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M. Krasnoshchok

MONOTONE ITERATIONS METHOD FOR FRACTIONAL DIFFUSION EQUATIONS

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In recent years, there has been a growing interest on non-local models because of their relevance in many practical applications. A widely studied class of non-local models involves fractional order operators. They usually describe anomalous diffusion. In particular, these equations provide a more faithful representation of the long-memory and nonlocal dependence of diffusion in fractal and porous media, heat flow in media with memory, dynamics of protein in cells etc.

For $a \in (0,1)$, we investigate the nonautonomous fractional diffusion equation:

$$D^a_{*,t}u - Au = f(x,t,u),$$

where $D^a_{*,t}$ is the Caputo fractional derivative and A is a uniformly elliptic operator with smooth coefficients depending on space and time. We consider these equations together with initial and quasilinear boundary conditions.

The solvability of such problems in Hölder spaces presupposes rigid restrictions on the given initial data. These compatibility conditions have no physical meaning and, therefore, they can be avoided, if the solution is sought in larger spaces, for instance in weighted Hölder spaces.

We give general existence and uniqueness result and provide some examples of applications of the main theorem. The main tool is the monotone iterations method. Preliminary we developed the linear theory with existence and comparison results. The principle use of the positivity lemma is the construction of a monotone sequences for our problem. Initial iteration may be taken as either an upper solution or a lower solution. We provide some examples of upper and lower solution for the case of linear equations and quasilinear boundary conditions. We notice that this approach can also be extended to other problems and systems of fractional equations as soon as we will be able to construct appropriate upper and lower solutions.

Introduction. Let $D^a_{*,t}u(x,t)$ $(a \in (0,1))$ be the Caputo derivative of order a (see Definition 3.2 in [16])

$$D^a_{*,t}u(x,t) = \frac{1}{\Gamma(1-a)}\frac{\partial}{\partial t}\int\limits_0^t (t-s)^{-a}(u(x,s)-u(x,0))ds,$$

here $\Gamma(\cdot)$ is Euler's Gamma function.

Let Q be bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth measure $S \in C^{2+\alpha}$, $\alpha \in (0,1)$ and $\nu = \nu(x)$ be the unit outward normal to S. Denote $Q_T = Q \times (0,T)$, $S_T = S \times (0,T)$, $T > 0$.

We need to find the function $u(x,t)$ satisfying the equation

$$D^a_{*,t}u(x,t) - A(x,t,\partial_x)u(x,t) = f(x,t,u(x,t)),\ (x,t) \in Q_T, \tag{1}$$

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with the initial and boundary conditions

$$u(x,0) = \psi(x),\ x \in Q, \tag{2}$$

$$B(x,t,\partial_x)u(x,t) = g(x,t,u(x,t)),\ (x,t) \in S_T, \tag{3}$$

where $f(x,t,u)$, $g(x,t,u)$ are some given functions, A are B strongly elliptic operator and conormal derivative operator respectively

$$A(x,t,\partial_x)u = \sum_{i,j=1}^{n} a_{i,j}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i} + a_0(x,t)u,$$

$$B(x,t,\partial_x)u = \sum_{i,j=1}^{n} \nu_j a_{i,j}(x,t)u_{x_i} + b(x,t)u.$$

In this paper we prove existence and uniqueness result to the problem (1)–(3) in weighted Hölder spaces. The main tool is a method of monotone iterations. The paper is organized as follows. In Section 1 we define the functional spaces and state the main result. Section 2 is devoted to some auxiliary results concerning the properties of the weighted Hölder spaces and of solutions to linear fractional equations, which will play a key role in the investigation. The main theorem is proved in Section 3. In section 4 we apply this theorem to some specific problems which involve nonlinear boundary conditions. Appendix contains the proof of some auxiliary assertion from Section 2.

Fractional partial differential equations received much attention in the literature because of numerous applications in physics, chemistry, hydrology and engineering (see for example [24], [45], [56]). The well-posedness of initial boundary problems for fractional diffusion equations established in [22], [29], [34], [55] in the spaces of continuous functions, in [41] in a spaces of generalized functions and in [67], [68],[4], [49], [58] in Sobolev spaces.

The solvability of parabolic problems in Hölder spaces presupposes rigid restrictions on the given initial data. These compatibility conditions have no physical meaning and, therefore, they can be avoided, if the solution is sought in larger spaces, for instance in weighted Hölder spaces (see [7], [7], [17], [61], [8], [9], [10]). We mention that the authors of [13], [51], [5], [47] consider fractional diffusion equation in weighted Hölder spaces with other characteristics and properties. Viscous solutions are investigated in [1], [53], [62]. We mention also papers [28], [19], studying equivalence of various definitions of solutions. Maximum and comparison principles were derived in [2], [11], [21], [40], [42], [43], [44], [32], [64], [3], [26]. Long time behaviour of solutions to fractional diffusion equations is the subject of the papers [15], [23], [64], [65]. Applications of monotone iterations method may be found for example in [60], [6], [12], [54], [39], [31], where classical parabolic equations and systems are considered. Papers [37], [25], [66] devoted to monotone iteration methods for ordinary equations with fractional derivatives. This method was developed in [50], [52], [21] for fractional diffusion equation where elliptic operators independent of time variable in Sobolev spaces.

Remark 1. To establish an existence theorem for our problem we use the method of upper and lower solutions and its associated monotone iteration. The basic idea of this method is that by using an upper solution or a lower solution as the initial iterations in a suitable iterative process. The resulting sequence of iterations is monotone and converges to a solution of our problem. We consider a linear problems on every step of iterations. One can evaluate the error of approximations by the difference of upper and lower iterations. We follow a standard approach and the main difficulty is a to derive comparison principle for fractional diffusion equations. On this way our proofs rely on [32], [34]. We consider only quasilinear conormal boundary condition since the papers we know are devored to the Dirichlet or homogeneous Neumann problems. These boundary conditions can be also investigated by the same approach.

1. Functional spaces. Main result. Let G be a bonded or unbounded domain $\mathbb{R}^n$, $G_T = G \times (0,T)$. We denote $D_x^p = \frac{\partial^{|p|}}{\partial_{x_1}^{p_1} \dots \partial_{x_N}^{p_N}}$ where $|p| = p_1 + p_2 + \dots + p_N$. We set

$$|w|_G = \sup_{x\in G} |w(x)|, \quad |w|_{G_T} = \sup_{(x,t)\in G_T} |w(x,t)|,$$

and for $\alpha \in (0,1)$

$$\langle w\rangle_{x,G_T}^{(\alpha)} = \sup_{\tau\in(0,T)} \sup_{x,y\in G} |w(x,\tau) - w(y,\tau)||x-y|^{-\alpha},$$

$$\langle w\rangle_{t,G_T}^{(\alpha)} = \sup_{\tau',\tau''\in(0,T)} \sup_{x\in G} |w(x,\tau') - w(x,\tau'')||\tau'-\tau''|^{-\alpha}, \quad \langle w\rangle_{a,G_T}^{(\alpha)} = \langle w\rangle_{x,G_T}^{(\alpha)} + \langle w\rangle_{t,G_T}^{(\frac{\alpha}{2}a)}.$$

Let r be positive noninteger number. The Hölder spaces $C_a^r(G_T)$ is defined by the norm

$$|w|_{\alpha,G_T}^{(r)} = \sum_{|p|+2m\le[r]} \left|\left(D_{*,t}^a\right)^m D_x^p w\right|_{G_T} + \langle w\rangle_{a,G_T}^{(r)},$$

where $[r]$ is the whole part of r and

$$\langle w\rangle_{a,G_T}^{(r)} = \langle w\rangle_{a,x,G_T}^{(r)} + \langle w\rangle_{t,G_T}^{(\frac{r}{2}a)}, \quad \langle w\rangle_{a,x,G_T}^{(r)} = \sum_{|p|+2m=[r]} \langle\left(D_{*,t}^a\right)^m D_x^p w\rangle_{x,G_T}^{(r-[r])},$$

$$\langle w\rangle_{t,G_T}^{(\frac{r}{2}a)} = \sum_{0<r-|p|-2m<2} \langle\left(D_{*,t}^a\right)^m D_x^p w\rangle_{t,G_T}^{(\frac{r-|p|-2m}{2}a)}.$$

Let $q \in [0,r]$. Similarly to [8], [9] we define the weighted Hölder space $C_{q,a}^r(G_T)$ with the norm

$$|w|_{q,a,G_T}^{(r)} = \sup_{t\in(0,T)} t^{\frac{r-q}{2}a} \langle w\rangle_{a,G_t'}^{(r)} + \sum_{q<|p|+2m<r} \sup_{t\in(0,T)} t^{\frac{2m+|p|-q}{2}a} \left|\left(D_{*,t}^a\right)^m D_x^p w(\cdot,t)\right|_G + |w|_{a,G_T}^{(q)},$$

where $G_T' = G \times (t/2, t)$.

If $q < 0$, then space $C_{q,a}^r(G_T)$ is the space with the norm

$$|w|_{q,a,G_T}^{(r)} = \sup_{t\in(0,T)} t^{\frac{r-q}{2}a} \langle w\rangle_{a,Q_t'}^{(r)} + \sum_{0\le|p|+2m<r} \sup_{t\in(0,T)} t^{\frac{2m+|p|-q}{2}a} \left|\left(D_{*,t}^a\right)^m D_x^p w(\cdot,t)\right|_Q.$$

Remark 2. Let $\varphi(x) \in C^\alpha(G)$ and $q \in (\alpha, 1)$. Then the function $\Phi(x,t) = t^{-s}\varphi(x)$ belongs to $C_{-q,a}^\alpha(G_T)$ for $s \in \left[\frac{\alpha}{2}a, \frac{\alpha+q}{2}a\right]$.

Next we give assumptions on the data of our problem. Operators A, B are such that

H_e) $\mu_1|\xi|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le \mu_2|\xi|^2$, $(x,t)\in Q_T$, $0<\mu_1\le\mu_2$;

H_d) $a_{ij} \in C_a^\alpha(Q_T)$, $i,j = 1,\dots,n$; $a_k \in C_a^\alpha(Q_T)$, $k = 0,1,\dots,n$;

H_b) $a_{ij} \in C_a^{1+\alpha}(S_T)$, $i,j = 1,\dots,n$; $b \in C_a^{1+\alpha}(S_T)$.

For any positive numbers ρ, T we set $\Omega_{\rho,T} = \{(x,t,u) : x \in \overline{Q}, t \in (0,T], u \in [-\rho,\rho]\}$, $S_{\rho,T} = \{(x,t,u) : x \in \overline{S}, t \in (0,T], u \in [-\rho,\rho]\}$. We also assume

H_c) functions $f(x,t,u)$ i $g(x,t,u)$ are continuous in $\Omega_{\rho,T}$ and $S_{\rho,T}$ for all $\rho > 0$, T;

H_p) partial derivative $f_u(x,t,u)$ is continuous in $\Omega_{\rho,T}$ and partial derivatives $g_x(x,t,u)$, $g_u(x,t,u)$, $g_{xu}(x,t,u)$, $g_{uu}(x,t,u)$ are continuous in $S_{\rho,T}$;

H_r) there are constants $M_{\rho,T}$, $L_{\rho,T}$ such that

$$\sup_{u\in[-\rho,\rho]} |f(\cdot,\cdot,u)|^{(\alpha)}_{\beta-2,a,Q_T} + \sup_{u\in[-\rho,\rho]} |g(\cdot,\cdot,u)|^{(\alpha)}_{\beta-2,a,S_T} \le M_{\rho,T}; \tag{4}$$

$$\sup_{u\in[-\rho,\rho]} |f_u(\cdot,\cdot,u)|_{Q_T} + \sup_{u\in[-\rho,\rho]} (|g_u(\cdot,\cdot,u)|_{S_T} +$$
$$+|g_u(\cdot,\cdot,u)|_{S_T} + |g_{xu}(\cdot,\cdot,u)|_{S_T} + |g_{uu}(\cdot,\cdot,u)|_{S_T}) \le L_{\rho,T}. \tag{5}$$

Definition 1. A function $\overline{u}(x,t) \in C^{2+\alpha}_{\beta,a}(Q_T)$ is said to be *an upper solution* of (1)–(3), if

$$D^a_{*,t}\overline{u}(x,t) - A(x,t,\partial_x)\overline{u}(x,t) \ge f(x,t,\overline{u}),\ (x,t) \in Q_T,$$
$$\overline{u}(x,0) \ge \psi(x),\ x \in Q,$$
$$B(x,t,\partial_x)\overline{u}(x,t) \ge g(x,t,\overline{u}),\ (x,t) \in S_T.$$

Definition 2. A function $\underline{u}(x,t) \in C^{2+\alpha}_{\beta,a}(Q_T)$ is said to be *a lower solution* of (1)–(3), if

$$D^a_{*,t}\underline{u}(x,t) - A(x,t,\partial_x)\underline{u}(x,t) \le f(x,t,\underline{u}),\ (x,t) \in Q_T,$$
$$\underline{u}(x,0) \le \psi(x),\ x \in Q,$$
$$B(x,t,\partial_x)\underline{u}(x,t) \le g(x,t,\underline{u}),\ (x,t) \in S_T.$$

Definition 3. An upper solution $\overline{u}(x,t)$ and a lower solution $\underline{u}(x,t)$ of (1)–(3) are *ordered*, if $\overline{u}(x,t) \ge \underline{u}(x,t),\ (x,t) \in \overline{Q}_T$.

We set

$$\underline{U} = \inf_{(x,t)\in Q_T} \underline{u}(x,t), \quad \overline{U} = \sup_{(x,t)\in Q_T} \overline{u}(x,t), \quad \rho = \max\{|\underline{U}|, |\overline{U}|\}. \tag{6}$$

For any u_1, u_2 such that $-\rho \le u_2 \le u_1 \le \rho$, we get by (4), (5) the following inequalities

$$-L(u_1 - u_2) \le f(t,x,u_1) - f(x,t,u_2) \le L(u_1 - u_2),\ (x,t) \in Q_T$$
$$-L(u_1 - u_2) \le g(t,x,u_1) - g(x,t,u_2) \le L(u_1 - u_2),\ (x,t) \in S_T, \tag{7}$$

here $L \equiv L_{\rho,T}$. We emphasize L that $\underline{U}$, $\overline{U}$.

The main result of this paper is as follows.

Theorem 1. *Assume that a) $\alpha \in (0,1)$, $\beta \in (\alpha,1)$, $T > 0$; b) $\psi \in C^{\beta}(\overline{Q}_T)$; c) hypothesis H_e, H_d, H_b, H_c, H_p, H_r are valid; d) there are exist ordered upper $\overline{u}$ and lower $\underline{u}$ solutions of* (1)–(3)*. Then classical solution $u \in C^{2+\alpha}_{\beta,a}(Q_T)$ of* (1)–(3) *exists such that*

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t), \quad (x,t) \in Q_T.$$

2. Preliminaries. For $\sigma > 0$ we set

$$\omega_\sigma(t) = \frac{t^{\sigma-1}}{\Gamma(\sigma)}, \quad (\omega_\sigma * f)(t) = \int_0^t \omega_\sigma(t-\tau) f(\tau)d\tau.$$

We denote by $\mathbf{D}^a_{*,t}u(x,t)$ the Marchaud derivative of the function $u(x,t) - u(x,0)$ (see §13 in [59])

$$\mathbf{D}^a_{*,t}u(x,t) = \frac{u(x,t) - u(x,0)}{\Gamma(1-a)\, t^a} + \frac{a}{\Gamma(1-a)} \int_0^t \frac{u(x,t) - u(x,\tau)}{(t-\tau)^{1+a}} d\tau.$$

The most comprehensive study of the Marchaud derivative in Hölder and Sobolev spaces may be found in [59] (see also [1], [33], [53], [63]). In the same way as in [59] we prove an equivalence of Caputo and Marchaud derivatives this time in $C^{2+\alpha}_{\beta,a}(Q_T)$ (see Lemma 4 below). The Marchaud derivative is more suitable in order to establish a comparison principle (see Theorem 3).

We study properties of weighted Hölder classes $C^{2+\alpha}_{\beta,a}(Q_T)$ in the next three lemmas. Throughout below we assume that $\beta \in (\alpha, 1)$ as in Theorem 1.

Lemma 1. *Let $u \in C^{2+\alpha}_{\beta,a}(G_T)$, $u_0(x) = u(x,0) \in C^{\beta}(G)$. Then we have*

$$u(x,t) - u_0(x) = (\omega_a * D^a_{*,t}u)(x,t). \tag{8}$$

Proof. It is easy to observe that $(\omega_a * \omega_{1-a})(t) = \omega_1(t) = 1$ when $a \in (0,1)$. For short we don't indicate the dependence on x. It is clear that

$$|(\omega_{1-a} * (u-u_0))(t)| \le c \sup_{Q_t} |u-u_0| t^{1-a}, \quad \left| \frac{\partial}{\partial t} (\omega_{1-a} * (u-u_0))(t) \right| = |D^a_{*,t}u(t)| \le |u|^{2+\alpha}_{\beta,a,Q_t} t^{\frac{\beta-2}{2}a},$$

and $(\omega_{1-a} * (u-u_0))(0) = 0$, $(\omega_{1-a} * (u-u_0)) \in W^{1,1}(Q_T)$, so

$$(\omega_{1-a} * (u-u_0))(t) = \int_0^t D^a_{*,\tau}u(\tau)d\tau = (\omega_1 * D^a_{*,\tau}u)(t).$$

We follow arguments of Lemma 1.3 in [35]. First we see that $(\omega_a * (\omega_{1-a} * (u-u_0)))(t) =$

$$= ((\omega_a * \omega_{1-a}) * (u-u_0))(t) = (\omega_1 * (u-u_0))(t) = \int_0^t (u-u_0)(\tau)d\tau.$$

On the other hand, one gets

$$(\omega_a * (\omega_{1-a} * (u-u_0)))(t) = (\omega_a * (\omega_1 * D^a_{*,t}u))(t) = ((\omega_a * \omega_1) * D^a_{*,t}u))(t) =$$
$$= ((\omega_1 * \omega_a) * D^a_{*,t}u))(t) = \int_0^t (\omega_a * D^a_{*,\tau}u))(\tau)d\tau$$

and $\int_0^t (u-u_0)(\tau)d\tau = \int_0^t (\omega_a * D^a_{*,\tau}u))(\tau)d\tau$.

The desired formula (8) is an immediately consequence of the last equality. □

Lemma 2. *Let $u \in C^{2+\alpha}_{\beta,a}(G_T)$. Then for all $x \in G$ we have*

$$t^{\frac{2+\alpha-\beta}{2}a} D^a_{*,t}u(x,t) \in C^{\frac{\alpha}{2}a}([0,T]). \tag{9}$$

Proof. Assume to be specific that $0 < \tau < t < T$. We obtain

$$t^{\frac{2+\alpha-\beta}{2}a} D^a_{*,t}u(x,t) - \tau^{\frac{2+\alpha-\beta}{2}a} D^a_{*,\tau}u(x,\tau) =$$
$$= \left(t^{\frac{2+\alpha-\beta}{2}a} - \tau^{\frac{2+\alpha-\beta}{2}a}\right) D^a_{*,t}u(x,t) + \tau^{\frac{2+\alpha-\beta}{2}a}\left(D^a_{*,t}u(x,t) - D^a_{*,\tau}u(x,\tau)\right) = \delta_1 + \delta_2.$$

Since $\alpha < \beta$ one has $|\delta_1| \le c(t-\tau)^{\frac{2+\alpha-\beta}{2}a} t^{\frac{\beta-2}{2}} |D^a_{*,t}u(x,t)| \le c|u|^{(2+\alpha)}_{\beta,a,Q_T}|(t-\tau)^{\frac{\alpha}{2}a}$.

In the case of $\frac{t}{2} < \tau < t$, $|\delta_2| \le |u|^{(2+\alpha)}_{\beta,a,Q_T}|(t-\tau)^{\frac{\alpha}{2}a}$.

Conversely, if $0 < \tau \le \frac{t}{2}$, then $\frac{t}{2} \le t - \tau$ and it follows

$$|\delta_2| \le \tau^{\frac{2+\alpha-\beta}{2}a}\left(|D^a_{*,t}u(x,t)| + D^a_{*,\tau}u(x,\tau)|\right)\left(\frac{t}{2}\right)^{-\frac{\alpha}{2}a} |t-\tau|^{\frac{\alpha}{2}a} \le$$
$$\le c\tau^{\frac{\alpha}{2}a} t^{-\frac{\alpha}{2}a} |u|^{(2+\alpha)}_{\beta,a,Q_T}|(t-\tau)^{\frac{\alpha}{2}a}.$$

□

Assume $\lambda_* = \min\left\{\frac{\alpha}{2}a, 1-a\right\}$, $\lambda \in (0, \lambda_*)$, so $\lambda + a < 1$.

One gets the following result immediately from Theorem 3.3 in [59] and Lemma 2.

Lemma 3. *If $u \in C^{2+\alpha}_{\beta,a}(G_T)$, then for all $x \in G$ we have*

$$t^{\frac{2+\alpha-\beta}{2}a}(\omega_a * D^a_{*,t}u)(x,t) \in C^{\lambda+a}([0,T]). \tag{10}$$

One can obtain as a consequence of Corollary of Lemma 13.2 in [59] and Lemmas 1, 3 the following lemma.

Lemma 4. *The Caputo and Marchaud derivatives are coincide on $C^{2+\alpha}_{\beta,a}(Q_T)$.*

Next we formulate two auxiliary assertions. We use first lemma (Lemma 5) below in the proof of the main result. Second lemma (Lemma 6) is of importance in its own right and may be useful in further studies.

Lemma 5 (interpolation inequalities)**.** *For any parameter $\varepsilon > 0$ there exists a constant C_ε such that*

$$|u|^{(\alpha)}_{\beta-2,a,G_T} \le \varepsilon|u|^{(2+\alpha)}_{\beta,a,G_T} + C_\varepsilon|u|_{G_T}, \tag{11}$$

$$|u|^{(1+\alpha)}_{\beta-1,a,G_T} \le \varepsilon|u|^{(2+\alpha)}_{\beta,a,G_T} + C_\varepsilon|u|_{G_T}. \tag{12}$$

Proof. One can easily see that higher seminorms in $C^{\alpha}_{\beta-2,a}$ and $C^{1+\alpha}_{\beta-1,a}$ have the same weight $t^{\frac{2+\alpha-\beta}{2}a}$, as higher seminorms in $C^{2+\alpha}_{\beta,a}$. Besides Hölder exponents of the function u (and its first derivatives with respect to x) in $C^{\alpha}_{\beta-2,a}$ and ($C^{1+\alpha}_{\beta-1,a}$) are less than corresponding exponents in $C^{2+\alpha}_{\beta,a}$. We apply the reasoning quite similar to §33 in [48]. For clarity we consider the Hölder constant of $u \in C^{\alpha}_{\beta-2,a}(G_T)$ with respect to t.

Let $\tau < \sigma$ and $\delta > 0$. If $\sigma - \tau < \delta$ then we have

$$\frac{|u(x,\tau)-u(x,\sigma)|}{|\sigma-\tau|^{\frac{\alpha}{2}a}} \le \frac{|u(x,\tau)-u(x,\sigma)|}{|\sigma-\tau|^{\frac{\beta}{2}a}}(\sigma-\tau)^{\frac{\beta-\alpha}{2}a},$$

and

$$\frac{|u(x,\tau)-u(x,\sigma)|}{|\sigma-\tau|^{\frac{\alpha}{2}a}} \le |u|^{(2+\alpha)}_{\beta,a,G_T}\delta^{\frac{\beta-\alpha}{2}a}.$$

Otherwise if $\sigma - \tau > \delta$, we obtain

$$\frac{|u(x,\tau)-u(x,\sigma)|}{|\sigma-\tau|^{\frac{\alpha}{2}a}} \le 2|u|_{G_T}\delta^{-\frac{\alpha}{2}a}.$$

We take $\delta^{\frac{\beta}{2}a} = \frac{|u|^{(2+\alpha)}_{\beta,a,G_T}}{|u|_{Q_T}}$ and get

$$\sup_{t\in(0,T)} t^{\frac{2+\alpha-\beta}{2}a}\langle u\rangle^{(\frac{\alpha}{2}a)}_{t,G_T} \le C(T)\left(|u|^{(2+\alpha)}_{\beta,a,G_T}\right)^{\frac{\alpha}{\beta}}\left(|u|_{G_T}\right)^{1-\frac{\alpha}{\beta}}.$$

The rest of Hölder constants are studied by similar arguments. Then we apply Young's inequality to obtained inequalities. □

Lemma 6. *If $u \in C^{2+\alpha}_{\beta,a}(G_T)$ then for all $x \in G$ we have*

$$\int_0^t \frac{|u(x,t)-u(x,\tau)|}{(t-\tau)^{1+a}}d\tau \le c\,t^{-\frac{2-\beta}{2}a}.$$

The last estimate validates that Marchaud derivative in $C^{2+\alpha}_{\beta,a}(Q_T)$ have the same singularity near $t=0$ as Caputo derivative. This lemma is proved in Appendix.

We consider a linear problem

$$\begin{gathered} D^{a}_{*,t}\mathrm{u}(x,t) - A(x,t,\partial_x)\mathrm{u}(x,t) = \mathrm{f}(x,t),\ (x,t)\in Q_T,\\ \mathrm{u}(x,0)=\psi(x),\ x\in Q,\\ B(x,t,\partial_x)\mathrm{u}(x,t)=\mathrm{g}(x,t),\ (x,t)\in S_T. \end{gathered} \tag{13}$$

The following result is true (see Theorem 3.3 in [34]).

Theorem 2. *Let a) the assumptions H_e), H_d), H_b) are valid; b) $\alpha\in(0,1)$, $\beta\in(\alpha,1)$; c) $\psi\in C^{\beta}(Q)$, $\mathrm{f}\in C^{\alpha}_{\beta-2,a}(Q_T)$. Then there exists a unique solution $\mathrm{u}\in C^{2+\alpha}_{\beta,a}(Q_T)$. This solution satisfies the following estimate*

$$|\mathrm{u}|^{(2+\alpha)}_{\beta,a,Q_T} \le C(T)\left(|\psi|^{(\beta)}_{Q} + |\mathrm{f}|^{(\alpha)}_{\beta-2,a,Q_T} + |\mathrm{g}|^{(1+\alpha)}_{\beta-1,a,S_T}\right). \tag{14}$$

By Lemma 4 we can repeat the same arguments as in Theorem 4.3 in [32].

Theorem 3. *Let the constants A_0, B_0 are such that*

$$a_0(x,t)\ge -A_0,(x,t)\in Q_T,\quad b(x,t)\ge B_0,(x,t)\in S_T$$

and besides $\mathrm{f}(x,t)\ge 0,\ (x,t)\in Q_T,\quad \mathrm{g}(x,t)\ge 0,(x,t)\in S_T,\quad \psi(x)\ge 0,\ x\in Q.$
The solution $\mathrm{u}\in C^{2+\alpha}_{\beta,a,}(Q_T)$ *of* (13) *is nonnegative*$\mathrm{u}(x,t)\ge 0,\quad (x,t)\in Q_T$.

3. Monotone iterations method. We return to the problem (1)–(3) and Theorem 1. We are following to the approaches of Chapter 4 in [54] and Lection 25 in [31].

We denote $M\equiv M_{\rho,T}$ (see (5)), where the parameter ρ is chosen in (6).

We set

$$\begin{gathered} -\mathcal{A}(x,t,\partial_x)u = -A(x,t,\partial_x)u + Lu,\quad \mathcal{F}(x,t,u)=Lu+f(x,t,u),\\ \mathcal{B}(x,t,\partial_x)u = B(x,t,\partial_x)u + Lu,\quad \mathcal{G}(x,t,u)=Lu+g(x,t,u). \end{gathered} \tag{15}$$

We define the successive terms of the approximation sequences u_k as solutions of the following initial-boundary problems

$$\begin{gathered} D^{a}_{*,t}u_k(x,t) - \mathcal{A}(x,t,\partial_x)u(x,t) = \mathcal{F}(x,t,u_{k-1}),\ (x,t)\in Q_T,\\ u_k(x,0)=\psi(x),\ x\in Q,\\ \mathcal{B}(x,t,\partial_x)u_k(x,t)=\mathcal{G}(x,t,u_{k-1}),\ (x,t)\in S_T. \end{gathered} \tag{16}$$

Denote the sequence with the initial iteration $u_0=\overline{u}$ by $\{\overline{u}_k\}$ and the sequence with $u_0=\underline{u}$ by $\{\underline{u}_k\}$, and refer to them as upper and lower sequences, respectively. Theorem 2 is sequentially applied on every step of iterations, so $\overline{u}_k$, $\underline{u}_k\in C^{2+\alpha}_{\beta,a}(Q_T)$ for all k.

Then we establish monotonocity of the upper and lower sequences.

Lemma 7. *Let $\underline{u}$, $\overline{u}$ be ordered lower and upper solutions of* (1)–(3)*. Assume that function f, g satisfy* (4), (5) *i* (7)*. Then the sequences $\{\underline{u}_k\}$, $\{\overline{u}_k\}$ are monotone, i.e.*

$$\underline{u}=\underline{u}_0\le\underline{u}_k\le\underline{u}_{k+1}\le\overline{u}_{k+1}\le\overline{u}_k\le\overline{u}_0=\overline{u},$$

namely lower sequence $\{\underline{u}_k\}$ is increasing and upper sequence $\{\overline{u}_k\}$ is decreasing.

Proof. We compare zeroth and first iterations. Consider $\{\overline{u}_0\}$ i $\{\overline{u}_1\}$. We set

$$w(x,t) = \overline{u}_0(x,t) - \overline{u}_1(x,t) = \overline{u}(x,t) - \overline{u}_1(x,t),$$

then

$$D^a_{*,t} w(x,t) - \mathcal{A}(x,t,\partial_x) w(x,t) \geq \mathcal{F}(x,t,\overline{u}_0) - \mathcal{F}(x,t,\overline{u}_0) = 0,\ (x,t) \in Q_T,$$
$$w(x,0) \geq \overline{u}_0(x,0) - \psi(x) \geq 0,\ x \in Q,$$
$$\mathcal{B}(x,t,\partial_x) w(x,t) \geq \mathcal{G}(x,t,\overline{u}_0) - \mathcal{G}(x,t,\overline{u}_0) = 0,\ (x,t) \in S_T.$$

In view of Theorem 3, $w(x,t) \geq 0$ which leads to $\overline{u}_1(x,t) \leq \overline{u}_0(x,t),\ (x,t) \in Q_T$. In a similar way we get $\underline{u}_1(x,t) \geq \underline{u}_0(x,t), \quad (x,t) \in Q_T$.

Then we compare first lower and upper iterations. For $w_1(x,t) = \overline{u}_1(x,t) - \underline{u}_1(x,t)$, we derive that

$$D^a_{*,t} w_1(x,t) - \mathcal{A}(x,t,\partial_x) w_1(x,t) = \mathcal{F}(x,t,\overline{u}_0) - \mathcal{F}(x,t,\underline{u}_0) \geq 0,\ (x,t) \in Q_T,$$
$$w_1(x,0) = 0,\ x \in Q,$$
$$\mathcal{B}(x,t,\partial_x) w_1(x,t) = \mathcal{G}(x,t,\overline{u}_0) - \mathcal{G}(x,t,\underline{u}_0) \geq 0,\ (x,t) \in S_T.$$

By Theorem 3 $w_1(x,t) \geq 0$, thus

$$\underline{u}(x,t) = \underline{u}_0(x,t) \leq \underline{u}_1(x,t) \leq \overline{u}_1(x,t) \leq \overline{u}_0(x,t) = \overline{u}(x,t), \quad (x,t) \in Q_T.$$

Finally we suppose that

$$\underline{u}_0(x,t) \leq ... \leq \underline{u}_{k-1}(x,t) \leq \underline{u}_k(x,t) \leq \overline{u}_k(x,t) \leq \overline{u}_{k-1}(x,t) \leq ... \leq \overline{u}_0(x,t), \quad (x,t) \in Q_T.$$

and consider $k+1$ step of iterations. The function $w_k(x,t) = \overline{u}_k(x,t) - \underline{u}_{k+1}(x,t)$, satisfies

$$D^a_{*,t} w_k(x,t) - \mathcal{A}(x,t,\partial_x) w_k(x,t) = \mathcal{F}(x,t,\overline{u}_{k-1}) - \mathcal{F}(x,t,\underline{u}_k) \geq 0,\ (x,t) \in Q_T,$$
$$w_k(x,0) = 0,\ x \in Q,$$
$$\mathcal{B}(x,t,\partial_x) w_k(x,t) = \mathcal{G}(x,t,\overline{u}_{k-1}) - \mathcal{G}(x,t,\underline{u}_k) \geq 0,\ (x,t) \in S_T.$$

Theorem 3 allows to conclude that $w_k(x,t) \geq 0$, i.e. $\overline{u}_k(x,t) \geq \overline{u}_{k+1}(x,t), \quad (x,t) \in Q_T$. In the same way we ensure that

$$\underline{u}_k(x,t) \leq \underline{u}_{k+1}(x,t), \quad \underline{u}_{k+1}(x,t) \leq \overline{u}_{k+1}(x,t) \quad (x,t) \in Q_T.$$

This completes the proof of Lemma 7. □

By monotone convergence theorem there exist pointwise limits $\underline{w}$, $\overline{w}$:

$$\lim_{k\to\infty} \underline{u}_k(x,t) = \underline{w}(x,t), \quad \lim_{k\to\infty} \overline{u}_k(x,t) = \overline{w}(x,t),$$
$$\underline{u}(x,t) \leq \underline{w}(x,t) \leq \overline{w}(x,t) \leq \overline{u}(x,t), \quad (x,t) \in Q_T.$$

Lemma 8. *The function* $\overline{w} \in C^{2+\alpha}_{\beta,a}(Q_T)$ *is a classical solution of* (1)–(3).

Proof. Monotonicity of $\{\overline{u}_k\}$ implies monotonicity of $\{\mathcal{F}(x,t,\overline{u}_k))\}$, $\{\mathcal{G}(x,t,\overline{u}_k))\}$. It is clear that there limits $\mathcal{F}(x,t,\overline{u})$ and $\mathcal{G}(x,t,\overline{u})$ respectively.

By estimate (14) it follows

$$|\overline{u}_k|^{(2+\alpha)}_{\beta,a,Q_T} \leq C(T) \left(|\psi|^{(\beta)}_Q + |\mathcal{F}(x,t,\overline{u}_{k-1})|^{(\alpha)}_{\beta-2,a,Q_T} + |\mathcal{G}(x,t,\overline{u}_{k-1})|^{(1+\alpha)}_{\beta-1,a,S_T} \right). \qquad (17)$$

Then (see (4), (5)) we obtain

$$|\mathcal{F}(x,t,\overline{u}_{k-1})|^{(\alpha)}_{\beta-2,a,Q_T} \le L|\overline{u}_{k-1}|^{(\alpha)}_{\beta-2,a,Q_T} + N_1 \le$$
$$\le \varepsilon L|\overline{u}_{k-1}|^{(2+\alpha)}_{\beta,a,Q_T} + C_\varepsilon|\overline{u}_{k-1}|_{Q_T} + N_1 \le \varepsilon L|\overline{u}_{k-1}|^{(2+\alpha)}_{\beta,a,Q_T} + C_\varepsilon\rho + N_1, \quad (18)$$

and similarly (see (12))

$$|\mathcal{G}(x,t,\overline{u}_{k-1})|^{(1+\alpha)}_{\beta-1,a,S_T} \le \varepsilon L|\overline{u}_{k-1}|^{(2+\alpha)}_{\beta,a,Q_T} + C_\varepsilon\rho + N_2. \quad (19)$$

Now we take ε so small that $\varepsilon = 2C(T)L\varepsilon < 1$.

Thus we see from (17)–(19) that there exists the constant N_3 such that

$$|\overline{u}_k|^{(2+\alpha)}_{\beta,a,Q_T} \le \varepsilon|\overline{u}_{k-1}|^{(2+\alpha)}_{\beta,a,Q_T} + N_3,$$

and

$$|\overline{u}_k|^{(2+\alpha)}_{\beta,a,Q_T} \le \varepsilon(\varepsilon|\overline{u}_{k-2}|^{(2+\alpha)}_{\beta,a,Q_T} + N_3) + N_3 \le \dots$$
$$\dots \le \varepsilon^k|\overline{u}_0|^{(2+\alpha)}_{\beta,a,Q_T} + N_3\sum_{m=1}^{k-1}\varepsilon^m \le |\overline{u}_0|^{(2+\alpha)}_{\beta,a,Q_T} + N_3\sum_{m=0}^{\infty}\varepsilon^m.$$

Eventually we have $|\overline{u}_k|^{(2+\alpha)}_{\beta,a,Q_T} \le N_4$.

We recall $C^\beta_a(Q_T) \subseteq C^{2+\alpha}_{\beta,a}(Q_T)$. By Arzelà–Ascoli theorem there exists subsequence $\{\overline{u}_k\}$ (we keep the same notation) such that

$$\overline{u}_k \rightrightarrows \overline{w}, \text{ in } \overline{Q} \times [0,T],$$
$$D^p_x\overline{u}_{k_l} \rightrightarrows D^p_x\overline{w} \text{ in } \overline{Q} \times [t_0,T] \text{ with } |p| \le 2,$$
$$D^a_{*.t}\overline{u}_{k_l} \rightrightarrows D^a_{*.t}\overline{w} \text{ in } \overline{Q} \times [t_0,T], \text{ for any } t_0 > 0,$$

here by $\rightrightarrows$ we denote uniform convergence.

Remark 3. We emphasize that one can consider the Caputo derivatives $D^a_{*,t}\overline{u}_k$ as ordinary derivatives of convolutions $\hat{u}_k = (\omega_{1-a} * (\overline{u}_k - \psi))$. Thus we can also apply the Arzelà–Ascoli theorem to the sequence $\{\hat{u}_k\}$.

Then we prove that $\overline{w} \in C^{2+\alpha}_{\beta,a}(Q_T)$ in the same way as in [18, Ch. 3, Theorem 3]. □

We use similar proofs for the function $\underline{w}$.

Lemma 9. *Limit function $\underline{w} \in C^{2+\alpha}_{\beta,a}(Q_T)$ is a classic solution of the problem* (1)–(3)*.*

Lemma 10. *We have*

$$\underline{w}(x,t) = \overline{w}(x,t), \text{ for all } (x,t) \in Q_T. \quad (20)$$

Proof. By construction

$$\underline{w}(x,t) \le \overline{w}(x,t), \text{ for all } (x,t) \in Q_T, \quad (21)$$

thus by (7) we obtain

$$f(t,x,\overline{w}) - f(x,t,\underline{w}) \le L(\overline{w} - \underline{w}), \quad g(t,x,\overline{w}) - g(x,t,\underline{w}) \le L(\overline{w} - \underline{w}),$$

and

$$0 \le f(x,t,\underline{w}) - f(t,x,\overline{w}) - L(\underline{w} - \overline{w}), \quad 0 \le g(x,t,\underline{w}) - g(t,x,\overline{w}) - L(\underline{w} - \overline{w}).$$

Denote $w(x,t) = \underline{w}(x,t) - \overline{w}(x,t)$. We have

$$D^a_{*,t} w(x,t) - A(x,t,\partial_x) w(x,t) = f(x,t,\underline{w}) - f(t,x,\overline{w}) - L(\underline{w} - \overline{w}) \geq 0, \ (x,t) \in Q_T,$$
$$w(x,0) = 0, \ x \in Q,$$
$$B(x,t,\partial_x) w(x,t) = g(x,t,\underline{w}) - g(t,x,\overline{w}) - L(\underline{w} - \overline{w}) \geq 0, \ (x,t) \in S_T.$$

By Theorem 3 one get $w \geq 0$, i.e. $\underline{u}(x,t) \geq \overline{u}(x,t)$, for all $(x,t) \in Q_T$. Hence, by (21), we obtain (20). □

Theorem 1 is an immediate consequence of Lemmas 7–10.

4. Examples. We consider two examples with linear fractional equations and quasilinear boundary conditions 1) Stefan-Boltzman condition; 2) boundary condition arising in fermentation process (see Examples on p.176–177 in [54]).

Example 1. Denote

$$A'(x,t,\partial_x)u = \sum_{i,j=1}^{n} a_{i,j}(x,t) u_{x_i x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i}, \quad B'(x,t,\partial_x)u = \sum_{i,j=1}^{n} \nu_j a_{i,j}(x,t) u_{x_i}.$$

Let b_*, m be the constants such that $b_* > 0, \ m \geq 2$.

We assume that functions f,g, θ, ψ are such that

$$f \in C^{\alpha}_{\beta-2,a}(Q_T), g \in C^{1+\alpha}_{\beta-1,a}(S_T), \theta^m \in C^{1+\alpha}_{\beta-1,a}(S_T), \psi \in C^{\beta}(Q), \tag{22}$$
$$a_* \geq 0, \ 0 \leq f(x,t) \leq m_1 \exp(\gamma_1 t) \ \text{ in } Q_T, \ 0 \leq g(x,t) \leq m_2 \exp(\gamma_2 t) \ \text{ in } S_T, \tag{23}$$
$$0 \leq \theta(x,t) \leq \overline{\theta} \text{ in } Q_T, , \quad 0 \leq \psi(x) \leq \overline{\theta} \text{ in } Q, \tag{24}$$

here $\overline{\theta} > 0$, $m_i \geq 0$, $\gamma_i \geq 0$, $i = 1, 2$.

We consider a problem

$$\begin{gathered} D^a_{*,t} u(x,t) - A'(x,t,\partial_x) u(x,t) + a_* u(x,t) = f(x,t), \ (x,t) \in Q_T, \\ u(x,0) = \psi(x), \ x \in Q, \\ B'(x,t,\partial_x) u(x,t) = -b_*(u^m(x,t) - \theta^m(x,t)) + g(x,t), \ (x,t) \in S_T. \end{gathered} \tag{25}$$

Function $\underline{u} = 0$ is a lower solution of (25), since

$$0 \leq f(x,t), \ (x,t) \in Q_T, \quad 0 \leq \psi(x), \ x \in Q, \quad 0 \leq b_* \theta^m(x,t) + g(x,t), \ (x,t) \in S_T.$$

Then we use one-parametric Mittag-Leffler functions ([46])

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+ak)}.$$

with the following properies (see [46], [20])

$$\begin{gathered} D^a_{*,t} E_a(\lambda t^a) = \lambda E_a(\lambda t^a), \qquad E_a(0) = 1, \\ M_1 \exp\left(\frac{1}{2} z^a\right) \leq E_a(z) \leq M_2 \exp(z^a), \qquad z \geq 0, \\ \frac{1}{1+\Gamma(1-\alpha) z} \leq E_a(-z) \leq \frac{1}{1+\frac{z}{\Gamma(1+a)}} \qquad z \geq 0. \end{gathered} \tag{26}$$

We are looking for an upper solution in the form $\overline{u} = \overline{\theta} + \varrho E_a(\lambda t^a)$. By (25), (26) it suffices to find ϱ and λ such that

$$(\lambda + a_*)\varrho E_a(\lambda t^a) \geq m_1 \exp(\gamma_1 t), \quad b_*(\overline{\theta} + \varrho E_a(\lambda t^a))^m - \theta^m \geq m_2 \exp(\gamma_2 t),\ (x,t) \in S_T.$$

or otherwise

$$(\lambda + a_*)\varrho M_1 \exp\left(\frac{1}{2}\lambda^{\frac{1}{a}} t\right) \geq m_1 \exp\left(\gamma_1 t\right)^{\alpha}), \quad b_*\varrho^m M_1^m \exp\left(\frac{m}{2}\lambda^{\frac{1}{a}} t\right) \geq m_2 \exp\left((\gamma_2 t\right).$$

First we take λ so large that $\lambda^{\frac{1}{a}} \geq 2\gamma_1^a$, $\frac{m}{2}\lambda^{\frac{1}{a}} \geq \gamma_2$ and then we choose ϱ by inequalities $(a_* + \lambda)\varrho M_1 \geq m_1$, $\quad b_*\varrho^m M_1^m \geq m_2$.

If we replace assumptions (23) on

$$0 < \kappa \leq a_*, \quad 0 \leq f \leq \frac{1}{1 + m_1 t^a}, \quad 0 \leq g \leq \frac{1}{(1 + m_2 t^a)^m},$$

one can construct an upper solution in the form $\overline{u} = \overline{\theta} + \varrho E(-\lambda t^a)$ for sufficiently small λ and sufficiently small ϱ.

Example 2. Let σ_1, σ_2 be arbitrary positive parameters. We consider the problem

$$\begin{gathered} D^a_{*,t}u(x,t) - A(x,t,\partial_x)u(x,t) = f(x,t),\ (x,t) \in Q_T, \\ u(x,0) = \psi(x),\ x \in Q, \\ B(x,t,\partial_x)u(x,t) = \frac{\sigma_1 u(x,t)}{1 + \sigma_2 u(x,t)} + g(x,t),\ (x,t) \in S_T. \end{gathered} \tag{27}$$

It is clear that $0 \leq \frac{\sigma_1 u(x,t)}{1+\sigma_2 u(x,t)} \leq \frac{\sigma_1}{\sigma_2}$ if $u \geq 0$. This implies a choice of lower and upper solutions. We assume

$$\begin{gathered} \psi \in C^{\beta}(Q), \quad f \in C^{\alpha}_{\beta-2,a}(Q_T), \quad g \in C^{1+\alpha}_{\beta-1,a}(s_T), \\ \psi(x) \geq 0,\ x \in Q \quad f(x,t) \geq 0,\ (x,t) \in Q_T,\ g(x,t) \geq 0(x,t) \in S_T. \end{gathered}$$

Similarly to [54], we take $\underline{u} = 0$, as an lower solution. As an upper solution $\overline{u}$ we consider a solution of linear problem

$$\begin{gathered} D^a_{*,t}u(x,t) - L(x,t,\partial_x)u(x,t) = f(x,t),\ (x,t) \in Q_T, \\ u(x,0) = \psi(x),\ x \in Q, \\ B(x,t,\partial_x)u(x,t) = \frac{\sigma_1}{\sigma_2} + g(x,t),\ (x,t) \in S_T. \end{gathered}$$

5. Appendix. In this Appendix we prove Lemma 6. For short we don't indicate a dependence on x. We write

$$\int_0^t \frac{u(t) - u(\tau)}{(t-\tau)^{1+a}} d\tau = \int_0^{t/2} \frac{u(t) - u(\tau)}{(t-\tau)^{1+a}} d\tau + \int_{t/2}^t \frac{u(t) - u(\tau)}{(t-\tau)^{1+a}} d\tau = J_1 + J_2. \tag{28}$$

In J_1 we have $t - \tau > \frac{t}{2}$ thus

$$|J_1| \leq ct^{-a} \int_0^{t/2} (t-\tau)^{\frac{\beta}{2}a - 1} d\tau \langle u \rangle^{\frac{\beta}{2}a}_{t,Q_T} \leq c\langle u \rangle^{\frac{\beta}{2}a}_{t,Q_T} t^{\frac{\beta-2}{2}a}. \tag{29}$$

For the second integral J_2 we obtain by (11)

$$u(t) - u(\tau) = (\omega_a * D^a_{*,t}u)(t) - (\omega_a * D^a_{*,\tau}u)(\tau) =$$

$$= \int_0^t \omega_a(t-\sigma)D^a_{*,\sigma}u(\sigma)d\sigma - \int_0^\tau \omega_a(\tau-\sigma)D^a_{*,\sigma}u(\sigma)d\sigma =$$

$$\int_\tau^t \omega_a(t-\sigma)D^a_{*,\sigma}u(\sigma)d\sigma + \int_0^\tau \left(\omega_a(t-\sigma) - \omega_a(\tau-\sigma)\right) D^a_{*,\sigma}u(\sigma)d\sigma =$$

$$= \int_\tau^t \omega_a(t-\sigma)\left[D^a_{*,\sigma}u(\sigma) - D^a_{*,\tau}u(\tau)\right]d\sigma + D^a_{*,\tau}u(\tau)\Big[\int_0^t \omega_a(t-\sigma)d\sigma - \int_0^\tau \omega_a(\tau-\sigma)d\sigma\Big] +$$

$$+ \int_0^\tau (\omega_a(t-\sigma) - \omega(\tau-\sigma))\left[D^a_{*,\sigma}u(\sigma) - D^a_{*,\tau}u(\tau)\right]d\sigma = K_1 + K_2 + K_3.$$

Since we estimate the integral J_2 we suppose $\frac{t}{2} < \tau < t$ in K_1, K_2, K_3. For K_1 we have

$$|K_1| \le \int_\tau^t (t-\sigma)^{a-1} t^{\frac{\beta-2-\alpha}{2}a}(\sigma-\tau)^{\frac{\alpha}{2}a}d\sigma |u|^{(2+\alpha)}_{\beta,a,Q_T} \le c|u|^{(2+\alpha)}_{\beta,a,Q_T} t^{\frac{\beta-2-\alpha}{2}a}(t-\tau)^{\frac{\alpha}{2}a+a}.$$

In K_2 we use inequality $t^a - \tau^a \le a\tau^{a-1}(t-\tau)$. Thus

$$|K_2| \le c|D^a_{*,\tau}u(\tau)|(t^a - \tau^a) \le c|u|^{2+\alpha}_{\beta,a,Q_T}\tau^{\frac{\beta-2}{2}a}\tau^{a-1}(t-\tau) \le c|u|^{(2+\alpha)}_{\beta,a,Q_T} t^{\frac{\beta}{2}a-1}(t-\tau).$$

For the last integral K_3 we obtain

$$|K_3| \le c|u|^{(2+\alpha)}_{\beta,a,Q_T} \int_0^\tau \frac{(t-\sigma)^{1-a} - (\tau-\sigma)^{1-a}}{(t-\sigma)^{1-a}(\tau-\sigma)^{1-a}} \sigma^{\frac{\beta-2-\alpha}{2}a}(\tau-\sigma)^{\frac{\alpha}{2}a}d\sigma \le$$

$$\le c|u|^{(2+\alpha)}_{\beta,a,Q_T} \int_0^\tau \frac{(t-\tau)(\tau-\sigma)^{-a+\frac{\alpha}{2}a}\sigma^{\frac{\beta-2-\alpha}{2}a}}{(t-\sigma)^{1-a}(\tau-\sigma)^{1-a}}d\sigma,$$

We introduce a parameter $\rho \in (0, \frac{\alpha}{2}a)$ (for example $\rho = \frac{\alpha}{4}a$) and continue

$$|K_3| \le |u|^{(2+\alpha)}_{\beta,a,Q_T} \int_0^\tau \frac{(t-\tau)(\tau-\sigma)^{\frac{\alpha}{2}a-1}\sigma^{\frac{\beta-2-\alpha}{2}a}}{(t-\tau)^{1-a-\rho}(\tau-\sigma)^{\rho}}d\sigma \le$$

$$\le c|u|^{(2+\alpha)}_{\beta,a,Q_T}(t-\tau)^{a+\rho}\int_0^\tau (\tau-\sigma)^{\frac{\alpha}{2}a-\rho-1}\sigma^{\frac{\beta-2-\alpha}{2}a}d\sigma \le$$

$$\le c|u|^{(2+\alpha)}_{\beta,a,Q_T}(t-\tau)^{a+\rho}\tau^{\frac{\alpha}{2}a-\rho+\frac{\beta-2-\alpha}{2}a} \le c|u|^{(2+\alpha)}_{\beta,a,Q_T}(t-\tau)^{a+\rho}t^{\frac{\beta-2}{2}a-\rho}.$$

Using estimates K_1, K_2, K_3 we get

$$|J_2| \le c|u|^{(2+\alpha)}_{\beta,a,Q_T}\Big[\int_{t/2}^t (t-\tau)^{\frac{\alpha}{2}a-1}d\tau t^{-\frac{\alpha}{2}a} + \int_{t/2}^t (t-\tau)^{-a}d\tau t^{a-1} +$$

$$+ \int_{t/2}^t (t-\tau)^{\rho-1}d\tau t^{-\rho}\Big] t^{\frac{\beta-2}{2}a} \le c|u|^{(2+\alpha)}_{\beta,a,Q_T} t^{\frac{\beta-2}{2}a}. \tag{30}$$

The statement of Lemma 6 follows by (28)–(30).

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REFERENCES

1. M. Allen, L. Caffarelli, A. Vasseur, *A parabolic problem with a fractional time derivative*, Arch. Ration. Mech. Anal., **221** (2016) 603–630.
2. A. Alsaedi, B. Ahmad, M. Kirane, *Maximum principle for certain generalized time and space fractional diffusion equations*, Quart. Appl. Math., **73** (2015), 163–175.
3. A. Alsaedi, M. Kirane, R. Lassoued, *Global existence and asymptotic behavior for a time fractional reaction–diffusion system*, Computers and Mathematics with Applications, **73** (2017), 951–958.
4. E. Bazhlekova, *Subordination principle for fractional evolution equations*, Fractional Calculus Appl. Anal., **3** (2000), 213–230.
5. E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, *An analysis of the Rayleigh–Stokes problem for a generalized second-grade fluid*, Numerische Math., **131** (2015), 1–31.
6. J.W. Bebernes, K. Schmitt, *On the existence of maximal and minimal solutions for parabolic partial differential equations*, Proc. Amer. Math. Soc., **73** (1979), 211–218.
7. V.S. Belonosov, *Estimates of the solutions of parabolic systems in Holder weight classes and some of their applications*, Mat. Sb., **110(152)** (1979), 163–188.
8. G.I. Bizhanova, *Solution in a weighted Holder space of an initial-boundary value problem for a second-order parabolic equation with a time derivative in the conjugation condition*, St. Petersburg Math. J., **6** (1995), 51–75.
9. G.I. Bizhanova, V.A. Solonnikov, *On the solvability of an initial-boundary value problem for a second-order parabolic equation with a time derivative in the boundary condition in a weighted Holder space of functions*, St. Petersburg Math. J., **5** (1994), 97-–124.
10. G.I. Bizhanova, V.A. Solonnikov, *On problems with free boundaries for second-order parabolic equations*, St. Petersburg Math. J., **12** (2001), 949–981.
11. H. Brunner, H. Han, D. Yin, *The maximum principle for time-fractional diffusion equations and its applications*, Numer. Funct. Anal. Optim., **36** (2015), 1307–1321.
12. S. Brzychczy, *Monotone iterative methods for infinite systems of reaction-diffusion-convection equations with functional dependence*, Opuscula Mathematica, **25** (2005), 29–99.
13. Ph. Clement, S.-O. Londen, G. Simonett, *Quasilinear evolutionary equations and continuous interpolation spaces*, J. Differ. Equ., **196** (2004), 418–447.
14. C.Y. Chan, H.T. Liu, *A maximum principle for fractional diffusion differential equations*, Quarterly Appl. Math., **74** (2016), 421–427.
15. J.I. Diaz, T.T. Pierantozzi, L. Vazquez, *Finite time extinction for nonlinear fractional evolution equations and related properties*, Electronic Journal of Differential Equations, **2016** (2016), 1–13.
16. K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin. 2010.
17. E.V. Domanova, *Estimates of the solutions of parabolic systems in Holder weight classes without the compatibility conditions*, Proc. of S.L. Sobolev seminar, (1989), 70–85.
18. A. Friedman, Partial differential equations of parabolic type, Englewood Cliffs, N.J., Prentice-Hall, 1964.
19. Y. Giga, H. Mitake, S. Sato, *On the equivalence of viscosity solutions and distributional solutions for the time-fractional diffusion equation*, Journal of Differential Equations, **316** (2022), 364–386.
20. R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications, Springer-Verlag, Berlin, Heidelberg, 2014.
21. H. Gou, Y. Li, *The method of lower and upper solutions for impulsive fractional evolution equations*, Annals of Functional Analysis, **11** (2020), 350–369.
22. D. Guidetti, *Time fractional derivatives and evolution equations*, Bruno Pini Mathematical Analysis Seminar, **9** (2017), 142–152.
23. A. Heibig, L.I. Palade, *On the rest state stability of an objective fractional derivative viscoelastic fluid model*, Journal of mathematical physics, **49** (2008), 43101, 1–22.

24. R. Hermann, Fractional Calculus: an introduction for Physicists, World Scientific, Singapore, 2014.
25. T. Jankowski, *Systems of nonlinear fractional differential equations*, Fractional Calculus and Applied Analysis, **18** (2013), 122–131.
26. J. Janno, K. Kasemets *Uniqueness for an inverse problem for a semilinear time-fractional diffusion equation*, Inverse problems and imaging, **11** (2017), 125–149.
27. J. Kemppainen, K. Ruotsalainen *Boundary integral solution of the time-fractional diffusion equation*, Integr. Equ. Oper. Theory, **64** (2009), 239–249.
28. Y. Kyan, *Equivalence of definitions of solutions for some class of fractional diffusion equations*, (2021), arXiv:2111.06168v1.
29. A.N. Kochubei, *Diffusion of fractional order*, Differential Equations, **26** (1990), 485–492.
30. A.N. Kochubei, *Fractional parabolic systems,* Potential analysis, **37** (2012), 1–30.
31. M. Korpusov, A.Ovchinnikov, A. Panin, Lectures in Nonlinear Functional Analysis, World Scientific, Singapore, 2022.
32. M. Krasnoschok, V. Pata, S. Siryk, N. Vasylyeva, *Equivalent definitions of Caputo derivatives and applications to subdiffusion equations*, Dynamics of Partial Differential Equations, **11** (2020), 383–402.
33. M. Krasnoschok, N. Vasylyeva, *On a solvability of a nonlinear fractional reaction-diffusion system in the Hölder spaces*, Nonlinear Stud., **20** (2013), 591–621.
34. M. Krasnoschok, N. Vasylyeva, *Linear subdiffusion in weighted fractional Holder spaces*, Evolution Equations and Control Theory, (2021), DOI: 10.3934/eect.2021050.
35. A. Kubica, K. Ryszewska, M. Yamamoto, Time-fractional differential equations, Springer, Singapore, 2020.
36. O.A. Ladyzhenskaja, V.A. Solonnikov, N.I. Uraltseva, Linear and quasi-linear equations of parabolic type, Translation of Mathematical Monographs, V.23, Amer. Math. Soc., Providence, RI, 1968.
37. V. Lakshmikantham, S. Leola, J. Vasundhara, Theory of fractional dynamics systems, Cambridge Academic Press, Cambridge, 2008.
38. V. Lakshmikantham, A.S. Varsala, *General inequalities and monotone iterative technique for fractional differential methods*, Applied Mathematics Letters, **21** (2001), 828–834.
39. A.W. Leung, Systems of Nonlinear Partial Differential Equations: Applications to Biology and Engineering, Springer, Dordrecht, 1989.
40. H.T. Liu, *Strong maximum principles for fractional diffusion differential equations*, Dynamic. Systems Appl., **28** (2016), 365–376.
41. H.P. Lopushanska, A.O. Lopushanskyj, E.V. Pasichnik, *The Cauchy problem in a space of generalized functions for the equations possessing the fractional time derivative*, Siberian Mathematical Journal, **52** (2011), 1022–1031.
42. Y. Luchko, *Maximum principle for the generalized time-fractional diffusion equation*, J. Math. Anal. Appl., **351** (2009), 218–233.
43. Y. Luchko, M. Yamamoto, *A survey on the recent results regarding maximum principles for the time fractional diffusion equations*, Current Development Math. Sci., **1** (2018), 33–69.
44. Y. Luchko, M. Yamamoto, *Maximum principle for the time fractional PDEs*, Fractional Differential Equations, A. Kochubei, Yu. Luchko (Eds.), Berlin, Boston: De Gruyter, (2019), 299–326.
45. F. Mainardi, Fractional calculus and waves in linear viscoelasticity, Imperial College Press, London, 2010.
46. F. Mainardi, *On some properties of the Mittag-Leffler functiona* $E_\alpha(-t^\alpha)$ *competely monotone for* $t > 0$ *with* $0 < t < \alpha$, Discrete and Continuous Dynamical Systems, Series B, **19** (2014), 2267–2278.
47. W. McLean, *Regularity of solutions to a time-fractional diffusion equation*, ANZIAM Journal, **52** (2010), 123–138.
48. C. Miranda, Partial differential equations of elliptic type, Springer, Berlin, 1970.
49. G. Mophou, G. N'Guerekata, *On a class of fractional differential equations in a Sobolev space*, Applicable Analysis, **91** (2012), 15–34.
50. J. Mu, *Monotone iterative technique for fractional evolution equations in Banach spaces*, Journal of Applied Mathematics, **5** (2011), ID:767186.
51. J. Mu, B. Ahmad, S. Huang, *Existence and regularity of solutions to time-fractional diffusion equations*, Computers and Mathematics with Applications, **73** (2017), 985–996.
52. J. Mu, Y.X. Li, *Periodic boundary value problems for semilinear fractional differential equations*, Mathematical Problems in Engineering, (2012), 1–16.

53. T. Namba, *On existence and uniqueness of viscosity solutions for second order fully nonlinear PDEs with Caputo time fractional derivatives*, Nonlinear Differential Equations Appl., **25** (2018), №23, 39 pp.
54. C.V. Pao, Nonlinear parabolic and elliptic equations, New York, Plenum Press, 1992.
55. R. Ponce, *Hölder continuous solutions for fractional differential equations and maximal regularity*, J. Differ. Equat., **255** (2013), 3284–3304.
56. Yu. Povstenko, Fractional thermoelasticity, Springer-Verlag, Cham, 2015.
57. A.V. Pskhu, *The fundamental solution of a diffusion-wave equation of fractional order,* Izvestiya: Mathematics, **73** (2009), 351–392.
58. K. Sakamoto, M. Yamamoto, *Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl., **382** (2011), 426-–447.
59. S.S. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach, New York, 1993.
60. D.H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J., **21** (1972), 979–1000.
61. V.A. Solonnikov, A.G. Khachatryan, *Estimates for solutions of parabolic initial-boundary value problems in weighted Holder norm*, Proceedings of the Steklov Institute of Mathematics, **147** (1981), 153–162.
62. C. Topp, M. Yangari, *Existence and uniqueness for parabolic problems with Caputo time derivative*, J. Differential Equations, **262** (2017), 6018–6046.
63. G. Vainikko, *Which Functions are Fractionally Differentiable?*, Zeitschrift Fur Analysis Und Ihre Anwendungen, **35** (2016), 465–487.
64. V. Vergara, R. Zacher, *Stability, instability, and blowup for time fractional and other nonlocal in time semilinear subdiffusion equations*, Journal of Evolution Equations, **17** (2017), 599–626.
65. V. Vergara, R. Zacher *Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods*, SIAM J. Math. Anal., **47** (2013), 210–239.
66. G. Wang, R. Agarwal, A. Cabada, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, Applied Mathematics Letters, **25** (2012), 1019–1024.
67. R. Zacher, Quasilinear parabolic problems with nonlinear boundary conditions, Ph.D. Thesis, Halle-Wittenberg: Martin-Luther-Universität, 2003.
68. R. Zacher, *Time fractional diffusion equations: solution concepts, regularity, and long-time behavior*, Handbook of fractional calculus with applications, De Gruyter, Berlin, **2** (2019), 159–179.

Institute of applied mathematics and mechanics of NAS of Ukraine
Sloviansk, Ukraine
iamm012@ukr.net

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