\|_{H^{1/2}(\Sigma)} \le C\|g_-\|_{H^{1/2}(\Sigma)}, \quad C > 0.$$

Different types of equations in Hilbert spaces which have not unique solutions were considered in [12].

6. Neumann boundary value problems of transmission type. Let us consider the interior N_+ and exterior N_- Neumann problems when we present their solutions using potential of the simple layer.

Problem NT_+: find function $u \in H^1(\Omega', L)$ which satisfies Laplace equation (3) in Ω', Neumann boundary condition (15) $\gamma_1^+ u = f_+ \in H^{-1/2}(\Sigma)$, boundary condition of transmission type on Σ

$$\gamma_0^+ u = \gamma_0^- u \tag{30}$$

and condition at infinity (22).

Theorem 16. *Problem NT_+ is equivalent to equation of the second kind*

$$B^+\tau \equiv \frac{1}{2}\tau + N\tau = f_+, \tag{31}$$

where $\tau = f_+ - \gamma_1^- u$ *with condition* $\langle \tau, 1\rangle = 0$*, i.e. the solution* u *of the problem* NT_+ *has the form*

$$u(x) = V\tau(x), \quad x \in \Omega', \tag{32}$$

where τ *is a solution of equation (31) with condition* $\langle \tau, 1\rangle = 0$*. And vice versa if* τ *is a solution of equation (31) with condition* $\langle \tau, 1\rangle = 0$ *then function* u *given by (32) is a solution of the problem* NT_+*.*

Proof. If solution of the problem NT_+ exists from integral representation $u(x) = V\tau(x) - W\mu(x)$, $x \in \Omega'$. where $\tau = [\gamma_1]u$, $\mu = [\gamma_0]u$ it follows that $u(x) = V\tau(x)$, $x \in \Omega'$. Here $\tau = f_+ - \gamma_1^- u$ and $\langle \tau, 1\rangle = 0$. From the boundary conditions and jump relations (11) we obtain equation (31).

Let now function u be given by (32) where $\tau \in H^{-1/2}(\Sigma)$ is a solution of equation (31). Then function $u \in H^1(\Omega')$ and satisfies Laplace equation $Lu = 0$ in Ω'. So far as τ is a solution of equation (31) then from the jump relations (11) we have boundary conditions $\gamma_1^+ u = f_+$ and $\gamma_0^+ u = \gamma_0^- u$. If we tend $x \to \infty$ we obtain conditions at infinity (22). □

Theorem 17. *Problem* NT_+ *with boundary condition* $\gamma_1^+ u = 0$ *has only trivial solution.*

Proof. From the first Green formula (1) in Ω_+ we get $\int_{\Omega_+} |\nabla u|^2 dx = 0$. Thus $u(x) = \mathrm{const}$, $x \in \Omega_+$, and the solution of the problem NT_+ with boundary condition $\gamma_1^+ u = 0$ has the following integral representation: $u(x) = cV\tau_1(x)$, $x \in \Omega'$. Here c is an arbitrary constant, $\tau_1 = -\gamma_1^- u$ and τ_1 is a solution of equation $K_d\tau_1 = 1$. If we take to attention condition at infinity (22) then $c\langle \tau_1, 1\rangle = 0$ or $c = 0$ and we have $u(x) = 0$ in $\mathbb{R}^2$. □

Theorem 18. *Problem* NT_+ *has unique solution for functional* $f_+ \in H^{-1/2}(\Sigma)$ *with condition* $\langle f_+, 1\rangle = 0$*.*

Proof. From Theorem 5 it follows that there exists solution $u(x) = v(x) + c$, $x \in \Omega_+$, of the problem N_+, where $\gamma_1^+ v = f_+$, $\langle f_+, 1\rangle = 0$ and c is an arbitrary constant. We can choose this constant in such a manner that $\gamma_0^+ u = g \in H^{1/2}(\Sigma)$ and $\langle \tau_1, g\rangle = 0$, where τ_1 is a solution of equation $K_d\tau_1 = 1$. From Corollary 3 it follows that we can present $u(x)$, $x \in \Omega_+$, as $u(x) = V\tau(x)$, where τ is unique solution of equation $K\tau = g$ and satisfies $\langle \tau, 1\rangle = 0$. Then function $u(x) = V\tau(x)$, $x \in \Omega'$, is unique solution of the problem NT_+. □

We denoted above $\mathcal{Z} = \{f \in H^{-1/2}(\Sigma) \colon \langle f, 1\rangle = 0\}$. As a consequence of Theorems 16, 17 and 18 we have the following assertion.

Theorem 19. *Operator* $B^+ \colon \mathcal{Z} \to \mathcal{Z}$ *is an isomorphism, i.e. equation* $B^+\tau = f_+$ *has unique solution which satisfies condition* $\langle \tau, 1\rangle = 0$ *for functional* $f_+ \in H^{-1/2}(\Sigma)$ *with condition* $\langle f_+, 1\rangle = 0$*. Homogeneous equation* $B^+\tau_1 = 0$ *has solution* τ_1 *where* $K_d\tau_1 = 1$*.*

Let us consider equation

$$B_1^+\sigma \equiv \frac{1}{2}\sigma + N\sigma + \langle \sigma, 1\rangle = g. \tag{33}$$

Here $\langle \sigma, 1\rangle$ is a function that equals $\langle \sigma, 1\rangle$ on Σ.

Theorem 20. *Operator* $B_1^+ \colon H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ *is an isomorphism, i.e. equation (33) has unique solution* $\sigma \in H^{-1/2}(\Sigma)$ *for arbitrary functional* $g \in H^{-1/2}(\Sigma)$ *and*

$$\|\sigma\|_{H^{-1/2}(\Sigma)} \le C\|g\|_{H^{-1/2}(\Sigma)}, \quad C > 0.$$

Proof. Let $g \in H^{-1/2}(\Sigma)$ and $f = g - c_g$, where $c_g = \langle g, 1\rangle/|\Sigma|$, $|\Sigma|$ is a length of Σ. Then $\langle f, 1\rangle = 0$. Let now τ be unique solution of equation $B^+\tau = f$, $\langle \tau, 1\rangle = 0$ and $\sigma = \tau + c\tau_1$ where τ_1 is a solution of homogeneous equation $B^+\tau_1 = 0$ and c is an arbitrary constant. If we take $c = c_g/\langle \tau_1, 1\rangle$ it's easy to verify that $\sigma = \tau + c\tau_1$ is a solution of equation $B_1^+\sigma = g$. Thus operator B_1^+ is surjective.

Now we show injectivity of B_1^+, i.e. homogeneous equation $B_1^+\tau = 0$ has only trivial solution. Let τ_0 be a solution of $B_1^+\tau_0 = 0$. Then $B^+\tau_0 = \langle \tau_0, 1\rangle$. We consider function $u_0 = V\tau_0$. It's obviously that $\gamma_1^+ u_0 = \langle \tau_0, 1\rangle = \text{const}$. Inasmuch $\langle \gamma_1^+ u_0, 1\rangle = 0$ then $\langle \tau_0, 1\rangle = 0$ and $B^+\tau_0 = 0$. It means that $\tau_0 = 0$.

Continuity of operator $B_1^+ \colon H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ is obvious. □

The solution of the problem NT_+ we search in the form $u = V\tau$ where $\tau = \sigma - \tau_1$ and σ is a solution of equation $B_1^+\sigma = f_+ + \langle \tau_1, 1\rangle$. Since $B^+\tau_1 = 0$ we have $B^+\tau = B_1^+\sigma - \langle \sigma, 1\rangle = f_+ - \langle \tau, 1\rangle$. So far as $\langle B^+\tau, 1\rangle = 0$ for $\tau \in H^{-1/2}(\Sigma)$ and $\langle f_+, 1\rangle = 0$ we obtain $B^+\tau = f_+$ or $\gamma_1^+ u = f_+$ and $\langle \tau, 1\rangle = 0$.

As a consequence we have the following theorem.

Theorem 21. *We can present solution of the problem N_+ in the form $u = V\tau + c$ where c is an arbitrary constant and $\tau = \sigma - \tau_1$. Here σ is a solution of equation $B_1^+\sigma = f_+ + \langle \tau_1, 1\rangle$, $K_d\tau_1 = 1$, and $\langle \tau, 1\rangle = 0$.*

If $g \in L_2(\Sigma)$ equation $B_1^+\sigma = g$ we have in the next integral form

$$\frac{1}{2}\sigma(x) + \int_\Sigma \left(\frac{\partial Q(x, y)}{\partial n_x} + 1 \right) \sigma(y) ds_y = g(x), \quad x \in \Sigma.$$

Problem NT_-: find function $u \in H^1(\Omega', L)$ which satisfies Laplace equation (3) in Ω', Neumann boundary condition (16), boundary condition of transmission type (30) on Σ $\gamma_0^+ u = \gamma_0^- u$ and condition at infinity (4) where $\alpha = -\langle f_-, 1\rangle$ and $c_\infty = 0$.

Theorem 22. *Problem NT_- is equivalent to equation of the second kind*

$$B^-\tau \equiv -\frac{1}{2}\tau + N\tau = f_-, \tag{34}$$

where $B^-\tau = \gamma_1^- V\tau$, $\tau = \gamma_1^+ u - f_-$, i.e. the solution u of the problem NT_- has the form

$$u(x) = V\tau(x), \quad x \in \Omega', \tag{35}$$

where τ is a solution of equation (34). And vice versa if τ is a solution of equation (34) then function u given by (35) is a solution of the problem NT_-.

Proof we can get in the same way as for Theorem 16.

Theorem 23. *Problem NT_- has unique solution for arbitrary functional $f_- \in H^{-1/2}(\Sigma)$.*

Proof. For arbitrary functional $f_- \in H^{-1/2}(\Sigma)$ there exists unique solution u of the problem N_- and in condition at infinity (4) $\alpha = -\langle f_-, 1\rangle$ (theorem 6). Analogously for function $g = \gamma_0^+ u = \gamma_0^- u \in H^{1/2}(\Sigma)$ there exists unique solution of the problem D_+. □

From Theorems 22 and 23 we can obtain the next conclusion.

Theorem 24. *Operator $B^-: H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ is an isomorphism, i.e. equation $B^-\tau = f_-$ has unique solution $\tau \in H^{-1/2}(\Sigma)$ for arbitrary functional $f_- \in H^{-1/2}(\Sigma)$ and*

$$\|\tau\|_{H^{-1/2}(\Sigma)} \leq c\|f_-\|_{H^{-1/2}(\Sigma)},$$

where $c > 0$ is some constant.

As a consequence we have the following corollary

Corollary 7. *Problem N_- has unique solution for arbitrary functional $f_- \in H^{-1/2}(\Sigma)$. We can represent this solution in the form $u(x) = V\tau(x)$, $x \in \Omega_-$, where τ is a unique solution of equation $B^-\tau = f_-$.*

Acknowledgement. The author thanks Prof. R.S. Chapko for attention and helpful discussions, and also to an anonymous reviewer for valuable recommendations that helped the author to significantly improve the substantive part of the article.

REFERENCES

1. M.S. Agranovich, Sobolev spaces, their generalizations and elliptic boundary value problems in domains with smooth and Lipschitz boundary, M., 2013, 379 p. (in Russian)
2. J.-P. Aubin, Approximation of elliptic boundary-value problems, Wiley-Interscience, New York, 1972.
3. M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results*, SIAM J. Math. Anal., **19** (1988), 613–626.
4. E. Gagliardo, *Caratterizazioni delle trace sullo frontiera relative ad alcune classi de funzioni in „n" variabili*, Rendiconti del Seminario Matematico della Universita di Padova, **27** 1957, 284–305.
5. G.C. Hsiao, W.L. Wendland, Boundary integral equations, Springer-Verlag, Berlin, 2008, 640 p.
6. R. Kress, Linear integral equations, Springer–Verlag, Berlin, 1998.
7. V.G. Maz'ya, *Boundary integral equations*, In Encyclopaedia of Mathematical Sciences. V.27, Analysis IV. Maz'ya, V.G., Nikolskii eds. Springer-Verlag, Berlin, 1991, 127–228.
8. W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, 2000, 357 p.
9. S.E. Mikhailov, *Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains*, J. Math. Analysis Appl., **378** (2011), 324–342.
10. S.G Mikhlin, Course of mathematical phisycs, M., 1968, 575 p. (in Russian)
11. T. von Petersdorff, *Boundary integral equations for the mixed, Dirichlet, Neumann and transmission problems*, Math. Meth. in Appl. Sci., **11** (1989), 185–213.
12. Yu. Sybil, *On some problems of solving linear equations that have a non-unique solution*, Herald of LNU, ser. Applied mathematics and informatics, **28** (2020), 73–81. (in Ukrainian)
13. V.S. Vladimirov, Equations of mathematical physics, M., 1981, 512 p. (in Russian)
14. W.L. Wendland, E.P. Stephan, G.C. Hsiao, *On the integral equation method for the plane mixed boundary value problem of the Laplacian*, Math. Meth. Appl. Sci., **1** (1979), 265–321.

Ivan Franko National University of Lviv
Lviv, Ukraine
sybil.yuri@gmail.com

Received 12.09.2022
Revised 11.08.2024