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LINEAR EXPAND-CONTRACT PLASTICITY OF ELLIPSOIDS REVISITED

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This work is aimed to describe linearly expand-contract plastic ellipsoids given via quadratic form of a bounded positively defined self-adjoint operator in terms of its spectrum.

Let Y be a metric space and $F\colon Y \to Y$ be a map. F is called non-expansive if it does not increase distance between points of the space Y. We say that a subset M of a normed space X is linearly expand-contract plastic (briefly an LEC-plastic) if every linear operator $T\colon X \to X$ whose restriction on M is a non-expansive bijection from M onto M is an isometry on M.

In the paper, we consider a fixed separable infinite-dimensional Hilbert space H. We define an ellipsoid in H as a set of the following form $E = \{x \in H\colon \langle x, Ax\rangle \le 1\}$ where A is a self-adjoint operator for which the following holds: $\inf_{\|x\|=1} \langle Ax, x\rangle > 0$ and $\sup_{\|x\|=1} \langle Ax, x\rangle < \infty$.

We provide an example which demonstrates that if the spectrum of the generating operator A has a non empty continuous part, then such ellipsoid is not linearly expand-contract plastic.

In this work, we also proof that an ellipsoid is linearly expand-contract plastic if and only if the spectrum of the generating operator A has empty continuous part and every subset of eigenvalues of the operator A that consists of more than one element either has a maximum of finite multiplicity or has a minimum of finite multiplicity.

1. Introduction. Let M be a metric space and $F\colon M \to M$ be a map. F is called *non-expansive* if it does not increase distance between points of the space M. M is called *expand-contract plastic* (or just plastic for short) if every non-expansive bijection $F\colon M \to M$ is an isometry.

There is a number of relatively recent publications devoted to plasticity of the unit balls of Banach spaces (see [1, 3, 4, 6, 12]). Here we give only one theorem which is a simple consequence of Theorem 1 in [7] or Theorem 3.8 in [12].

Theorem 1. *Let X be a finite-dimensional Banach space. Then its unit ball is plastic.*

However, the question about plasticity of the unit ball of an arbitrary infinite-dimensional Banach space is open. At least, there are no counterexamples. On the other hand, an example of non-plastic ellipsoid in separable Hilbert space was built in [3]. In [14], this example was generalized and the following definition was introduced.

Definition 1. Let M be a subset of a normed space X. We say that M is *linearly expand-contract plastic (briefly an LEC-plastic)* if every linear operator $T\colon X \to X$ whose restriction on M is a non-expansive bijection from M onto M is an isometry on M.

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In the mentioned article [14] the ellipsoids of the following form were considered

$$E = \Big\{ x = \sum_{n\in\mathbb{N}} x_n e_n \in H \colon \sum_{n\in\mathbb{N}} \left| \frac{x_n}{a(n)} \right|^2 \le 1 \Big\},$$

where H is a separable Hilbert space with basis $\{e_n\}_1^\infty$ and $a(n) > 0$. There was given a description of the LEC-plastic ellipsoids of such a form.

In what follows, we use the notations from [5]. The letter H denotes a fixed *separable infinite-dimensional Hilbert space* (real or complex), the symbol $\langle x, y\rangle$ stays for the *scalar product* of elements $x, y \in H$. We use the symbol Lin to denote the *linear span*, and the symbol $\overline{\mathrm{Lin}}$ to denote the *closed linear span*.

In the present paper, we will consider a more general definition of an *ellipsoid*.

Definition 2. An *ellipsoid* in H is a set of the form

$$E = \{x \in H \colon \langle x, Ax\rangle \le 1\},$$

where A is a self-adjoint operator such that $\inf_{\|x\|=1} \langle Ax, x\rangle > 0$ and $\sup_{\|x\|=1} \langle Ax, x\rangle < \infty$.

We will denote the boundary of E by

$$S = \{x \in H \colon \langle x, Ax\rangle = 1\}.$$

In what follows, $\sigma(A)$ will stand for the spectrum of A. Note that in this case $\sigma(A)$ is bounded from below and above by some positive constants.

In this paper we will show, that in fact the description of LEC-plastic ellipsoids in [14] was complete. In other words, there is no other LEC-plastic ellipsoids, except for those already described.

2. Basic facts. For our purpose, we will need some results related to measure theory (see, e.g., [5], [9], [13]). Let us collect these results.

Recall that a distribution function of a given positive finite Borel measure on the real numbers μ is given by

$$F_\mu(t) = \mu([0, t]).$$

Notice that this function is non-decreasing, and hence its generalized inverse

$$F_\mu^{-1}(t) = \sup\{x \colon F_\mu(x) \le t\}$$

is well-defined and also non-decreasing.

Notice that any normalized atomless Borel measure μ can be mapped into Lebesgue measure λ on $[0,1]$ (Indeed, performing a simple computation, we obtain

$$\mu([0, F_\mu^{-1}([0,t])]) = t = \lambda([0,t]),\ t \in [0,1]).$$

Hence, we get the following theorem.

Theorem 2 ([13] or Theorem 9.2.2 [2])**.** *All atomless standard probability spaces are mutually almost isomorphic.*

Corollary 1. *Let μ and ν be (finite and compactly supported) atomless Borel measures on $\mathbb{R}$ with $M_\nu = \nu(\mathbb{R})$ and $M_\mu = \mu(\mathbb{R})$. Then there exists a map $G_{\mu,\nu}$ such that $\nu = \frac{M_\nu}{M_\mu}\mu \circ G_{\mu,\nu}$.*

Remark 1. *Observe that:*

[1.] *$G_{\mu,\nu}$ can be written explicitly in terms of corresponding distribution functions, namely, $G_{\mu,\nu} = F_\mu^{-1} \circ \dfrac{M_\mu}{M_\nu} F_\nu$.*

[2.] $G_{\mu,\nu} \colon \operatorname{supp}(\nu) \longrightarrow \operatorname{supp}(\mu)$.

The next theorem can be found in [9], but since this source was not published yet, we will provide the proof.

Theorem 3 ([9], Theorem 2.16). *For given measure spaces (X, Σ) and (Y, Ω), a measure μ on Σ and a function $f\colon X \longrightarrow Y$ which is measurable w.r.t. μ, we define a measure $f_*\mu$ on Ω as $f_*\mu(B) = \mu(f^{-1}(B))$ for $B \in \Omega$. Let $g\colon Y \longrightarrow \mathbb{C}$ be a Borel function. Then the function $g \circ f\colon X \longrightarrow \mathbb{C}$ is integrable w.r.t. μ if and only if g is integrable w.r.t $f_*\mu$. Moreover*

$$\int_Y g d(f_*\mu) = \int_Y g \circ f d\mu.$$

Proof. It suffices to check this formula for simple functions g, which follows since $\chi_B \circ f = \chi_{f^{-1}(B)}$, where χ_B is the characteristic function of the set B. □

Furthermore, we will need some results related to operator theory (see, e.g., [8], [11]). Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then we can introduce continuous functions of A, as it is shown in the following theorem. Before formulating the theorem, let us recall that $C(\sigma(A))$ denotes the set of continuous functions on $\sigma(A)$ and $\mathcal{L}(H)$ stands for the space of linear operators acting from H to H.

Theorem 4 ([8], Theorem VII.1; [11], Theorem 3.1). *Let A be a bounded self-adjoint operator on a Hilbert space H. Then there is a unique map $\phi_A\colon C(\sigma(A)) \to \mathcal{L}(H)$ with the following properties: 1. $\phi_A(fg) = \phi_A(f)\phi_A(g)$, $\phi_A(\lambda f) = \lambda\phi_A(f)$, $\phi_A(1) = I$, $\phi_A(\bar{f}) = \phi_A(f)^*$; 2. $||\phi_A(f)||_{\mathcal{L}(H)} \leq C||f||_\infty$; 3. if $f(x) = x$, then $\phi_A(f) = A$.*

Moreover, 4. if $A\psi = \lambda\psi$, than $\phi_A(f)\psi = f(\lambda)\psi$; 5. $\sigma(\phi_A(f)) = \{f(\lambda)|\lambda \in \sigma(A)\}$; 6. if $f \geq 0$, then $\phi_A(f) \geq 0$; 7. $||\phi_A(f)|| = ||f||_\infty$.
Then we define $f(A) := \phi_A(f)$.

For every $\psi \in H$ we can define a corresponding linear functional on $C(\sigma(A))$ mapping $f \to \langle\psi, f(A)\psi\rangle$. Then by Riesz theorem, there exist a unique measure μ_ψ on $\sigma(A)$ such that $\langle\psi, f(A)\psi\rangle = \int_{\sigma(A)} f(\lambda)d\mu_\psi$. The measure μ_ψ is called the spectral measure associated with the vector ψ.

The next important result (spectral theorem) states that every bounded self-adjoin operator can be realized as multiplication operator on a suitable measure space. Let us specify that in the following theorem and further in the text we use the notion $L_2(\mathbb{R}, d\mu)$ for the space of measurable scalar-valued functions f on $\mathbb{R}$ for which the integral $\int_{\mathbb{R}} \|f(t)\|^2 d\mu$ exists and $\|f\| = (\int_{\mathbb{R}} \|f(t)\|^2 d\mu)^{1/2}$.

Theorem 5 ([8], Theorem VII.3; [11], Lemma 3.4 and Theorem 3.6). *Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then, there exist measures $\{\mu_n\}_{n=1}^N (N \in \mathbb{N}$ or $N = \infty)$ on $\sigma(A)$ and a unitary operator*

$$U\colon H \to \bigoplus_{n=1}^{N} L_2(\mathbb{R}, d\mu_n)$$

so that $(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda)$, where we write an element $\psi \in \oplus_{n=1}^N L_2(\mathbb{R}, d\mu_n)$ as an N-tuple $(\psi_1(\lambda), ..., \psi_N(\lambda))$. This realization of A is called a spectral representation.

Let us define $H_{pp} = \{\psi \in H|\mu_\psi$ is pure point$\}$, $H_{ac} = \{\psi \in H|\mu_\psi$ is absolutely continuous$\}$, $H_{sc} = \{\psi \in H|\mu_\psi$ is singularly continuous$\}$.

Theorem 6 ([8], Theorem VII.4; [11], Lemma 3.19)**.** *$H = H_{pp} \oplus H_{ac} \oplus H_{sc}$. Each of these subspaces is invariant under A. $A|_{H_{pp}}$ has a complete set of eigenvectors, $A|_{H_{ac}}$ has only absolutely continuous spectral measures and $A|_{H_{sc}}$ has only singularly continuous spectral measures.*

We will use the following notations:

$$\sigma_{pp} = \sigma(A|_{H_{pp}}), \quad \sigma_{cont} = \sigma(A|_{H_{cont}}), \quad \text{where } H_{cont} = H_{ac} \oplus H_{sc},$$
$$\sigma_{ac} = \sigma(A|_{H_{ac}}), \quad \sigma_{sc} = \sigma(A|_{H_{sc}}), \quad \sigma_p = \{\lambda \colon \lambda \text{ is an eigenvalue of } A\}.$$

Note that

$$\sigma_{cont} = \sigma_{ac} \cup \sigma_{sc}, \quad \sigma(A) = \overline{\sigma_p} \cup \sigma_{cont}.$$

The following useful results can be found in [11] and [10].

Theorem 7 ([11], Theorem 2.20)**.** *Let A be bounded self-adjoint. Then*

$$\inf\{\sigma(A)\} = \inf_{||x||=1} \langle x, Ax\rangle\,, \quad \sup\{\sigma(A)\} = \sup_{||x||=1} \langle x, Ax\rangle$$

Note that

$$\inf_{||x||=1} \langle x, Ax\rangle = \inf_{x\in H,\ x\neq 0} \frac{\langle x, Ax\rangle}{||x||^2}, \quad \sup_{||x||=1} \langle x, Ax\rangle = \sup_{x\in H,\ x\neq 0} \frac{\langle x, Ax\rangle}{||x||^2}.$$

Moreover, one can show the following:

Theorem 8 (see Problem 13.1 in [10])**.** *Let A be bounded self-adjoint (particularly, $\sigma(A) \subset [a, b]$). Then $\lambda_0 := \inf\{\sigma(A)\}$ is an eigenvalue iff $\inf_{||x||=1} \langle x, Ax\rangle$ is a minimum. In this case, eigenvectors are precisely the minimizers.*

Since [10] also was not published yet, we provide the proof.

Proof. Consider the functional $F\colon H \to \mathbb{R}$ given by $F(x) = \langle x, (A - \lambda_0)x\rangle$.
By assumption we have $F(x) \geq 0$ and $F(x_0) = 0$, where x_0 is the minimizer of $\langle x, Ax\rangle$. Then one may calculate the Gateaux derivative $\delta F(x_0, x) = 2\,\mathrm{Re}\,(\langle (A - \lambda_0)x_0, x\rangle) = 0$ for $x \in H$. Replacing x by ix we also get $2\,\mathrm{Im}\,(\langle (A - \lambda_0)x_0, x\rangle) = 0$. Hence $\langle (A - \lambda_0)x_0, x\rangle = 0$, for $x \in H$, which means $(A - \lambda_0)x_0 = 0$. □

3. Main result.

Proposition 1. *Suppose the spectrum $\sigma(A)$ of the self-adjoint operator A contains a set of eigenvalues B possessing the following properties:*
1. B has at least two elements;
2. either B doesn't have minimum or the multiplicity of the minimum is infinite;
3. either B doesn't have maximum or the multiplicity of the maximum is infinite.
Then E is not LEC-plastic.

Proof. Denote $r = \inf B$, $R = \sup B$; according to (1) $r < R$. The property (2) ensures the existence of distinct $n_k \in \mathbb{N}$, $k = 1, 2, \ldots$ such that eigenvalues $\lambda_{n_k} \in B$, $\lambda_{n_k} < \frac{1}{2}(r + R)$ and

$$\lambda_{n_1} \geq \lambda_{n_2} \geq \lambda_{n_3} \geq \ldots, \quad \lim_{k\to\infty} \lambda_{n_k} = r.$$

Analogously, the property (3) gives us the existence of distinct $n_k \in \mathbb{N}$, $k = 0, -1, -2, \dots$ such that $\lambda_{n_k} \in B$ and

$$\lambda_{n_1} < \lambda_{n_0} \le \lambda_{n_{-1}} \le \lambda_{n_{-2}} \le \dots, \quad \lim_{k \to -\infty} \lambda_{n_k} = R.$$

Take in H the orthonormalized eigenvectors e_{n_k} corresponding to λ_{n_k} (in case of infinite multiplicity of the minimum we are choosing them to be ONB of $Ker(A-\lambda_{min})$, analogously for the maximum) and extend to an orthonormal basis e_n in H. Define the linear operator T as follows: $Te_n = e_n$ for $n \in \mathbb{N} \setminus \{n_k\}_{k\in\mathbb{Z}}$, and $Te_{n_k} = \sqrt{\frac{\lambda_{n_k}}{\lambda_{n_{k-1}}}} e_{n_{k-1}}$, for $k \in \mathbb{Z}$. Using the fact that for a self-adjoint operator A and $x = \sum x_k e_{n_k}$ (here e_{n_k} are eigenvectors of A): $\langle x + x^\perp, A(x + x^\perp)\rangle = \langle x, Ax\rangle + \langle x^\perp, Ax^\perp\rangle$, we get that the linear non-expansive operator T maps E onto itself bijectively but not isometrically. □

Now, let us consider the case when the operator has purely continuous spectrum.

Proposition 2. *Let $\mu(t)$ be an (finite) atomless Borel measure with* $\operatorname{supp}(\mu) \subset (\alpha, \beta)$, $\alpha > 0$. *Let $A\colon L_2(\mathbb{R}, d\mu(t)) \to L_2(\mathbb{R}, d\mu(t))$ be an operator acting by the rule*

$$Af(t) = tf(t).$$

Consider an ellipsoid $E \in L_2(\mathbb{R}, d\mu(t))$ generated by this operator A, i.e.

$$E = \{f \in L_2(\mathbb{R}, \mu(t)dt)\colon \langle Af, f\rangle = \int_\alpha^\beta t|f(t)|^2 d\mu(t) \le 1\}.$$

E is not LEC-plastic.

Proof. Note that under our assumptions $L_2(\mathbb{R}, d\mu(t))$ can be identified with $L_2((\alpha, \beta), d\mu(t))$.

Consider some infinite partition of the segment $[\alpha, \beta) = \bigsqcup_{k=-\infty}^{\infty} \Delta_k$, where$\Delta_k = [a_k, a_{k+1})$, $a_{-\infty} = \alpha$, $a_\infty = \beta$, and such that $\mu(\Delta_k) > 0$ for all k (we can do this e.g. using any convergent series). Note that $L_2((\alpha, \beta), d\mu(t)) = \oplus_2 L_2(\Delta_k, d\mu_k(t))$ with $\mu_k := \mu|_{\Delta_k}$. We will use the following notation $M_{\mu_k} := \mu_k(\Delta_k) = \mu(\Delta_k)$. By Theorem 2, the spaces $(\Delta_k, \mu_k(t))$ are mutually almost isomorphic and we will denote the corresponding isomorphism by $G_k := G_{\mu_k,\mu_{k+1}}\colon \operatorname{supp}(\mu_{k+1}) \longrightarrow \operatorname{supp}(\mu_k)$.

Introduce the associated operators $H_k\colon L_2(\Delta_k, d\mu_k) \to L_2(\Delta_{k+1}, d\mu_{k+1})$ acting as

$$H_k f_k = f_k \circ G_k \cdot \sqrt{\frac{M_{\mu_{k+1}}}{M_{\mu_k}}}.$$

Then using Theorem 3 in the first step and Corollary 1 on the second step, we get

$$\int_{\Delta_{k+1}} |f_k \circ G_k|^2 \frac{M_{\mu_{k+1}}}{M_{\mu_k}} d\mu_{k+1} = \int_{\Delta_k} |f_k|^2 \frac{M_{\mu_{k+1}}}{M_{\mu_k}} d(G_{k*}\mu_{k+1}) = \int_{\Delta_k} |f_k|^2 d\mu_k.$$

For the second step note that $G_{k*}\mu_{k+1} = \mu_{k+1} \circ G_k = \frac{M_{\mu_k}}{M_{\mu_{k+1}}}\mu_k$. This means that

$$\|H_k f_k\|_{L_2(\Delta_{k+1}, d\mu_{k+1})} = \|f_k\|_{L_2(\Delta_k, d\mu_k)}.$$

Now, consider an operator $T\colon L_2((\alpha, \beta), d\mu(t)) \to L_2((\alpha, \beta), d\mu(t))$ acting as

$$Tf_k = g_k H_k f_k,$$

where g_k will be defined later.

Then we have

$$\langle f, Af\rangle = \sum_{k\in\mathbb{Z}} \int_{\Delta_k} t|f(t)|^2 d\mu_k(t),$$

and using again Theorem 3 and Corollary 1, we get

$$\langle Tf, ATf\rangle = \sum_{k\in\mathbb{Z}} \int_{\Delta_{k+1}} t|g_k(t)|^2 \left|(f_k \circ G_k)(t)\right|^2 \frac{M_{\mu_{k+1}}}{M_{\mu_k}} d\mu_{k+1}(t) =$$
$$= \sum_{k\in\mathbb{Z}} \int_{\Delta_k} G_k^{-1}(s) \left|(g_k \circ G_k^{-1})(s)\right|^2 |f_k(s)|^2 d\mu(s).$$

Let us denote $\hat{g}_k(s) := g_k G_k^{-1}(s)$.

We will choose g_k ($\hat{g}_k(s)$) in such way that $\langle f, Af\rangle = \langle Tf, ATf\rangle$ (then $T\colon E \longrightarrow E$ is bijective), i.e.

$$\hat{g}_k(s) := \sqrt{\frac{s}{G_k^{-1}(s)}}, \quad s \in \Delta_k.$$

It remains to show that T is non-expansive but not an isometry. Notice that $\hat{g}_k(s)^2 \le 1$ implies that T is non-expansive. Then to show that T is not an isometry it suffices to show that there are $k \in \mathbb{N}$ and $t \in \Delta_k$ such that $\hat{g}_k(s)^2 < 1$ in some neighbourhood of this t. Indeed, for $s \in \Delta_k$ by construction $G_k^{-1}(s) \in \Delta_{k+1}$, which means $\hat{g}_k(s)^2 < 1$ for $s \in \Delta_k$. Thus, T is non-expansive but not an isometry. □

Corollary 2. *Let A be a bounded self-adjoint operator. Let $\sigma_{cont} \neq \{\varnothing\}$. Then the ellipsoid generated by this operator is not LEC-plastic.*

Proof. Indeed, due to the Theorem 6 we can split an operator into two pieces: the first piece corresponds to the continuous measure, the second piece corresponds to the rest. Then we can apply Theorem 5 to the first piece and get that there is a spectral representation of $A|_{\sigma_{cont}}$. One may chose one of those parts where A acts as multiplication on the independent variable and the construction described in the previous proposition allows us to obtain an operator T which is a non-expansive bijection, but not an isometry. □

Three following lemmas are, in fact, building blocks for the proof of Theorem 9.

Lemma 1. *Let $T\colon H \to H$ be a linear operator which maps E bijectively onto itself. Then T maps the whole H bijectively onto itself and $T(S) = S$. If, moreover, T is non-expansive on E, then $\|T\| \le 1$.*

Proof. E is absorbing since it contains a ball (the spectrum is bounded from below by positive constant). The rest of the proof is as in the article [14]. For convenience of the reader we give it here.

E is an absorbing set, so $H = \cup_{t>0} tE$. By linearity T is injective on every set tE, consequently it is injective on the whole H. Also, $T(H) = \cup_{t>0} T(tE) = \cup_{t>0} tE = H$ which gives the surjectivity on H. Finally, $S = E \setminus \cup_{t\in(0,1)} tE$, so $T(S) = T(E) \setminus \cup_{t\in(0,1)} T(tE) = E \setminus \cup_{t\in(0,1)} tE = S$. If, moreover, T is non-expansive on E, then for every $x \in H$ there exists $t > 0$ such that $tx \in E$ and we have $\|T(tx)\| = \|(T(0) - T(tx))\| \le \|tx - 0\| = \|tx\|$. It remains to divide by t to obtain that $\|Tx\| \le \|x\|$ for all $x \in H$.

□

Before moving to the next result, let us introduce the following notation:

$$H_t = Ker(A - t).$$

Lemma 2. *Let $\sigma(A) = \sigma_{pp}(A) \subset (0, +\infty)$ and let the set of eigenvalues of an operator A contain the minimal element r and let r have finite multiplicity. Let $T\colon H \to H$ be a linear operator which maps E bijectively onto itself and whose restriction on E is non-expansive. Then $T(H_r) = H_r$, $T(H_r \cap E) = H_r \cap E$ and the restriction of T onto H_r is a bijective isometry.*

Proof. Theorem 7 implies that $r = \inf_{||x||=1} \langle x, Ax\rangle = \inf_{x\in H,\ x\neq 0} \frac{\langle x,Ax\rangle}{||x||^2}$.

Recall that for $x \in S$, we have $\langle x, Ax\rangle = 1$. Hence, $r = \inf_{x\in H,\ x\neq 0} \frac{\langle x,Ax\rangle}{||x||^2} = \inf_{x\in S} \frac{\langle x,Ax\rangle}{||x||^2} = \inf_{x\in S} \frac{1}{||x||^2}$ (second equality: renormalization using $\langle x, Ax\rangle$ instead of norm). That is, for $x \in S$ we have $||x|| \le \sqrt{\frac{1}{r}}$.

Let $x \in S$ with $||x|| = \sqrt{\frac{1}{r}}$ (i.e. by Theorem 8 it is an eigenvector corresponding to r. Moreover, all eigenvectors can be written in this form after a suitable renormalisation). Since T is non-expansive on E, we have $||T^{-1}(x)|| \ge ||x|| = \sqrt{\frac{1}{r}}$ and using $T^{-1}(S) = S$, we get $||T^{-1}(x)|| = \sqrt{\frac{1}{r}}$. Hence, by Theorem 8, $T^{-1}(x)$ is an eigenvector corresponding to r. Hence, $T^{-1}(H_r) \subset H_r$.

Note that H_r is finite dimensional, hence $T^{-1}|_{H_r}\colon H_r \mapsto H_r$ is surjective iff $T^{-1}\colon H_r \mapsto H_r$ is injective. Injectivity of $T^{-1}|_{H_r}$ follows from Lemma 1 (i.e. injectivity of T^{-1}). Hence, $T^{-1}|_{H_r}\colon H_r \mapsto H_r$ is bijection and $T^{-1}(H_r) = H_r$ (i.e. $T(H_r) = H_r$). Hence, also $T(H_r \cap E) = H_r \cap E$ (combining $T(H_r) = H_r$ with $T(E) = E$).

We obtain the last claim using linearity and the fact that for all eigenvectors x corresponding to r with norm $\sqrt{\frac{1}{r}}$, we have that $||T^{-1}(x)|| = \sqrt{\frac{1}{r}}$. (We can also observe that $S \cap H_r$ is a sphere in H_r and map T from $S \cap H_r$ onto $S \cap H_r$ is bijective.)

□

Lemma 3. *Let $\sigma(A) = \sigma_{pp}(A) \subset (0, +\infty)$ and let the set of eigenvalues of an operator A contain the maximal element R and let R have finite multiplicity. Let $T\colon H \to H$ be a linear operator which maps E bijectively onto itself and whose restriction on E is non-expansive. Then $T(H_R) = H_R$, $T(H_R \cap E) = H_R \cap E$ and the restriction of T onto H_R is a bijective isometry.*

Proof. The proof is similar to the previous one (alternatively consider T^{-1} instead of T and apply the previous result). □

The proof of the next theorem partially repeats the proof of Theorem 1 in [14]. To make this work self-contained we provide all details.

Theorem 9. *Let A be a bounded self-adjoint operator. Then an ellipsoid E generated by A is LEC-plastic if and only if the following two conditions hold:*

1. *$\sigma_{cont} = \varnothing$;*
2. *every subset of $\sigma_p(A)$ that consists of more than one element either has a maximum of finite multiplicity or has a minimum of finite multiplicity.*

Proof. We only need to prove the "if"part of the statement. Note that under our assumptions the spectrum $\sigma(A) = \overline{\sigma_p(A)} = \sigma_{pp}(A)$ (i.e. consists of eigenvalues and their limiting points). Moreover A cannot contain more than one element of infinite multiplicity.

Note that in this case there exists a basis of eigenvectors of A (see Problem 3.26 in [11]).

Claim 1. There is a $\tau > 0$ such that $A^+ = \sigma_p(A) \cap (\tau, +\infty)$ is well-ordered with respect to the ordering $\geq$ (that is every non empty subset of A^+ has a maximal element), $A^- = \sigma_p(A) \cap (0, \tau)$ is well-ordered with respect to the ordering $\leq$ (that is every non empty subset of A^- has a minimal element), and neither A^+ nor A^- contain elements of infinite multiplicity.

Indeed, if there is an element $a_\infty \in \sigma_p(A)$ of infinite multiplicity, let us take $\tau = a_\infty$. Let us demonstrate that $(A^+, \geq)$ is well-ordered. If $A^+ = \varnothing$ the statement is clear. In the other case for every non empty subset D of A^+ consider $B = \{\tau\} \cup D$. Then the minimal element of B is τ , which has infinite multiplicity so B must have a maximum of finite multiplicity. This maximum will be also the maximal element of D. The demonstration of well ordering for $(A^-, \leq)$ works in the same way.

Now, consider the remaining case of $\sigma_p(A)$ consisting only of finite multiplicity elements. Consider the set U of all those $t \in (0, +\infty)$ that $\sigma_p(A) \cap (t, +\infty)$ is not empty and well-ordered with respect to the ordering $\geq$. If U is not empty, take $\tau = \inf U$, if $U = \varnothing$, take $\tau = \sup \sigma_p(A)$. Let us demonstrate that this τ is what we need. In the first case $A^+ = \sigma_p(A) \cap (\tau, +\infty)$ and for every $t > \tau$ we have $\sigma_p(A) \cap (t, +\infty)$ is not empty and well-ordered with respect to the ordering $\geq$. This implies that $(A^+, \geq)$ is well-ordered. In the second case $A^+ = \varnothing$, which is also well- ordered. So, it remains to demonstrate that $A^- = \sigma_p(A) \cap (0, \tau)$ is well-ordered with respect to the ordering $\leq$. Assume this is not true. Then, there is a non empty subset $B \subset A^-$ with no minimal element. According to the conditions of our theorem B has a maximal element b. Since $b < \tau$ and by definition of τ the set $\sigma_p(A) \cap (b, +\infty)$ is not well-ordered with respect to the ordering $\geq$. Consequently, there is a non empty $D \subset \sigma_p(A) \cap (b, +\infty)$ with no maximal element. Then, $B \cup D$ satisfies neither condition (1) nor condition (2) of our theorem. This contradiction completes the demonstration of Claim 1.

We introduce the following three subspaces:

- H^- is the closed linear span of the set of all those eigenvectors, for which the corresponding eigenvalue lies in A^-;
- H^τ is the closed linear span of the set of all those eigenvectors, for which the corresponding eigenvalue is τ if τ is eigenvalue or empty otherwise;
- H^+ is the closed linear span of the set of all those eigenvectors, for which the corresponding eigenvalue lies in A^+.

Since eigenvectors of a self-adjoint operator corresponding to different eigenvalues are orthogonal, using continuity of scalar product, we get that these subspaces are mutually orthogonal and $H = H^- \oplus H^\tau \oplus H^+$ (we have equality here since the set of all eigenvectors of A spans H). Let $T \colon H \mapsto H$ be a linear operator which maps E bijectively onto itself and whose restriction on E is non-expansive.

Claim 2. $T(H^-) = H^-$, $T(H^+) = H^+$ and the restrictions of T onto H^- and H^+ are bijective isometries.

We will demonstrate the part of our claim that speaks about H^+: the reasoning about H^- will differ only in the usage of Lemma 2 instead of Lemma 3.

Let us define a subspace $H(t)$ as the closed linear span of the set of all those eigenvectors, for which the corresponding eigenvalue lies in $A^+ \cap [t, +\infty)$.

If $A^+ = \varnothing$ there is nothing to do. In the case of $A^+ \neq \varnothing$ we are going to demonstrate by transfinite induction in $t \in (A_+, \geq)$ the validity for all $t \in A_+$ of the following statement $\mathfrak{U}(t)$: the subspace $H(t)$ is T-invariant and T maps $H(t)$ onto $H(t)$ isometrically. Since the collection of subspaces $H(t)$, $t \in A_+$ is a chain whose union is dense in H_+, the continuity of T will imply the desired Claim 2.

The base of induction is the statement $\mathfrak{U}(t)$ for $t = \max A$. This is just the statement of Lemma 3. We assume now as inductive hypothesis the validity of $\mathfrak{U}(t)$ for all $t > t_0 \in A^+$, and our goal is to prove the statement $\mathfrak{U}(t_0)$. For every x, y in H let us introduce a modified scalar product $\langle\langle x, y\rangle\rangle$ as follows:

$$\langle\langle x, y\rangle\rangle = \langle x, Ay\rangle .$$

Then the norm on H induced by this modified scalar product is

$$|||x||| = \sqrt{\langle\langle x, x\rangle\rangle} = \sqrt{\langle x, Ax\rangle}.$$

The ellipsoid E is the unit ball in this new norm and since T is linear and maps E onto E bijectively, T is a bijective isometry of $(H, |||\cdot|||)$ onto itself. Due to [5, Theorem 2, p. 353] T is a unitary operator in the modified scalar product and thus T preserves the modified scalar product. In particular, it preserves the orthogonality in the modified scalar product. Denote

$$X = \bigcup_{t>t_0} H(t).$$

In other words, X is the linear span of the set of all those eigenvectors, for which the corresponding eigenvalue lies in $A^+ \cap (t_0, +\infty)$.

The orthogonal complement to X in the modified scalar product $X^\perp$ is the closed linear span of the set of all those eigenvectors, for which the corresponding eigenvalue lies in $A^+ \cap (0, t_0]$.

Occasionally the orthogonal complement to X in the original scalar product is the same. Our inductive hypothesis implies that $T(X) = X$, consequently $T(X^\perp) = X^\perp$ and $T(X^\perp \cap E) = X^\perp \cap E$.

$X^\perp$ equipped with the original scalar product is a Hilbert space, $X^\perp \cap E$ is an ellipsoid in $X^\perp$, t_0 is the maximal eigenvalue of A and the multiplicity of t_0 is finite because $t_0 \in A^+$. The application of Lemma 3 gives us that $T(H_{t_0}) = H_{t_0}$ and the restriction of T onto H_{t_0} is a bijective isometry in the original norm. Now, T maps X onto X isometrically, maps H_{t_0} onto H_{t_0} isometrically and $H(t_0)$ is the orthogonal direct sum of subspaces H_{t_0} and the closure of X. This implies that T maps $H(t_0)$ onto $H(t_0)$ isometrically, and the inductive step is done. This completes the demonstration of Claim 2.

From Claim 2 and mutual orthogonality of H^- and H^+ we deduce that $T(H^- \oplus H^+) = H^- \oplus H^+$ and T is an isometry on $H^- \oplus H^+$. Recalling again that T preserves the modified scalar product and the orthogonal complement to X in the modified scalar product is H_τ we obtain that $T(H_\tau) = H_\tau$ and consequently $T(H_\tau \cap E) = H_\tau \cap E$. But $H_\tau \cap E$ is equal to the closed ball (in the original norm) of radius $\frac{1}{\sqrt{\tau}}$ centered at 0, so the equality $T(H_\tau \cap E) = H_\tau \cap E$ and linearity of T implies that T is an isometry on H_τ. Finally, as we know, $H = H^- \oplus H_\tau \oplus H^+$, so T is an isometry on the whole H. □

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