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ASYMPTOTIC ESTIMATES FOR ANALYTIC FUNCTIONS IN STRIPS AND THEIR DERIVATIVES

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Let $-\infty \le A_0 < A \le +\infty$, Φ be a continuous function on $[a, A)$ such that for every $x \in \mathbb{R}$ we have $x\sigma - \Phi(\sigma) \to -\infty$ as $\sigma \uparrow A$, $\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) : \sigma \in [a, A)\}$ be the Young-conjugate function of Φ, $\Phi_*(x) = \widetilde{\Phi}(x)/x$ for all sufficiently large x, and F be an analytic function in the strip $\{s \in \mathbb{C}\colon A_0 < \operatorname{Re} s < A\}$ such that the quantity $S(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}$ is finite for all $\sigma \in (A_0, A)$ and $F(s) \not\equiv 0$. It is proved that if

$$\ln S(\sigma, F) \le (1 + o(1)\Phi(\sigma) \text{ as } \sigma \uparrow A,$$

then

$$\varlimsup_{\sigma\uparrow A} \frac{S(\sigma, F')}{S(\sigma, F)\Phi_*^{-1}(\sigma)} \le c_0,$$

where $c_0 < 1{,}1276$ is an absolute constant. From previously obtained results it follows that c_0 cannot be replaced by a constant less than 1.

1. Introduction. Denote by Λ the class of all nonnegative increasing to $+\infty$ sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$. For a sequence $\lambda = (\lambda_n)_{n=0}^{\infty}$ from the class Λ we put

$$n(t, \lambda) = \sum_{\lambda_n \le t} 1, \quad \tau(\lambda) = \varlimsup_{t\to+\infty} \frac{\ln n(t, \lambda)}{t}.$$

Let $A \in (-\infty, +\infty]$ be a constant and $\lambda \in \Lambda$. We will write $F \in \mathcal{D}_A(\lambda)$, if $F(s)$ is the sum of an absolutely convergent in the half-plane $\{s \in \mathbb{C}\colon \operatorname{Re} s < A\}$ Dirichlet series with the system of exponents λ, i.e.

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad \operatorname{Re} s < A, \tag{1}$$

and in addition $F(s) \not\equiv 0$. For each function $F \in \mathcal{D}_A(\lambda)$ of the form (1) and every $\sigma < A$ let

$$S(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}, \quad K(\sigma, F) = \frac{S(\sigma, F')}{S(\sigma, F)}, \quad G(\sigma, F) = \sum_{n=0}^{\infty} |a_n| e^{\sigma\lambda_n}. \tag{2}$$

Put $\mathcal{D}_A = \bigcup_{\lambda\in\Lambda} \mathcal{D}_A(\lambda)$.

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Let $\Phi\colon D_\Phi \to \mathbb{R}$ be a real function. We say that $\Phi \in \Omega_A$ if the domain D_Φ of Φ is an interval of the form $[a, A)$, Φ is continuous on D_Φ, and the following condition

$$\forall x \in \mathbb{R}\colon \quad \lim_{\sigma\uparrow A}(x\sigma - \Phi(\sigma)) = -\infty \tag{3}$$

holds. It is easy to see that in the case $A < +\infty$ condition (3) is equivalent to the condition $\Phi(\sigma) \to +\infty$, $\sigma \to A - 0$, and in the case $A = +\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$. For $\Phi \in \Omega_A$ by $\widetilde{\Phi}$ we denote the Young-conjugate function of Φ, i.e.

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma)\colon \sigma \in D_\Phi\}, \quad x \in \mathbb{R}.$$

Note (see Lemma 1 below), that the function $\Phi_*(x) = \widetilde{\Phi}(x)/x$ is continuous and increasing to A on some interval of the form $(x_0, +\infty)$. Hence the inverse function Φ_*^{-1} is defined on some interval of the form (A_0, A) and Φ_*^{-1} is continuous and increasing to $+\infty$ on (A_0, A).

We say that $\Phi \in \Omega'_A$, if Φ is a continuously differentiable on D_Φ function from the class Ω_A such that Φ' is a positive increasing function on D_Φ.

For functions $F \in \mathcal{D}_A$ and $\Phi \in \Omega_A$ put

$$T_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln S(\sigma, F)}{\Phi(\sigma)}, \quad \mathcal{T}_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln G(\sigma, F)}{\Phi(\sigma)}. \tag{4}$$

Suppose that f is an entire function and let

$$M(r, f) = \max\{|f(z)|\colon |z| = r\} \tag{5}$$

for all $r \geq 0$. S. Bernstein ([1, p. 76]) proved that if f has order $\rho \in (0, +\infty)$ and type $T \in (0, +\infty)$, i.e.

$$\varlimsup_{r\to+\infty} \frac{\ln M(r, f)}{r^\rho} = T,$$

then the following inequality

$$\varlimsup_{r\to+\infty} \frac{M(r, f')}{M(r, F) r^{\rho-1}} \leq eT\rho$$

holds. The exactness of this inequality was proved by T. Kövari [2]: for any $\rho \in (0, +\infty)$ and $T \in (0, +\infty)$ there exists an entire function f of order ρ and type T such that

$$\varlimsup_{r\to+\infty} \frac{M(r, f')}{M(r, F) r^{\rho-1}} = eT\rho.$$

Let $\lambda^0 = (n)_{n=0}^\infty$. For each entire function f with the sequence of Maclaurin coefficients $(a_n)_{n=0}^\infty$ we can put in a one-to-one correspondence a function $F \in \mathcal{D}_{+\infty}$ presented by Dirichlet series with the same sequence of coefficients and the system of exponents λ^0. With such correspondence we will have

$$S(\sigma, F) = M(e^\sigma, f), \quad S(\sigma, F') = e^\sigma M(e^\sigma, f') \tag{6}$$

for all $\sigma \in \mathbb{R}$. Thus, we can give the following equivalent formulations to the above results of S. Bernstein and T. Kövari.

Theorem A. *Let $F \in \mathcal{D}_{+\infty}(\lambda^0)$ and $\rho, T \in (0, +\infty)$. If*

$$\varlimsup_{\sigma\to+\infty} \frac{\ln S(\sigma, F)}{e^{\rho\sigma}} = T, \tag{7}$$

then

$$\varlimsup_{\sigma\to+\infty} \frac{K(\sigma, F)}{e^{\rho\sigma}} \le T\rho e. \tag{8}$$

Theorem B. *Let $\rho, T \in (0, +\infty)$. Then there exists a function $F \in \mathcal{D}_{+\infty}(\lambda^0)$ such that* (7) *holds and*

$$\varlimsup_{\sigma\to+\infty} \frac{K(\sigma, F)}{e^{\rho\sigma}} = T\rho e.$$

Some analogues of the results of S. Bernstein and T. Kövari for classes of entire functions and classes of analytic in half-planes functions presented by Dirichlet series, which are defined by general conditions on the growth of functions from these classes, were obtained in the articles [3, 4, 5, 6]. In particular, for an arbitrary $A \in (-\infty, +\infty]$, the following theorem obtained in [4] gives an estimate of the growth of the quantity $K(\sigma, F)$ as $\sigma \uparrow A$ for every function $F \in \mathcal{D}_A(\lambda)$ by some conditions on $\lambda \in \Lambda$ and $\Phi \in \Omega_A$.

Theorem C. *Let $A \in (-\infty, +\infty]$, $\lambda \in \Lambda$, α be a positive increasing to $+\infty$ on $[0, +\infty)$ function such that $\alpha(t) = o(t)$ as $t \to +\infty$, $F \in \mathcal{D}_A(\lambda)$, $\Phi \in \Omega'_A$, and $\gamma(\sigma) = 2/\alpha(\Phi_*^{-1}(\sigma))$ for every $\sigma \in (A_0, A)$. Suppose that $\sigma + \gamma(\sigma) < A$, $\sigma \in [\sigma_0, A)$, and*

$$\ln n(t, \lambda) \le t/\alpha(t), \quad t \ge t_0.$$

If $T_\Phi(F) = 1$, then

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\Phi_*^{-1}(\sigma + \gamma(\sigma))} \le 1. \tag{9}$$

It is shown in [4] that in many cases estimate (9) is sharp. To substantiate the exactness of inequality (9), in [4] it is proved that if $A \in (-\infty, +\infty]$ and $\Phi \in \Omega'_A$ is a twice continuously differentiable function on D_Φ for which

$$\Phi((1 + o(1))\sigma) \sim (1 + o(1))\Phi(\sigma), \quad \sigma \uparrow A,$$

and $t^2\varphi'(t) \uparrow +\infty$ as $t \uparrow +\infty$, where φ is the inverse function of Φ', then for every sequence $\lambda \in \Lambda$ there exists a function $F \in \mathcal{D}_A(\lambda)$ such that $T_\Phi(F) = 1$ and

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\Phi_*^{-1}(\sigma)} = 1. \tag{10}$$

The following general result is proved in [5].

Theorem D. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$. Then for every sequence $\lambda \in \Lambda$ there exists a function $F \in \mathcal{D}_A(\lambda)$ such that $T_\Phi(F) = \mathcal{T}_\Phi(F) = 1$ and equality* (10) *holds.*

It is easily seen that the conditions of Theorem C imply the equality $\tau(\lambda) = 0$. Therefore, in the case $\tau(\lambda) > 0$ Theorem C does not give any information about the growth of the quantity $K(\sigma, F)$. Moreover, if $A < +\infty$, then even in the case $\tau(\lambda) = 0$ the conclusion of Theorem C is true only by some conditions on Φ.

Let $F \in \mathcal{D}_A$ be a function of the form (1) with nonnegative coefficients a_n. Then $M(\sigma, F) = G(\sigma, F) = F(\sigma)$, $\sigma < A$. Hence, $T_\Phi(F) = \mathcal{T}_\Phi(F)$ and $K(\sigma, F) = (\ln M(\sigma, F))'$,

$\sigma < A$. Therefore, as is easy to prove (see, for example, Lemma 3 below), for the function F, without any conditions on the sequence $\lambda = (\lambda_n)_{n=0}^{\infty}$ in (1) and on a function $\Phi \in \Omega_A$, the equality $T_\Phi(F) = 1$ (or the identical equality $\mathcal{T}_\Phi(F) = 1$) implies the inequality

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\Phi_*^{-1}(\sigma)} \le 1. \tag{11}$$

The following theorem [5] shows that inequality (11) follows from the equality $\mathcal{T}_\Phi(F) = 1$ for any other function $F \in \mathcal{D}_A$.

Theorem E. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $F \in \mathcal{D}_A$. If $\mathcal{T}_\Phi(F) \le 1$, then inequality* (11) *holds.*

Let $\lambda \in \Lambda$, $\Phi \in \Omega_A$, and $\varphi(x) = \widetilde{\Phi}'_+(x)$ for all $x \in \mathbb{R}$. Theorem E supplements Theorem C in a certain part. From results obtained in [7] it follows that the condition

$$\forall t > 0\colon \quad \ln n = o(\Phi(\varphi(\lambda_n/t))), \quad n \to \infty,$$

is sufficient in order that $T_\Phi(F) = \mathcal{T}_\Phi(F)$ for every function $F \in \mathcal{D}_A(\lambda)$. Under this condition, of course, the inequality $T_\Phi(F) \le 1$ implies inequality (11) for each function $F \in \mathcal{D}_A(\lambda)$.

Note that in the general situation the growth of the function $\ln S(\sigma, F)$ may differ significantly from the growth of the function $\ln G(\sigma, F)$ as $\sigma \uparrow A$ (see, for example, [8, 9, 10]), in particular, there exist functions $\Phi \in \Omega_A$ and $F \in \mathcal{D}_A(\lambda^0)$ such that $T_\Phi(F) = 0$, but $\mathcal{T}_\Phi(F) = +\infty$.

Suppose that $\rho, T \in (0, +\infty)$ and $\Phi(\sigma) = Te^{\rho\sigma}$ for all $\sigma \ge \sigma_1$. Then $\Phi \in \Omega_{+\infty}$ and, as is easy to evaluate, $\Phi_*^{-1}(\sigma) = T\rho e^{\rho\sigma+1}$ for all $\sigma \ge \sigma_2$. This and the above results imply that in Theorem B the sequence λ^0 can be replaced by an arbitrary sequence $\lambda \in \Lambda$. From Theorems C and E it follows that under the condition $\tau(\lambda) = 0$ the same substitution is also possible in Theorem A. The following result of O.V. Shapovalovs'kyi [11] shows that the condition $\tau(\lambda) = 0$ can be removed if instead of (8) we will require the fulfilment of a slightly weaker inequality.

Theorem F. *Let $F \in \mathcal{D}_{+\infty}$ and $\rho, T \in (0, +\infty)$. If* (7) *holds, then*

$$\varlimsup_{\sigma\to+\infty} \frac{K(\sigma, F)}{T\rho e^{\rho\sigma+1}} \le \frac{e}{2} = 1,35914\ldots.$$

In connection with the above results the following question arises: is it possible to replace the condition $\mathcal{T}_\Phi(F) \le 1$ in Theorem E by the condition $T_\Phi(F) \le 1$ if instead of (11) we will require the fulfilment of the following inequality

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\Phi_*^{-1}(\sigma)} \le \frac{e}{2}?$$

Below we will give a positive answer to this question. Moreover, we will show that in the last inequality the constant $e/2$ can be replaced even by some smaller absolute constant $c_0 < 1,1276$.

2. Main results. We put $\mathbb{S}_{A_1,A_2} = \{s \in \mathbb{C}\colon A_1 < \operatorname{Re} s < A_2\}$ for arbitrary constants $A_1, A_2 \in [-\infty, +\infty]$, $A_1 < A_2$. If $A_0, A \in [-\infty, +\infty]$, $A_0 < A$, then let $\mathcal{B}_{A_0,A}$ be the class of all analytic in the strip $\mathbb{S}_{A_0,A}$ functions, that are not identically zero and are bounded in each strip $\mathbb{S}_{A_1,A_2}$, where $A_0 < A_1 < A_2 < A$.

Let $F \in \mathcal{B}_{A_0,A}$. Note that either $F'(s) \equiv 0$, or, as follows from Cauchy's integral formula, $F' \in \mathcal{B}_{A_0,A}$. Define $S(\sigma, F)$ and $K(\sigma, F)$ for all $\sigma \in (A_0, A)$ as in (2). It is well known ([12]) that the function $\ln S(\sigma, F)$ is convex on (A_0, A).

Put $\mathcal{B}_A = \cup_{A_0<A}\mathcal{B}_{A_0,A}$. For functions $F \in \mathcal{B}_A$ and $\Phi \in \Omega_A$ we define $T_\Phi(F)$ as in (4).

We say that $F \in \mathcal{C}_{A_0,A}$, if $F \in \mathcal{B}_{A_0,A}$ and the function $S(\sigma, F)$ is nondecreasing on (A_0, A). Put $\mathcal{C}_A = \cup_{A_0<A}\mathcal{C}_{A_0,A}$.

Note that if $A_1 < A_2 < A$, then $\mathcal{B}_{A_1,A} \subset \mathcal{B}_{A_2,A} \subset \mathcal{B}_A$, $\mathcal{C}_{A_1,A} \subset \mathcal{C}_{A_2,A} \subset \mathcal{C}_A$. It is clear also that $\mathcal{D}_A \subset \mathcal{C}_{-\infty,A} \subset \mathcal{C}_A \subset \mathcal{B}_A$.

Let $\Pi = (0, \pi/2) \times (0, +\infty)$. Consider the function

$$u(\theta, y) = \frac{1}{\pi y \sin\theta}\left(\cos\theta + \left(\frac{e^{2y}-1}{2y}\left(\frac{\pi}{2\sin 2\theta}+1\right)\right)^{1/2}\right), \quad (\theta, y) \in \Pi. \tag{12}$$

It is obvious that the function $u(\theta, y)$ is continuous in Π. If (θ_0, y_0) is a point on the boundary of the half-strip Π and $(\theta, y) \in \Pi$, then $u(\theta, y) \to +\infty$ as $(\theta, y) \to (\theta_0, y_0)$. In addition, $u(\theta, y) \to +\infty$ as $y \to +\infty$ uniformly with respect to $\theta \in (0, \pi/2)$. It follows that $u(\theta, y)$ attains a minimum value on Π. Calculations with the help of computer programs show that

$$c_0 := \min_{(\theta,y)\in\Pi} u(\theta, y) = u(1,169821\ldots, 1,50853\ldots) = 1,12755\ldots.$$

Theorem 1. *Let $A \in (-\infty, +\infty]$, $F \in \mathcal{B}_A$, and $\Phi \in \Omega_A$. If $T_\Phi(F) \le 1$, then*

$$\varlimsup_{\sigma\uparrow A} \frac{K(\sigma, F)}{\Phi_*^{-1}(\sigma)} \le c_0. \tag{13}$$

The following theorem will play an important role in the proof of Theorem 1.

Theorem 2. *Let $-\infty \le A_0 < \sigma_0 < \sigma < A \le +\infty$ and $F \in \mathcal{C}_{A_0,A}$. Then for any $h \in (0, A-\sigma)$ and $\theta \in (0, \pi/2)$ we have*

$$S(\sigma, F') \le \frac{S(\sigma, F)}{2(\sigma-\sigma_0)} + \frac{S(\sigma, F)}{\pi}\left|\frac{1}{\sigma-\sigma_0} - \frac{1}{h\operatorname{tg}\theta}\right| + \frac{1}{\pi}\int_0^{h/\cos\theta} \frac{S(\sigma+h-t\cos\theta, F)}{|h-te^{i\theta}|^2}dt. \tag{14}$$

Theorem 1 can be used to establish Berstein-type estimates for analytic functions in an annulus.

Suppose that $0 \le R_0 < R \le +\infty$, and let f be an analytic function in the annulus $\{z \in \mathbb{C}\colon R_0 < |z| < R\}$, which is not identically zero. For each $r \in (R_0, R)$ we define $M(r, f)$ by (5). Put $F(s) = f(e^s)$ for all $s \in \mathbb{S}_{\ln R_0, \ln R}$. Then $F \in \mathcal{B}_{\ln R}$ and equalities (6) hold for every $\sigma \in (\ln R_0, \ln R)$. Thus, from Theorem 1 we obtain immediately the following statement: if $\Phi \in \Omega_{\ln R}$ and

$$\ln M(r, f) \le (1 + o(1)\Phi(\ln r), \quad r \uparrow R,$$

then

$$\varlimsup_{r\uparrow R} \frac{rM(r, f')}{M(r, f)\Phi_*^{-1}(\ln R)} \le c_0.$$

In addition to Theorem 2, to prove Theorem 1 we will need other auxiliary results, which are given in the next section.

3. Auxiliary results. The following lemma is well known (see, for example, [7]).

Lemma 1. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $\varphi(x) = \max\{\sigma \in D_\Phi \colon x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$, $x \in \mathbb{R}$. Then the following statements are true:*

(i) the function φ is nondecreasing on $\mathbb{R}$;

(ii) the function φ is continuous from the right on $\mathbb{R}$;

(iii) $\varphi(x) \to A$, $x \to +\infty$;

(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x \in \mathbb{R}$;

(v) if $x_0 = \inf\{x > 0 \colon \Phi(\varphi(x)) > 0\}$, then the function $\Phi_*(x) = \widetilde{\Phi}(x)/x$ increase to A on $(x_0, +\infty)$;

(vi) the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$.

In the following two lemmas, which are proved in [13] and [5] respectively, φ and x_0 are defined by Φ in the same way as in Lemma 1.

Lemma 2. *Let $\delta \in (0,1)$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \Phi_*(x_0 + 0)$, and $y(\sigma) = \varphi(\Phi_*^{-1}(\sigma))$ for all $\sigma \in (\sigma_0, A)$. Then*

$$\Phi_*^{-1}\left(\sigma + \frac{\delta\Phi(y(\sigma))}{\Phi_*^{-1}(\sigma)}\right) \le \frac{\Phi_*^{-1}(\sigma)}{1-\delta}, \quad \sigma \in (\sigma_0, A).$$

Lemma 3. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \Phi_*(x_0 + 0)$, $b \in [\sigma_0, A)$, Ψ be a convex function on (b, A) such that $\Psi(y) \le \Phi(y)$ for all $y \in (b, A)$, and*

$$E = \{\sigma \in (b, A) \colon \ \Psi(y) - \Psi(\sigma) \le \Phi(y) \text{ for all } y \in (\sigma, A)\}.$$

Then $\Psi'_+(\sigma) \le \Phi_*^{-1}(\sigma)$ for every $\sigma \in E$.

4. Proof of Theorems.

Proof of Theorem 2. Suppose that $-\infty \le A_0 < \sigma_0 < \sigma < A \le +\infty$, and let $F \in \mathcal{C}_{A_0,A}$. Let also $h \in (0, A - \sigma)$ and $\theta \in (0, \pi/2)$ be fixed constants.

We fix an arbitrary point s_0 on the straight line $\{s \in \mathbb{C} \colon \ \operatorname{Re} s = \sigma\}$. Put $t_0 = \operatorname{Im} s_0$ and let $H(s) = F(s + it_0)$ for all $s \in \mathbb{S}_{A_0,A}$. It is clear that $H \in \mathcal{C}_{A_0,A}$ and $H'(s) = F'(s + it_0)$ for all $s \in \mathbb{S}_{A_0,A}$. We set

$$C_1 = [\sigma - i(\sigma - \sigma_0); \sigma - ih \operatorname{tg}\theta], \quad C_2 = [\sigma - ih\operatorname{tg}\theta; \sigma + h], \quad C_3 = [\sigma + h; \sigma + ih\operatorname{tg}\theta],$$
$$C_4 = [\sigma + ih\operatorname{tg}\theta; \sigma + i(\sigma - \sigma_0)], \quad C_5 = \left\{ z = \sigma + (\sigma - \sigma_0)e^{it} \colon \frac{\pi}{2} \le t \le \frac{3\pi}{2} \right\}.$$

It is easy to see that the segments C_1, C_2, C_3, C_4 and the semicircle C_5 constitute a simply closed contour C. Since the point σ is inside this contour, Cauchy's integral formula gives

$$F'(s_0) = F'(\sigma + it_0) = H'(\sigma) = \frac{1}{2\pi i}\int_C \frac{H(w)}{(w-\sigma)^2}dw = \frac{1}{2\pi i}\sum_{j=1}^{5} I_j, \quad I_j := \int_{C_j} \frac{H(w)}{(w-\sigma)^2}dw.$$

We next estimate each of the integrals I_j, $j = \overline{1,5}$.

Suppose that $\sigma-\sigma_0\neq h\operatorname{tg}\theta$. If $w\in C_1$, then $w=\sigma-it$, where t varies from $\sigma-\sigma_0$ to $h\operatorname{tg}\theta$. Hence,

$$|I_1|=\left|\int_{\sigma-\sigma_0}^{h\operatorname{tg}\theta}\frac{H(\sigma-it)(-i)}{(-it)^2}dt\right|\le S(\sigma,H)\left|\int_{\sigma-\sigma_0}^{h\operatorname{tg}\theta}\frac{dt}{t^2}\right|=S(\sigma,H)\left|\frac{1}{\sigma-\sigma_0}-\frac{1}{h\operatorname{tg}\theta}\right|.$$

The obtained estimate is also correct in the case when $\sigma-\sigma_0=h\operatorname{tg}\theta$, because in this case the segment C_1 degenerates into a point and therefore $I_1=0$.

If $w\in C_3$, then $w=\sigma+h-te^{i\theta}$, where $0\le t\le h/\cos\theta$. Hence,

$$|I_3|=\left|\int_0^{h/\cos\theta}\frac{H(\sigma+h-te^{i\theta})(-e^{i\theta})}{(h-te^{i\theta})^2}dt\right|\le\int_0^{h/\cos\theta}\frac{S(\sigma+h-t\cos\theta,H)}{|h-te^{i\theta}|^2}dt.$$

It is easy to see that the estimates obtained for $|I_1|$ and $|I_3|$ are also correct for $|I_4|$ and $|I_2|$, respectively.

Let $w\in C_5$. Then $w=\sigma+(\sigma-\sigma_0)e^{it}$, where $\pi/2\le t\le 3\pi/2$, and hence

$$|I_5|=\left|\int_{\pi/2}^{3\pi/2}\frac{H(\sigma+(\sigma-\sigma_0)e^{it})i(\sigma-\sigma_0)e^{it}}{(\sigma-\sigma_0)^2e^{2it}}dt\right|\le\frac{\pi S(\sigma,H)}{\sigma-\sigma_0}.$$

Noting that $S(x,H)=S(x,F)$ for all $x\in(A_0,A)$ and using the above estimates, we obtain

$$|F'(s_0)|\le\frac{1}{2\pi}\sum_{j=1}^5|I_j|\le$$

$$\le\frac{1}{2\pi}\left(2S(\sigma,F)\left|\frac{1}{\sigma-\sigma_0}-\frac{1}{h\operatorname{tg}\theta}\right|+2\int_0^{h/\cos\theta}\frac{S(\sigma+h-t\cos\theta,F)}{|h-te^{i\theta}|^2}dt+\frac{\pi S(\sigma,F)}{\sigma-\sigma_0}\right).$$

Since s_0 is an arbitrary point of the straight line $\{s\in\mathbb{C}:\operatorname{Re}s=\sigma\}$, this implies (14). □

Proof of Theorem 1. Suppose that $A\in(-\infty,+\infty]$, $F\in\mathcal{B}_A$, i.e. $F\in\mathcal{B}_{A_1,A}$ for some $A_1<A$, $\Phi\in\Omega_A$, and $T_\Phi(F)\le 1$. We prove that inequality (13) holds.

For each $\sigma\in(A_0,A)$ put

$$L(\sigma,F)=\frac{S'_+(\sigma,F)}{S(\sigma,F)}.$$

Since the function $\ln S(\sigma,F)$ is convex in (A_1,A), the function $L(\sigma,F)=(\ln S(\sigma,F))'_+$ is well defined and nondecreasing on (A_0,A). In addition, $L(\sigma,F)\le K(\sigma,F)$ for all $\sigma\in(A_0,A)$.

Suppose first that $F\notin\mathcal{C}_A$. Then there exists $\lim_{\sigma\uparrow A}L(\sigma,F)=l\le 0$. Let σ and h be numbers such that $A_1<\sigma-h<\sigma<\sigma+h<A$. For an arbitrary point s_0 on the straight line $\{s\in\mathbb{C}\colon\ \operatorname{Re}s=\sigma\}$ Cauchy's integral formula gives

$$|F'(s_0)|=\frac{1}{2\pi}\left|\int_{|w-\sigma_0|=h}\frac{F(w)}{(w-\sigma_0)^2}dw\right|\le\frac{S(\sigma-h,F)}{h}\le\frac{S(\sigma,F)}{h}e^{-hL(\sigma-h,F)}.$$

This implies

$$K(\sigma,F)\le e^{h|L(\sigma-h,F)|}/h. \tag{15}$$

Let $A=+\infty$. Then, using (15) with $\sigma>A_1+1$ and $h=1$, we see that $K(\sigma,F)=O(1)$ as $\sigma\uparrow A$. Since $\Phi_*^{-1}(\sigma)\to+\infty$ as $\sigma\uparrow A$, we obtain

$$\overline{\lim_{\sigma\uparrow A}}\frac{K(\sigma,F)}{\Phi_*^{-1}(\sigma)}=0. \tag{16}$$

Let $A<+\infty$. If $(A_1+A)/2<\sigma<A$, then, letting $h\uparrow A-\sigma$, we see from (15) that $(A-\sigma)K(\sigma,F)=O(1)$ as $\sigma\uparrow A$. On the other hand, if $\varphi(x)=\widetilde{\Phi}'_+(x)$ for all $x\in\mathbb{R}$, then for every $x>0$ we have

$$x\Phi_*(x)=\widetilde{\Phi}(x)=x\varphi(x)-\Phi(\varphi(x))<xA-\Phi(\varphi(x)).$$

This implies that $(A-\sigma)\Phi_*^{-1}(\sigma)>\Phi(\varphi(\Phi_*^{-1}(\sigma)))$ for each σ sufficiently close to A. Hence $(A-\sigma)\Phi_*^{-1}(\sigma)\to+\infty$ as $\sigma\uparrow A$ and we have again (16).

Therefore, in the case when $F\notin\mathcal{C}_A$ inequality (13) holds.

Suppose now that $F\in\mathcal{C}_A$, i.e. $F\in\mathcal{C}_{A_0,A}$ for some $A_0\in[A_1,A)$.

Fix an arbitrary number $\varepsilon>0$ and let $\theta_0\in(0,\pi/2)$ and $y_0>0$ be numbers such that $u(\theta_0,y_0)=c_0$, where $u(\theta,y)$ is defined by (12).

Since $T_\Phi(F)\le 1$, we have

$$\ln S(\sigma,F)\le(1+\varepsilon)\Phi(\sigma)\ \text{for all}\ \sigma\in(b,A).$$

Using Lemma 3 with $\Psi(\sigma)=\ln S(\sigma,F)/(1+\varepsilon)$, $\sigma\in(A_0,A)$, we see that there exists $\sigma_0\in(A_0,A)$ such that the function $\Phi_*^{-1}(\sigma)$ is definite and positive on the interval (σ_0,A), and in this interval the inequality $L(\sigma,F)\le(1+\varepsilon)\Phi_*^{-1}(\sigma)$ holds. We put $H(\sigma)=(1+\varepsilon)\Phi_*^{-1}(\sigma)$, $\sigma\in(\sigma_0,A)$.

Let $\sigma\in(\sigma_0,A)$ be a fixed point. Then $H(\sigma+x)$ as a function of the variable x is positive, continuous, increasing to $+\infty$ on the interval $(0,A-\sigma)$, and therefore in this interval the equation $H(\sigma+x)=y_0/x$ has a unique solution $h=h(\sigma)$. Put

$$I(\sigma)=\int_0^{h/\cos\theta_0}\frac{S(\sigma+h-t\cos\theta_0,F)}{S(\sigma,F)((t-h\cos\theta_0)^2+h^2\sin^2\theta_0)}dt.$$

Applying (14) with $\theta=\theta_0$, we have

$$K(\sigma,F)=\frac{S(\sigma,F')}{S(\sigma,F)}\le\frac{1}{2(\sigma-\sigma_0)}+\frac{1}{\pi}\left(\left|\frac{1}{\sigma-\sigma_0}-\frac{1}{h\operatorname{tg}\theta_0}\right|+I(\sigma)\right). \tag{17}$$

Let us estimate the integral $I(\sigma)$. Since

$$\ln S(\sigma+h-t\cos\theta_0,F)-\ln S(\sigma,F)\le(h-t\cos\theta_0)L(\sigma+h,F)\le(h-t\cos\theta_0)H(\sigma+h),$$

by applying the Cauchy-Bunyakovsky inequality, we obtain

$$I(\sigma)\le\int_0^{h/\cos\theta_0}\frac{e^{(h-t\cos\theta_0)H(\sigma+h)}}{(t-h\cos\theta_0)^2+h^2\sin^2\theta_0}dt\le$$

$$\le\left(\int_0^{h/\cos\theta_0}e^{2(h-t\cos\theta_0)H(\sigma+h)}dt\right)^{1/2}\left(\int_0^{h/\cos\theta_0}\frac{dt}{((t-h\cos\theta_0)^2+h^2\sin^2\theta_0)^2}\right)^{1/2}=$$

$$=\left(\frac{e^{2hH(\sigma+h)}-1}{2H(\sigma+h)\cos\theta_0}\right)^{1/2}\left(\frac{\pi}{4h^3\sin^3\theta_0}+\frac{\cos\theta_0}{h^3\sin^2\theta_0}\right)^{1/2}.$$

Using the equality $hH(\sigma+h)=y_0$, inequality (17), and the above estimate for $I(\sigma)$, we have

$$K(\sigma,F)\le\frac{1}{2(\sigma-\sigma_0)}+\frac{1}{\pi(\sigma-\sigma_0)}+\frac{1}{\pi h\operatorname{tg}\theta_0}+\frac{I(\sigma)}{\pi}\le$$
$$\le\frac{\pi+2}{2\pi(\sigma-\sigma_0)}+\frac{1}{\pi h\operatorname{tg}\theta_0}+\frac{1}{\pi}\left(\frac{e^{2hH(\sigma+h)}-1}{2H(\sigma+h)\cos\theta_0}\right)^{1/2}\left(\frac{\pi}{4h^3\sin^3\theta_0}+\frac{\cos\theta_0}{h^3\sin^2\theta_0}\right)^{1/2}=$$
$$=\frac{\pi+2}{2\pi(\sigma-\sigma_0)}+\frac{H(\sigma+h)}{\pi y_0\sin\theta_0}\left(\cos\theta_0+\left(\frac{e^{2y_0}-1}{2y_0}\left(\frac{\pi}{2\sin 2\theta_0}+1\right)\right)^{1/2}\right)=$$
$$=\frac{\pi+2}{2\pi(\sigma-\sigma_0)}+H(\sigma+h)u(\theta_0,y_0).$$

Therefore, for all $\sigma\in(\sigma_0,A)$ we obtain

$$K(\sigma,F)\le\frac{\pi+2}{2\pi(\sigma-\sigma_0)}+(1+\varepsilon)\Phi_*^{-1}(\sigma+h(\sigma))c_0. \tag{18}$$

Since for all $\sigma\in(\sigma_0,A)$ we have

$$h(\sigma)=\frac{y_0}{(1+\varepsilon)\Phi_*^{-1}(\sigma+h(\sigma))}\le\frac{y_0}{(1+\varepsilon)\Phi_*^{-1}(\sigma)},$$

by Lemma 2 we obtain $\Phi_*^{-1}(\sigma+h(\sigma))\sim\Phi_*^{-1}(\sigma)$ as $\sigma\uparrow A$. Therefore, as we see from (18),

$$\varlimsup_{\sigma\uparrow A}\frac{K(\sigma,F)}{\Phi_*^{-1}(\sigma)}\le(1+\varepsilon)c_0.$$

Since $\varepsilon>0$ is arbitrary, this implies (13). □

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