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V. V. Kravtsiv

ZEROS OF BLOCK-SYMMETRIC POLYNOMIALS ON BANACH SPACES

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We investigate sets of zeros of block-symmetric polynomials on the direct sums of sequence spaces. Block-symmetric polynomials are more general objects than classical symmetric polynomials. An analogue of the Hilbert Nullstellensatz for block-symmetric polynomials on $\ell_p(\mathbb{C}^n) = \ell_p \oplus \ldots \oplus \ell_p$ and $\ell_1 \oplus \ell_\infty$ is proved. Also, we show that if a polynomial P has a block-symmetric zero set then it must be block-symmetric.

1. Introduction. The Hilbert Nullstellensatz is a classical principle in Algebraic Geometry and actually its starting point. It provides a bijective correspondence between affine varieties, which are geometric objects and radical ideals in a polynomials ring, which are algebraic objects. For the proof and applications of the Hilbert Nullstellensatz we refer the reader to [6].

The question whether a bounded polynomial functional on a complex Banach space X is determined by its kernel set of zeros under the assumption that all the factors of its decomposition into irreducible factors are simple was posed by Mazur and Orlich (see also Problem 27 in [11]). A positive answer to this question was given in [14]. Moreover, this result remains valid even when the ring of bounded polynomial functionals is replaced by any ring of polynomials on X for which there exists a decomposition into irreducible factors satisfying the following condition: along with each polynomial $P(x)$ that it contains the ring also contains the polynomial $P_{\lambda;x_0}(x) = P(x_0 + \lambda x)$, where $x \in X$ and $\lambda \in \mathbb{C}$.

Let X and Y be vector spaces over the field $\mathbb{C}$ of complex numbers. A mapping $\overline{P}_k(x_1, \ldots, x_k)$ from the Cartesian product X^k into Y is *k-linear* if it is linear in each component. The restriction P_k of the k-linear operator $\overline{P}_k$ to the diagonal $\triangle = \{(x_1, \ldots, x_k) \in X^k \colon x_1 = \ldots = x_k\}$, which can be naturally identified with X, is a homogeneous polynomial of degree k (briefly, a k-monomial). A finite sum of k-monomials, $0 \le k \le n$, $P(x) = P_0(x) + P_1(x) + \ldots + P_n(x), P_n \ne 0$ is a polynomial of degree n. For general properties of polynomials on abstract linear spaces we refer the reader to [4].

In [1] it was proved the Nullstellensatz for algebras of symmetric polynomials on ℓ_p by R. Alencar, R. Aron, P. Galindo, A. Zagorodnyuk. In the case $\ell_1(\mathbb{C}^2)$ the Nullstellensatz was proved in [9]. This paper is devoted to generalizations of the Hilbert Nullstellensatz block-symmetric polynomials on $\ell_p(\mathbb{C}^n)$, where $1 \le p < +\infty$ and on $\ell_1 \oplus \ell_\infty$. Also, we show that if the kernel of a polynomial P on $\ell_p(\mathbb{C}^n)$ or $\ell_1 \oplus \ell_\infty$ is block-symmetric, then P must be block-symmetric. Block-symmetric polynomials on sequence Banach spaces are generalization of

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symmetric polynomials on ℓ_p which were studied, for example, in [2, 3, 5]. For the general information about polynomials and analytic functions on Banach spaces we refer the reader to [4].

2. The Hilbert Nullstelensatz. Let $X_1, \ldots, X_m$ be complex sequence spaces and $\mathcal{X} = X_1 \oplus \ldots \oplus X_m$. Every $x \in \mathcal{X}$ can be represented by $x = (x^{(1)}, \ldots, x^{(m)})$, where $x^{(j)} \in X_j$. A function $f\colon \mathcal{X} \to \mathbb{C}$ is called *block-symmetric* if $f(\sigma(x)) = f(x)$ for every $x \in \mathcal{X}$ and for every bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$, where $\sigma(x) = (\sigma(x^{(1)}), \ldots, \sigma(x^{(m)}))$ and $\sigma(x^{(j)}) = (x^{(j)}_{\sigma(1)}, x^{(j)}_{\sigma(2)}, \ldots, x^{(j)}_{\sigma(n)}, \ldots)$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X})$ the algebra of all block-symmetric continuous polynomials on $\mathcal{X}$. The group of all bijections (permutations) σ on $\mathbb{N}$ will be denoted by S.

Let $\mathcal{P}_0(X)$ be a subalgebra of polynomials on a Banach space X. Let us suppose that $\mathcal{P}_0(X)$ has the following property.

Property A: $\mathcal{P}_0(X)$ admits an algebraic basis $P_1, P_2, \ldots, P_n, \ldots$, that is, every polynomial in $\mathcal{P}_0(X)$ can be uniquely represented as an algebraic combination of polynomials P_i, and for every $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ there is $x \in X$ such that $P_1(x) = \xi_1, \ldots, P_n(x) = \xi_n$.

In this case the next theorem is true:

Theorem 1 (Nullstellensatz). *Let $\mathcal{P}_0(X)$ has the Property A and $R_1, \ldots, R_m \in \mathcal{P}_0(X)$ such that* $\ker R_1 \cap \ldots \cap \ker R_m = \varnothing$. *Then there are $Q_1, \ldots, Q_m \in \mathcal{P}_0(X)$ such that*

$$\sum_{i=1}^{m} R_i Q_i = 1.$$

Proof. For the proof we use the same method as in [1, p. 58]. Let $n = \max_i(\deg R_i)$. We may assume that $R_i(x) = q_i(P_1(x), \ldots, P_n(x))$ for some $q_i \in \mathcal{P}(\mathbb{C}^n)$, where $x \in X$. Let us suppose that at some point $\xi \in \mathbb{C}^n$, $\xi = (\xi_1, \ldots, \xi_n)$, $q_i(\xi) = 0$. Then by Property A there is $x_0 \in X$ such that $P_i(x_0) = \xi_i$ and $R_1(x_0) = R_2(x_0) = \ldots = R_n(x_0) = 0$. So the common set of zeros of all q_i is empty. Thus by the classic Hilbert Nullstellensatz there are polynomials $g_1, \ldots, g_m$ such that $\sum_i q_i g_i \equiv 1$. Put $Q_i(x) = g_i(P_1(x), \ldots, P_n(x))$. $\square$

Next we consider the cases when $X = \ell_p(\mathbb{C}^n)$ and $X = \ell_1 \oplus \ell_\infty$, and $\mathcal{P}_0(X) = \mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ and $\mathcal{P}_0(X) = \mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$, respectively. Note that if all spaces X_n are finite dimensional, then $\mathcal{P}_{vs}(\mathcal{X})$ has no algebraic basis [10].

2.1. The case of $\ell_p(\mathbb{C}^n)$. Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let us denote $\ell_p(\mathbb{C}^n)$ the vector space of all sequences

$$x = (x_1, x_2, \ldots, x_m, \ldots), \tag{1}$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(n)}) \in \mathbb{C}^n$ for $j \in \mathbb{N}$, such that the series $\sum\limits_{j=1}^{\infty} \sum\limits_{s=1}^{n} \left| x_j^{(s)} \right|^p$ is convergent. The space $\ell_p(\mathbb{C}^n)$ with norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty} \sum_{s=1}^{n} \left| x_j^{(s)} \right|^p \right)^{\frac{1}{p}} \tag{2}$$

is a Banach space. Note that, $\ell_p(\mathbb{C}^n)$ is isomorphic to $\underbrace{\ell_p \oplus \ldots \oplus \ell_p}_{n}$.

The algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ was considered in [8].

For a multi-index $k=(k_1,k_2,\dots,k_n)\in\mathbb{Z}_+^n$ let $|k|=k_1+k_2+\ldots+k_n$.

In [8] it was proved that polynomials

$$H^k(x)=\sum_{j=1}^{\infty}\prod_{\substack{s=1\\k_1+\ldots+k_n=|k|}}^{n}(x_j^{(s))})^{k_s}$$

form an algebraic basis of the algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$, where $|k|\geq\lceil p\rceil$ ($\lceil p\rceil$ is ceiling of p), $x=(x_1,\dots,x_m,\dots)\in\ell_p(\mathbb{C}^n)$, $x_j=(x_j^{(1)},\dots,x_j^{(n)})\in\mathbb{C}^n$.

For $m\in\mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the space of all sequences $x=(x_1,\dots,x_m,0,\dots)$, where $x_1,\dots,x_m\in\mathbb{C}^n$. Let $c_{00}(\mathbb{C}^n)=\bigcup\limits_{m=1}^{\infty}c_{00}^{(m)}(\mathbb{C}^n)$. For an arbitrary nonempty finite set $M\in\mathbb{Z}_+^n$ let us define a mapping $\pi_M\colon c_{00}(\mathbb{C}^n)\to\mathbb{C}^{|M|}$, where $|M|$ is the cardinality of M, by

$$\pi_M(x)=(H^k(x))_{k\in M}.$$

In [8] it was proved the following theorem.

Theorem 2. *Let M be a finite nonempty subset of $\mathbb{Z}_+^n$ such that $|k|\geq 1$ for every $k\in M$. Then*

1. *There exists $m\in\mathbb{N}$, such that for every $\xi=(\xi_k)_{k\in M}\in\mathbb{C}^{|M|}$ there exists $x_\xi\in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi)=\xi$;*
2. *There exists a constant $\rho_M>0$ such that if $||\xi||_\infty<1$, then $||x_\xi||_p\leq\rho_M$ for every $p\in[1,+\infty)$, where $||\xi||_\infty=\max\limits_{k\in M}|\xi_k|$.*

Corollary 1. *Let $P_1,\dots,P_m\in\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ such that $\ker P_1\cap\ldots\cap\ker P_m=\varnothing$. Then there are $Q_1,\dots,Q_m\in\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ such that*

$$\sum_{i=1}^{m}P_iQ_i=1.$$

Proof. From Theorem 2 we have that $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ satisfies Property A. So, we can apply Theorem 1. □

2.2. The case of $\ell_1\oplus\ell_\infty$. Let $\mathcal{X}=\ell_1\oplus\ell_\infty$. Each element of $\mathcal{X}$ can be represented by

$$x=\left(\begin{pmatrix}x_1^{(1)}\\x_1^{(2)}\end{pmatrix},\dots,\begin{pmatrix}x_m^{(1)}\\x_m^{(2)}\end{pmatrix},\dots\right),$$

where $(x_1^{(1)},x_2^{(1)},\dots,x_m^{(1)},\dots)\in\ell_1$, $(x_1^{(2)},x_2^{(2)},\dots,x_m^{(2)},\dots)\in\ell_\infty$. The space $\ell_1\oplus\ell_\infty$ with norm

$$||x||_{\ell_1\oplus\ell_\infty}=\sum_{i=1}^{\infty}|x_i^{(1)}|+\sup_{i\geq1}|x_i^{(2)}|$$

is a Banach space.

So, $P\in\mathcal{P}_{vs}(\ell_1\oplus\ell_\infty)$ if and only if

$$P\left(\begin{pmatrix}x_1^{(1)}\\x_1^{(2)}\end{pmatrix},\dots,\begin{pmatrix}x_m^{(1)}\\x_m^{(2)}\end{pmatrix},\dots\right)=P\left(\begin{pmatrix}x_{\sigma(1)}^{(1)}\\x_{\sigma(1)}^{(2)}\end{pmatrix},\dots,\begin{pmatrix}x_{\sigma(m)}^{(1)}\\x_{\sigma(m)}^{(2)}\end{pmatrix},\dots\right),$$

for every permutation σ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix} \in \mathbb{C}^2$.

Let us denote by $(\ell_1 \oplus \ell_\infty)^{(m)}$ the $2m$-dimensional subspace of $(\ell_1 \oplus \ell_\infty)$ consisting of all sequences

$$x_m = \left(\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} x_m^{(1)} \\ x_m^{(2)} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots \right),$$

where $(x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}, 0 \ldots) \in \ell_1$, $(x_1^{(2)}, x_2^{(2)}, \ldots, x_m^{(2)}, 0 \ldots) \in \ell_\infty$. Clearly, $(\ell_1 \oplus \ell_\infty)^{(m)}$ is isomorphic to $c_{00}^{(m)}(\mathbb{C}^2)$.

For an arbitrary nonempty finite set $M \in \mathbb{Z}_+^2$ let us define a mapping $\pi_M \colon \ell_1 \oplus \ell_\infty \longrightarrow \mathbb{C}^{|M|}$ by

$$\pi_M(x) = (H^{k_1,k_2}(x))_{(k_1,k_2) \in M}.$$

Corollary 2 ([7])**.** *Let M be a finite nonempty subset of $\mathbb{Z}_+^2$ such that $k_1 + k_2 \geq 1$ for every $(k_1, k_2) \in M$. Then*

1. *There exists $m \in \mathbb{N}$, such that for every $\xi = (\xi_{(k_1,k_2)})_{(k_1,k_2) \in M} \in \mathbb{C}^{|M|}$ there exists $x_\xi \in (\ell_1 \oplus \ell_\infty)^{(m)}$ such that $\pi_M(x_\xi) = \xi$;*
2. *There exists a constant $\rho_M > 0$ such that if $||\xi||_\infty < 1$, then $||x_\xi||_{\ell_1 \oplus \ell_\infty} \leq \rho_M$.*

In [7] the following theorem was proved.

Theorem 3 ([7])**.** *Polynomials*

$$H^{k_1,k_2}(x) = \sum_{i=1}^{\infty} (x_i^{(1)})^{k_1} (x_i^{(2)})^{k_2}, \tag{3}$$

form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$, where $k_1, k_2 \in \mathbb{Z}, k_1 \geq 1, k_2 \geq 0$.

Corollary 3. *Let $P_1, \ldots, P_m \in \mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ such that* $\ker P_1 \cap \ldots \cap \ker P_m = \varnothing$. *Then there are $Q_1, \ldots, Q_m \in \mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ such that*

$$\sum_{i=1}^{m} P_i Q_i = 1.$$

Proof. From Corollary 2 and Theorem 3 it follows that $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ has Property A. So, we can apply Theorem 1. □

3. Polynomials with block-symmetric zeros.

Definition 1. A subalgebra $\mathcal{P}_0(X)$ of the algebra of all polynomials $\mathcal{P}(X)$ is called *factorial* if for every $P \in \mathcal{P}_0(X)$ such that $P(x) = P_1(x)P_2(x)$, where $P_1, P_2 \in \mathcal{P}(X)$ we have that $P_1, P_2 \in \mathcal{P}_0(X)$.

Let V be a subset of $X_1 \oplus \ldots \oplus X_m$, where $X_1, \ldots, X_m$ are sequence Banach spaces. We say that V is block-symmetric if for every $x = (x^{(1)}, \ldots, x^{(m)}), x^{(i)} \in X_i$ and permutation $\sigma \in S$, $\sigma(x) = (\sigma(x^{(1)}), \ldots, \sigma(x^{(m)})) \in V$.

Note that for finite-dimensional space $\mathbb{C}^2 \oplus \mathbb{C}^2$ there exists a block-symmetric polynomial

$$P(x_1, x_2) = x_1^{(1)} x_2^{(1)} + x_1^{(1)} x_2^{(2)} + x_2^{(1)} x_1^{(2)} + x_1^{(2)} x_2^{(2)} = (x_1^{(1)} + x_1^{(2)})(x_2^{(1)} + x_2^{(2)}) \in \mathcal{P}_{vs}(\mathbb{C}^2 \oplus \mathbb{C}^2),$$

such that $(x_1^{(1)}+x_1^{(2)}),(x_2^{(1)}+x_2^{(2)})\notin\mathcal{P}_{vs}(\mathbb{C}^2\oplus\mathbb{C}^2)$.

So polynomial $Q(x)=(x_1^{(1)}+x_1^{(2)})^2(x_2^{(1)}+x_2^{(2)})$ is such that $\ker Q(x)$ is block-symmetric, but $Q(x)\notin\mathcal{P}_{vs}(\mathbb{C}^2\oplus\mathbb{C}^2)$.

Theorem 4. *The algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ is factorial.*

Proof. Note first that if $Q\in\mathcal{P}(\mathcal{X})$ is an irreducible polynomial, then $Q\circ\sigma$ is irreducible. Indeed, if $Q\circ\sigma=Q_1Q_2, \deg Q_1>0, \deg Q_2>0$, then $Q=Q_1Q_2\circ\sigma^{-1}=(Q_1\circ\sigma^{-1})(Q_2\circ\sigma^{-1})$, $\deg(Q_1\circ\sigma^{-1})=\deg Q_1$ and $\deg(Q_2\circ\sigma^{-1})=\deg Q_2$.

Let $P(x)\in\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ and

$$P(x)=P_1(x)\dots P_m(x), \tag{4}$$

where $P_1,\dots,P_m$ are irreducible polynomials. Now we show that $P_i(x)\in\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ for all $i\in\{1,\dots,m\}$. Let us assume that $P_k(x)\notin\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ for some $k\in\{1,\dots,m\}$. From (4) we obtain that

$$P(x)=P(\sigma(x))=P_1(\sigma(x))\dots P_k(\sigma(x))\dots P_m(\sigma(x))=P_1(x)\dots P_k(x)\dots P_m(x).$$

Since $P_k\circ\sigma\neq P_k$ for some σ, then there is $j\neq k, 1\leq j\leq m$ such that $P_k\circ\sigma=P_j$. In [14] it is proved that if a polynomial Q on a sequence space X is non-symmetric, then the cardinality of the set $\{Q\circ\sigma\colon \sigma\in S\}$ is greater or equal than $\dim X+1$. In particular, this set is infinite if $\dim X=\infty$. Let us show that it is still true for the block-symmetric case.

Let $x\in\ell_p(\mathbb{C}^n)$ such that $x=(x_m^{(s)})_{m=1}^\infty, s=1,\dots,n$ and $P_k(\sigma(x))\neq P_k(x)$ for some $\sigma\in S$. Set $\tilde{P}_k(t)=P_k(t_1x_1,t_2x_2,\dots), t\in\ell_p$. Then $\tilde{P}_k$ is a polynomial on ℓ_p and $\tilde{P}_k\circ\sigma\neq\tilde{P}_k$. As we observed above, the set $\{\tilde{P}_k\circ\sigma\colon \sigma\in S\}$ is infinite, so $\{P_k\circ\sigma, \sigma\in S\}$ is infinite too. Indeed, if $\tilde{P}_k\circ\sigma\neq\tilde{P}_k$, then there is $t^0\in\ell_p$ such that $\tilde{P}_k\circ\sigma(t^0)\neq\tilde{P}_k(t^0)$. So $P_k\circ\sigma(t_1^0x_1,t_2^0x_2,\dots)\neq P_k(t_1^0x_1,t_2^0x_2,\dots)$, that is, $P_k\circ\sigma\neq P_k$. Since the set $\{\tilde{P}_k\circ\sigma\colon \sigma\in S\}$ consists of infinite many mutually different polynomials, the same is true for the set $\{P_k\circ\sigma\colon \sigma\in S\}$. But in (4) we have just a finite number of $P_i(x), i\in\{1,\dots,m\}$. Therefore all $P_i(x), i\in\{1,\dots,m\}$ from (4) are block-symmetric. Hence, $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ is factorial. $\square$

Using the same arguments as in Theorem 4 we obtain the following result.

Proposition 1. *The algebra $\mathcal{P}_{vs}(\ell_1\oplus\ell_\infty)$ is factorial.*

Theorem 5. *Let P be a polynomial on $\mathcal{X}$, where $\mathcal{X}=\ell_p(\mathbb{C}^n)$ or $\ell_1\oplus\ell_\infty$. If $\ker P$ is block-symmetric, then $P\in\mathcal{P}_{vs}(\mathcal{X})$.*

Proof. Since $\ker P$ is block-symmetric, then $\ker P=\ker(P\circ\sigma)$ for every permutation σ on $\mathbb{N}$. From the Nullstellensatz for polynomials on Banach spaces [14] we have that $\operatorname{Rad}P=\operatorname{Rad}(P\circ\sigma)$, where $\operatorname{Rad}P$ is the radical of P. Let us recall that if $P=Q_1^{k_1}\dots Q_n^{k_n}$, where $Q_1,\dots,Q_n$ are irreducible polynomials, then $\operatorname{Rad}P=Q_1\dots Q_n$. So $\operatorname{Rad}P$ is block-symmetric. Since the algebra of all block-symmetric polynomials on $\mathcal{X}$ is factorial, all $Q_1,\dots,Q_n$ are block-symmetric. So P is block-symmetric. $\square$

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Vasyl Stefanyk Precarpathian National University,
maksymivvika@gmail.com

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