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SLICE HOLOMORPHIC FUNCTIONS IN THE UNIT BALL: BOUNDEDNESS OF L-INDEX IN A DIRECTION AND RELATED PROPERTIES

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Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a fixed direction. We consider slice holomorphic functions of several complex variables in the unit ball, i.e. we study functions which are analytic in intersection of every slice $\{z^0+t\mathbf{b}\colon t\in\mathbb{C}\}$ with the unit ball $\mathbb{B}^n=\{z\in\mathbb{C}\colon\ |z|:=\sqrt{|z|_1^2+\ldots+|z_n|^2}<1\}$ for any $z^0\in\mathbb{B}^n$. For this class of functions we consider the concept of boundedness of L-index in the direction $\mathbf{b}$, where $\mathbf{L}\colon\mathbb{B}^n\to\mathbb{R}_+$ is a positive continuous function such that $L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}$ and $\beta>1$ is some constant. For functions from this class we deduce analog of Hayman's Theorem. It is criterion useful in applications to differential equations. We introduce a concept of function having bounded value L-distribution in direction for the slice holomorphic functions in the unit ball. It is proved that a slice holomorphic function in the unit ball has bounded value L-distribution in a direction if and only if its directional derivative has bounded L-index in the same direction. Other propositions concern existence theorems. We show that for any slice holomorphic function F with bounded multiplicities of zeros on any slice in the fixed direction there exists such a positive continuous function L that the function F has bounded L-index in the direction.

1. Introduction and preliminaries. Here we continue our investigations initiated in [1, 2]. There was introduced a concept of L-index boundedness in direction for slice analytic functions of several complex variables and obtained many criteria of L-index boundedness in direction. Here we present some applications of these criteria to describe value distribution of slice analytic functions in the unit ball. We obtained analog of Hayman's theorem for this class of functions. This theorem is very important in theory of bounded index because it allows to study analytic solutions of differential equations and their systems. Applications of the theorem to differential equations in many cases [9, 17, 26] gave possibility to deduce sufficient conditions of index boundedness of their analytic solutions. Moreover, we examine value distribution of function belonging to this function class and prove existence theorem. It demonstrates an extent of the class, i.e. slice holomorphic functions of bounded L-index in direction.

As in [1, 2], we continue to consider the following general problem.

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Problem 1. *Is it possible to deduce main facts of theory of analytic functions having bounded L-index in the direction $\mathbf{b} \in \mathbb{B}^n \setminus \{0\}$ for functions which are holomorphic on the slices $\{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$ and are joint continuous?*

Let us introduce some notations from [1]. Let $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}^*_+ = [0, +\infty)$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{B}^n = \{z \in \mathbb{C}^n\colon |z| < 1\}$ be the unit ball, $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ be the unit disc, $L\colon\ \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that, for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1 - |z|},\ \beta = \text{const} > 1. \tag{1}$$

For a given $z \in \mathbb{B}^n$, we denote $S_z = \{t \in \mathbb{C}\colon z + t\mathbf{b} \in \mathbb{B}^n\}$. Clearly, $\mathbb{D} = \mathbb{B}^1$.

The slice functions on S_z for fixed $z^0 \in \mathbb{B}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$ for $t \in S_z$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be the class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b}\colon t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be the class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous.

The notation $\partial_{\mathbf{b}} F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for every $p \in \mathbb{N}$ $\partial^p_{\mathbf{b}} F(z) = g^{(p)}_z(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is an analytic function of complex variable $t \in S_z$ for given $z \in \mathbb{B}^n$. In this research, we will often call this derivative as directional derivative because if F is an analytic function in $\mathbb{B}^n$ then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function F. Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction do not imply holomorphy in whole n-dimensional unit ball. There were presented some examples to demonstrate it (see [1, 3]).

A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said [1] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{C}^n$ inequality

$$\frac{|\partial^m_{\mathbf{b}} F(z)|}{m! L^m(z)} \le \max_{0 \le k \le m_0} \frac{|\partial^k_{\mathbf{b}} F(z)|}{k! L^k(z)}, \tag{2}$$

is true. The least such integer number m_0, obeying (2), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F, L, \mathbb{B}^n)$. If such m_0 does not exist, then we put $N_{\mathbf{b}}(F, L, \mathbb{B}^n) = \infty$, and the function F is called of unbounded L-index in the direction $\mathbf{b}$ in this case. For $n = 1$, $\mathbf{b} = 1$, $L(z) = l(z)$, $z \in \mathbb{C}$ inequality (2) defines a function of bounded l-index with the l-index $N(F, l) \equiv N_1(F, l, \mathbb{C})$ [21, 22], and if in addition $l(z) \equiv 1$, then we obtain a definition of index boundedness with index $N(F) \equiv N_1(F, 1, \mathbb{C})$ [23, 24]. It is also worth to mention paper [32], which introduces the concept of generalized index. It is quite close to the bounded l-index.

In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an analytic function in the unit disc of bounded l-index [33]. Similarly, analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if it satisfies (2) for all $z \in \mathbb{B}^n$. If $z \in \mathbb{C}$ (instead $\mathbb{B}^n$), $n = 1$ and $L = l$ we obtain definition of bounded l-index for entire functions of one variable [21], and if in addition $l \equiv 1$ we have definition of entire function of bounded index [23].

Note that the positivity and continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function L.

For $z\in\mathbb{B}^n$ we denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{B}^n}\sup_{t_1,t_2\in S_z}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for the class of positive continuous functions $L\colon\mathbb{B}^n\to\mathbb{R}_+$, satisfying for every $\eta\in[0,\beta]$

$$\lambda_{\mathbf{b}}(\eta)<+\infty \tag{3}$$

and for all $z\in\mathbb{B}^n$ $L(z)>\frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta>1$ is some constant.

In our investigations we need the following propositions obtained in [1,2]. The first assertion indicates that multiplying of the function L by a constant does not change L-index boundedness in direction.

Proposition 1 ([1]). *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta}<\theta_1\le\theta_2<+\infty$, $\theta_1L(z)\le L^*(z)\le\theta_2L(z)$ for all $z\in\mathbb{B}^n$. A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L^*-index in the direction $\mathbf{b}$ if and only if F is of bounded L-index in the direction $\mathbf{b}$.*

Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that $\operatorname{dist}(D,\mathbb{B}^n)>0$. If the inequality (2) holds for all $z\in D$ instead $\mathbb{B}^n$, then the function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is called a *function of bounded L-index in the direction $\mathbf{b}$ in the domain D*. The least such integer m_0 is called the *L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ in the domain D* and is denoted by $N_{\mathbf{b}}(F,L,D)=m_0$. The notation $\overline{D}$ stands for the closure of the domain D.

Lemma 1 ([1]). *Let D be a bounded domain in $\mathbb{B}^n$ such that $d=\operatorname{dist}(\overline{D},\mathbb{B}^n)=\inf_{z\in D}(1-|z|)>0$, $\beta>1$, $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be an arbitrary direction. If $L\colon\mathbb{B}^n\to\mathbb{R}_+$ is a continuous function such that for all $z\in\mathbb{B}^n$*

$$L(z)\ge\frac{\beta|\mathbf{b}|}{d} \tag{4}$$

and a function $F\in\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is such that $(\forall z^0\in\overline{D})\colon\ F(z^0+t\mathbf{b})\not\equiv0$, then $N_{\mathbf{b}}(F,L,D)<\infty$.

In other words, Lemma 1 shows that every slice holomorphic in the unit ball function has bounded L-index in any direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ for any domain which is compactly embedded in the unit ball and for any continuous function $L\colon\mathbb{B}^n\to\mathbb{R}_+$ satisfying (4)

The next theorem describes local behavior of the slice holomorphic function in the unit ball (see also [4]). It presents sufficient conditions of boundedness of L-index in direction for this class of functions.

Theorem 1 ([2]). *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$. If there exist r_1 and r_2, $0<r_1<r_2\le\beta$, and $P_1\ge1$ such that for all $z^0\in\mathbb{B}^n$ inequality*

$$\max\left\{|F(z^0+t\mathbf{b})|\colon |t|{=}r_2/L(z^0)\right\}\le P_1\max\left\{|F(z^0{+}t\mathbf{b})|\colon\ |t|{=}r_1/L(z_0)\right\}.$$

is satisfied then the function F is of bounded L-index in the direction $\mathbf{b}$.

2. Analog of Hayman's Theorem. Below we formulate and prove criterion which is analog of Hayman's Theorem [20] (see also [6, 11–14, 29]).

Theorem 2. *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$. A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist $p\in\mathbb{Z}_+$ and $C>0$ such that for every $z\in\mathbb{B}^n$ one has*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)}\le C\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}\colon\ 0\le k\le p\right\}. \tag{5}$$

Proof. The proof use ideas and considerations from the proof for analytic in the unit ball functions of bounded L-index in direction [13]. Also, there are known analogs of Hayman's theorem for other classes of analytic functions [6, 14].
Necessity. If $N_{\mathbf{b}}(F, L) < +\infty$, then by definition of boundedness of L-index in direction we obtain (5) with $p = N_{\mathbf{b}}(F, L, \mathbb{B}^n)$ and $C = (N_{\mathbf{b}}(F, L, \mathbb{B}^n) + 1)!$
Sufficiency. Let inequality (5) be fulfilled, $z^0 \in \mathbb{B}^n$ and $K = \{t \in \mathbb{C}\colon |t| \leq 1/L(z^0)\}$. Since $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, for every $t \in K$ from (5) it follows

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0)} \leq \left(\frac{L(z^0+t\mathbf{b})}{L(z^0)}\right)^{p+1} \frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq$$
$$\leq (\lambda_{\mathbf{b}}(1))^{p+1} \frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq C(\lambda_{\mathbf{b}}(1))^{p+1} \max_{0\leq k\leq p} \left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{L^{k}(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq C(\lambda_{\mathbf{b}}(1))^{p+1} \max_{0\leq k\leq p} \left\{\left(\frac{L(z^0)}{L(z^0+t\mathbf{b})}\right)^k \frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{L^{k}(z^0)}\right\} \leq$$
$$\leq C(\lambda_{\mathbf{b}}(1))^{p+1} \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{L^{k}(z^0)}(\lambda_{\mathbf{b}}(1))^k\colon 0\leq k\leq p\right\} \leq Bg_{z^0}(t), \tag{6}$$

where $B = C(\lambda_{\mathbf{b}}(1))^{2p+1}$ and $g_{z^0}(t) = \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{L^{k}(z^0)}\colon 0 \leq k \leq p\right\}$.

Let us denote $\gamma_1 = \left\{t \in \mathbb{C}\colon |t| = \frac{1}{2\beta L(z^0)}\right\}$, $\gamma_2 = \left\{t \in \mathbb{C}\colon |t| = \frac{\beta}{L(z^0)}\right\}$. Choose arbitrarily two points $t_1 \in \gamma_1$, $t_2 \in \gamma_2$ and connect them by a piecewise analytic curve $\gamma = (t = t(s), 0 \leq s \leq T)$ such that $g_{z^0}(t) \neq 0$ for $t \in \gamma$. We construct the curve γ such that its length $|\gamma|$ does not exceed $\dfrac{2\beta^2+1}{\beta L(z^0)}$. Such a curve can be constructed.

The function $g_{z^0}(t(s))$ is continuous on $[0, T]$. Without loss of generality we may assume that the function $t = t(s)$ is analytic on $[0, T]$. Otherwise, one can consider each interval of analyticity of this function separately and repeat the corresponding considerations, which are given below on $[0, T]$. First, we show that the function $g_{z^0}(t(s))$ is continuously differentiable on $[0, T]$ except possibly a finite set of points. For arbitrary $k_1, k_2, 0 \leq k_1 \leq k_2 \leq p$, either $\frac{|\partial_{\mathbf{b}}^{k_1}F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)} \equiv \frac{|\partial_{\mathbf{b}}^{k_2}F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ or the equality $\frac{|\partial_{\mathbf{b}}^{k_1}F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)} = \frac{|\partial_{\mathbf{b}}^{k_2}F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ is true for a finite set of points $s_k \in [0, T]$. Then we can split the segment $[0, T]$ onto a finite number of segments such that on each of them $g_{z^0}(t(s)) \equiv \frac{|\partial_{\mathbf{b}}^{k}F(z^0+t(s)\mathbf{b})|}{L^{k}(z^0)}$ for some k, $0 \leq k \leq p$. It means that the function $g_{z^0}(t(s))$ is continuously differentiable with exception, perhaps, of a finite set of points. Taking into account (6), we obtain

$$\frac{dg_{z^0}(t(s))}{ds} \leq \max\left\{\frac{d}{ds}\left(\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t(s)\mathbf{b})|}{L^{k}(z^0)}\right)\colon 0\leq k\leq p\right\} \leq$$
$$\leq \max\left\{|\partial_{\mathbf{b}}^{k+1}F(z^0+t(s)\mathbf{b})||t'(s)|/L^{k}(z^0)\colon 0\leq k\leq p\right\} =$$
$$= L(z^0)|t'(s)| \max\left\{|\partial_{\mathbf{b}}^{k+1}F(z^0+t(s)\mathbf{b})|/L^{k+1}(z^0)\colon\ 0\leq k\leq p\right\} \leq Bg_{z^0}(t(s))|t'(s)|L(z^0).$$

Hence, we have

$$\left|\ln\frac{g_{z^0}(t_2)}{g_{z^0}(t_1)}\right| = \left|\int_0^T \frac{dg_{z^0}(t(s))}{g_{z^0}(t(s))}\right| \leq BL(z^0)\int_0^T |t'(s)|ds = BL(z^0)|\gamma| \leq \frac{2\beta^2+1}{\beta}B.$$

If we choose a point $t_2 \in \gamma_2$, such that $|F(z^0+t_2\mathbf{b})| = \max\{|F(z^0+t\mathbf{b})|\colon |t|=\beta/L(z^0)\}$, then we obtain

$$\max\left\{|F(z^0+t\mathbf{b})|\colon |t| = \frac{\beta}{L(z^0)}\right\} \le g_{z^0}(t_2)\le g_{z^0}(t_1)\exp\{(2\beta^2+1)/\beta\}. \qquad (7)$$

Applying Cauchy's inequality and using that $t_1 \in \gamma_1$ we obtain for all $j \in \{1,\dots,p\}$

$$|\partial_{\mathbf{b}}^j F(z^0+t_1\mathbf{b})|\le j!(2\beta L(z^0))^j \max\left\{|F(z^0+t\mathbf{b})\colon |t-t_1| = \frac{1}{2\beta L(z^0)}\right\}\le$$
$$\le j!(2\beta L(z^0))^j \max\left\{|F(z^0+t\mathbf{b})\colon |t-t_0| = \frac{1}{\beta L(z^0)}\right\},$$

$$\text{i.e. } g_{z^0}(t_1) \le p!(2\beta)^p \max\left\{|F(z^0+t\mathbf{b})\colon |t-t_0| = \frac{1}{\beta L(z^0)}\right\}.$$

Therefore, from (7) it follows that

$$|F(z^0+t_2\mathbf{b})| = \max\left\{|F(z^0+t\mathbf{b})|\colon |t| = \beta/L(z^0)\right\} \le$$
$$\le g_{z^0}(t_2)\le g_{z^0}(t_1)\exp\left\{(2\beta^2+1)/\beta\right\}\le p!(2\beta)^p\exp\left\{(2\beta^2+1)/\beta\right\}\times$$
$$\times\max\left\{|F(z^0+t\mathbf{b})|\colon |t| = 1/(\beta L(z^0))\right\}.$$

By Theorem 1, we conclude that the function F has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n$. Theorem 2 is proved. □

Using Theorem 2 we prove the following

Theorem 3. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $C \in (0,+\infty)$ and $N \in \mathbb{N}$ such that for all $z \in \mathbb{B}^n$*

$$\sum_{k=0}^{N}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} \ge C\sum_{k=N+1}^{\infty}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}. \qquad (8)$$

Proof. Proof of this theorem is similar to that of its analogs for slice entire functions of bounded L-index in direction [10] and for entire functions of bounded l-index [28].

Let $0 < \theta < 1$. If the function F is of bounded L-index in the direction $\mathbf{b}$, then by Proposition 1 the function F is also of bounded L^*-index in the direction $\mathbf{b}$, where $L^*(z) = \theta L(z)$. Denote $N^* = N_{\mathbf{b}}(F, L_*, \mathbb{B}^n)$ and $N = N_{\mathbf{b}}(F, L, \mathbb{B}^n)$. Thus,

$$\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}\colon 0\le k\le N^*\right\} = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L_*^k(z)}\theta^k\colon 0\le k\le N^*\right\}\ge$$
$$\ge\theta^{N^*}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L_*^k(z)}\colon 0\le k\le N^*\right\}\ge\theta^{N^*}\frac{|\partial_{\mathbf{b}}^j F(z)|}{j!L_*^j(z)} = \theta^{N^*-j}\frac{|\partial_{\mathbf{b}}^j F(z)|}{j!L^j(z)}$$

for all $j \ge 0$ and

$$\sum_{j=N^*+1}^{\infty}\frac{|\partial_{\mathbf{b}}^j F(z)|}{j!L^j(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}\colon 0\le k\le N^*\right\}\sum_{j=N^*+1}^{\infty}\theta^{j-N^*} =$$

$$= \frac{\theta}{1-\theta}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}\colon 0\le k\le N^*\right\} \le \frac{\theta}{1-\theta}\sum_{k=0}^{N^*}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)},$$

i.e. we obtain (8) with $N = N^*$ and $C = \frac{1-\theta}{\theta}$.

Now we prove the sufficiency. From (8) we obtain

$$\frac{|\partial_{\mathbf{b}}^{N+1}F(z)|}{(N+1)!L^{N+1}(z)} \le \sum_{k=N+1}^{\infty}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} \le \frac{1}{C}\sum_{k=0}^{N}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} \le \frac{N+1}{C}\max_{0\le k\le N}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}.$$

Applying Theorem 2, we obtain the desired conclusion. □

Using Lemma 1 and Theorem 2 we obtain this corollary.

Corollary 1. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$, G be a bounded domain in $\mathbb{B}^n$ such that $\forall z \in \overline{G}$ $F(z + t\mathbf{b}) \not\equiv 0$. The function F has bounded L-index in the direction* $\mathbf{b}$ *if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in \mathbb{C}^n \setminus G$ the inequality* (5) *holds.*

3. Functions having bounded value L-distribution in direction.

Let us remind the notion of function having bounded value distribution. An entire function $f(z)$ ($z \in \mathbb{C}$) is said ot be of bounded value distribution [18, 20, 27], if there exist $p \ge 0$, $R > 0$ such that the equation $f(z) = w$ has at least p roots in any disc of radius R.

One of the remarkable properties generating big interest to functions of bounded index is the following fact proved by W. Hayman [20]: an entire function has bounded value distribution if and only if its derivative has bounded index. Leter, there was introduced a concept of entire function of bounded value l-distribution [21], and this property was generalized for entire functions of bounded l-index [30]. For entire bivariate functions of bounded index in joint variables similar results are partially obtained in [25].

Definition 1. Function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said to be *of bounded value L-distribution in a direction* $\mathbf{b}$ if for all $p \in \mathbb{N}$, $w \in \mathbb{C}$ and $z_0 \in \mathbb{B}^n$ such that $F(z^0 + t\mathbf{b}) \not\equiv w$, the inequality holds $n\big(\frac{1}{L(z^0)}, z^0, \frac{1}{F-w}\big) \le p$, i.e. the equation $F(z^0 + t\mathbf{b}) = w$ has at most p solutions in the disc $\{t\colon\ |t| \le \frac{1}{L(z^0)}\}$. In other words, the function $F(z^0 + t\mathbf{b})$ is p-valent in $\{t\colon\ |t| \le \frac{1}{L(z^0)}\}$ for each fixed $z^0 \in \mathbb{B}^n$.

We will generalize the corresponding Sheremeta's result [30] for the functions from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, which have bounded value L-distribution in direction $\mathbf{b}$.

Proposition 2. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. An entire function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is a function of bounded value L-distribution in the direction* $\mathbf{b}$ *if and only if the directional derivative $\partial_{\mathbf{b}}F$ has bounded L-index in the same direction* $\mathbf{b}$.

Proof. Our proof is similar to the proof of the corresponding proposition for analytic functions in the unit ball [5].

Suppose that F is of bounded value L-distribution in direction $\mathbf{b}$, i.e. for all $z^0 \in \mathbb{B}^n$ such that $F(z^0 + t\mathbf{b}) \not\equiv$ *const* the function $F(z^0 + t\mathbf{b})$ is p-valent in every disc $\{t\colon\ |t| \le \frac{1}{L(z^0)}\}$.

To prove the proposition we need the following proposition from [29, p. 48, Theorem 2.8].

Theorem 4 ([29]). *Let $D_0 = \{t\colon\ |t-t_0| < R\}$, $0 < R < \infty$. If analytic in D_0 function f is p-valent in D_0, then for $j > p$*

$$\frac{|f^{(j)}(t_0)|}{j!}R^j \le (Aj)^{2p} \max\left\{\frac{|f^{(k)}(t_0)|}{k!}R^k\colon\ 1\le k\le p\right\}, \tag{9}$$

where $A = \sqrt[2p]{\frac{p+2}{2}}\sqrt{8e^{\pi^2}}$.

By Theorem 4, inequality (9) holds with $R = \frac{1}{L(z^0)}$ for the function $g_{z^0}(t) = F(z^0 + t\mathbf{b})$, as a function of single variable $t \in \mathbb{C}$ for every fixed $z^0 \in \mathbb{B}^n$. Then it is easy to deduce that for every $m \in \mathbb{N}$ the following equality $g_{z^0}^{(p)}(t) = \partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})$ holds. Take $j = p+1$ and $t_0 = 0$ in Theorem 4. From (9) it follows

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z^0)\right|}{(p+1)!L^{p+1}(z_0)} \le (A(p+1))^{2p} \max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z^0)\right|}{k!L^k(z_0)}\colon\ 1\le k\le p\right\} \Rightarrow$$

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z^0)\right|}{L^{p+1}(z_0)} \le (p+1)!(A(p+1))^{2p} \max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z^0)\right|}{L^k(z_0)}\colon 1\le k\le p\right\} \max\left\{\frac{1}{k!}\colon 1\le k\le p\right\} \Rightarrow$$

$$\frac{|\partial_{\mathbf{b}}^{p}\partial_{\mathbf{b}}F(z^0)|}{L^{p}(z_0)} \le L(z^0)\cdot(p+1)!A^{2p}(p+1)^{2p} \max\left\{\frac{\left|\partial_{\mathbf{b}}^{k-1}\partial_{\mathbf{b}}F(z^0)\right|}{L^k(z_0)}\colon 0\le k-1\le p-1\right\} \Rightarrow$$

$$\frac{|\partial_{\mathbf{b}}^{p}\partial_{\mathbf{b}}F(z^0)|}{L^{p}(z_0)} \le (p+1)!A^{2p}(p+1)^{2p} \max\left\{\frac{\left|\partial_{\mathbf{b}}^{k-1}\partial_{\mathbf{b}}F(z^0)\right|}{L^{k-1}(z_0)}\colon 0\le k-1\le p-1\right\}.$$

Now we will apply analog of Hayman's Theorem proved above. Thus, for $\partial_{\mathbf{b}}F$ inequality (5) holds with $p-1$ instead p and with $C = (p+1)!A^{2p}(p+1)^{2p}$. In Theorem 4, the constant $A \ge \max\limits_{j>p} \frac{p+2}{2}(8e^{\pi^2})^p(1-\frac{1}{j})^j$ does not depend on z^0, because the parameter p is independent of z^0. Hence, the quantity $C = (p+1)!A^{2p}(p+1)^{2p}$ does not depend on z^0. Therefore, by Theorem 2 the function $\partial_{\mathbf{b}}F$ has bounded L-index in the direction $\mathbf{b}$.

Conversely, let $\partial_{\mathbf{b}}F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a function of bounded L-index in the direction $\mathbf{b}$. By Theorem 2 there exist $p \in \mathbb{Z}_+$ and $C \ge 1$ such that for every $z \in \mathbb{B}^n$ the following inequality holds

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \le C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)}\colon\ 1\le k\le p\right\}. \tag{10}$$

Let us consider a disc $K_0 = \left\{t\in\mathbb{C}\colon |t| \le \frac{1}{L(z^0)}\right\}$, $z^0 \in \mathbb{B}^n$.

One should observe that if $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $z^0 \in \mathbb{B}^n$ then for all $r > 0$ the inequality $|t| \le \frac{r}{L(z^0)}$ and definition of class $Q_{\mathbf{b}}(\mathbb{B}^n)$ yield

$$L(z^0)/\lambda_{\mathbf{b}}(r) \le L(z^0+t\mathbf{b}) \le \lambda_{\mathbf{b}}(r)L(z^0). \tag{11}$$

Now from (10) and (11) for $z = z^0 + t\mathbf{b}$, $t \in K$ one has

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{(p+1)!(C\lambda_{\mathbf{b}}(1)L(z^0))^{p+1}} \le \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^k}\times\right.$$
$$\left.\times\left(\frac{L(z^0+t\mathbf{b})}{C\lambda_{\mathbf{b}}(1)L(z^0)}\right)^{p+1-k}\colon\ 1\le k\le p\right\} \le \frac{C}{p+1}\times$$

$$\times \max_{1\le k\le p}\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^k}\frac{1}{C^{p+1-k}}\right\}\le$$
$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^k}\colon\ 1\le k\le p\right\}. \qquad (12)$$

To prove the proposition we need such a statement from [29, p.44, Theorem 2.7].

Theorem 5 ([29, p. 44, Theorem 2.7])**.** *Let $D_0=\{t\in\mathbb{C}\colon\ |t-t_0|<R\}$, $0<R<+\infty$, and $f(t)$ is an analytic function in D_0. If for all $t\in D_0$*

$$\left(\frac{R}{2}\right)^{p+1}\frac{|f^{(p+1)}(t)|}{(p+1)!}\le \max\left\{\left(\frac{R}{2}\right)^{k}\frac{|f^{(k)}(t)|}{k!}\colon\ 1\le k\le p\right\}, \qquad (13)$$

then $f(t)$ is p-valent in $\{t\in\mathbb{C}\colon\ |t-t_0|\le\frac{R}{25\sqrt{p+1}}\}$, i.e., $f(t)$ attains every value at most p times.

From inequality (12) it follows inequality (13) with $R=\frac{2}{C\lambda_{\mathbf{b}}(1)L(z^0)}$ and $t_0=0$. By Theorem 5, the function $F(z^0+t\mathbf{b})$ is p-valent in the disc $\{t\in\mathbb{C}\colon\ |t|\le\frac{\rho}{L(z^0)}\}$, $\rho=\frac{2}{25C\lambda_{\mathbf{b}}(1)\sqrt{p+1}}$.

Let t_j be an arbitrary point in K_0 and $K_j^*=\{t\in\mathbb{C}\colon |t-t_i|\le\frac{\rho}{L(z^0+t_j\mathbf{b})}\}$. Since by the definition of the class $Q_{\mathbf{b}}(\mathbb{B}^n)$ $\quad L(z^0+t_j\mathbf{b})\le\lambda_{\mathbf{b}}(1)L(z^0)$, one has $K_j=\{t\in\mathbb{C}\colon |t-t_j|\le\frac{\rho}{\lambda_{\mathbf{b}}(1)L(z^0)}\}\subset K_j^*$. We will repeat the similar considerations for the set $\{t\in\mathbb{C}\colon |t-t_j|\le\frac{1}{L(z^0+t_j\mathbf{b})}\}$. As a consequence, we deduce that $F(z^0+t\mathbf{b})$ is p-valent in K_j^*. But $K_j\subset K_j^*$, then $F(z^0+t\mathbf{b})$ is p-valent in K_j.

Finally, we note that every closed disc of radius R_* can be covered by a finite number m_* of closed discs of radius $\rho_*<R_*$ with the centers in the disc. Moreover, $m_*<B_*(R_*/\rho_*)^2$, where $B_*>0$ is an absolute constant. Hence, K_0 can be covered finite number m of discs K_j, where $m\le 625B^*(p+1)C^2(\lambda_{\mathbf{b}}(1))^2/4$. Since the function $F(z^0+t\mathbf{b})$ in K_j is p-valent, it is mp-valent in K_0.

In view of arbitrariness of $z^0\in\mathbb{B}^n$, the statement is proved. □

3. Existence theorem for functions of bounded L-index in direction. For the one-dimensional case, some time ago mathematicians were interested in the following two problems: the problem of the existence of an entire function of bounded l-index for a given l, and the problem of the existence of a function l for a given entire function f such that f is of bounded l-index [15, 16, 19, 31]. It is clear that similar problems can be considered for the case of entire functions of several complex variables [7, 8]. We note that the solution of the first problem for the one-dimensional case is given by a canonical product. The solution of the first problem in the multidimensional case also exists in the class of canonical products with "planar" zeros.

We consider the function $F(z^0+t\mathbf{b})$ where $z^0\in\mathbb{B}^n$ is fixed. If $F(z^0+t\mathbf{b})\not\equiv 0$, then we denote by $p_{\mathbf{b}}(z^0+a_k^0\mathbf{b})$ the multiplicity of the zero a_k^0 of the function $F(z^0+t\mathbf{b})$. If $F(z^0+t\mathbf{b})\equiv 0$ for some $z^0\in\mathbb{B}^n$, then we put $p_{\mathbf{b}}(z^0+t\mathbf{b})=-1$.

Theorem 6. *In order that for a function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ there exist a continuous function $L\colon\mathbb{B}^n\to\mathbb{R}_+$ such that $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ it is necessary and sufficient that $\exists p\in\mathbb{Z}_+$ $\forall z^0\in\mathbb{B}^n$ $\forall k$ $p_{\mathbf{b}}(z^0+a_k^0\mathbf{b})\le p$.*

Proof. Necessity. To simplify the notation we denote everywhere in the proof $p_k^0 \equiv p_{\mathbf{b}}(z^0 + a_0^k\mathbf{b})$. One can prove the necessity using the definition of bounded L-index in direction. Indeed, assume on the contrary that $\forall p \in \mathbb{Z}_+ \ \exists z^0 \ \exists k \ p_k^0 > p$. It means that $\partial_{\mathbf{b}}^{p_k^0} F(z^0 + a_k^0\mathbf{b}) \neq 0$ and $\partial_{\mathbf{b}}^j F(z^0 + a_k^0\mathbf{b}) = 0$ for all $j \in \{1, \dots, p_k^0 - 1\}$. Therefore, the L-index in the direction b at the point $z^0 + a_k^0\mathbf{b}$ is not less than $p_k^0 > p$

$$N_{\mathbf{b}}(F, L, z^0 + a_k^0\mathbf{b}) > p.$$

If $p \to +\infty$, then we obtain that $N_{\mathbf{b}}(F, L, z^0 + a_k^0\mathbf{b}) \to +\infty$. But this contradicts the boundedness of L-index in the direction $\mathbf{b}$ of the function F.

Sufficiency. If $F(z^0 + t\mathbf{b}) \equiv 0$ for some $z^0 \in \mathbb{B}^n$, then inequality (2) is obvious for any positive function $L\colon \mathbb{B}^n \to \mathbb{R}_+$.

Let p be the least integer such that $\forall z^0 \in \mathbb{B}^n \ F(z^0 + t\mathbf{b}) \not\equiv 0$, and $\forall k \ p_k(z^0) \leq p$. For any point $z \in \mathbb{B}^n$ we choose $z^0 \in \mathbb{B}^n$ and $t_0 \in \mathbb{C}$ so that $z = z^0 + t_0\mathbf{b}$ and the point z^0 lies on the hyperplane $\langle z, m\rangle = 1$, where $\langle \mathbf{b}, m\rangle = 1$ (actually it is sufficient that $\langle \mathbf{b}, m\rangle \neq 0$, i.e. the hyperplane is not parallel to $\mathbf{b}$). Therefore, $t_0 = \langle z, m\rangle - 1$, $z^0 = z - (\langle z, m\rangle - 1)\mathbf{b}$. We put $K_R = \{t \in \mathbb{C}\colon \max\{0, R-1\} \leq |t| \leq R+1\}$ for all $R \geq 0$ and

$$m_1(z^0, R) = \min_{a_k^0 \in K_R} \max_{0 \leq s \leq p} \left\{ \frac{|\partial_{\mathbf{b}}^s F(z^0 + a_k^0\mathbf{b})|}{s!} \right\}.$$

Since F is a slice holomorphic function in the unit ball, there exists $\varepsilon = \varepsilon(z^0, R) > 0$ such that

$$\frac{|\partial_{\mathbf{b}}^{s_0} F(z^0 + t\mathbf{b})|}{s_0!} \geq \frac{m_1(z^0, R)}{2}$$

for some $s_0 = s(a_k^0) \in \{0, \dots, p\}$ and for all $t \in K_R \cap \{t \in \mathbb{C}\colon |t - a_k^0| < \varepsilon(R, z^0)\}$ and for all k. We denote $G_\varepsilon^0 = \bigcup_{a_k^0 \in K_R} \{t \in \mathbb{C}\colon \ |t - a_k^0| < \varepsilon\}$, $m_2(z^0, R) = \min\{|F(z^0 + t\mathbf{b})|\colon |t| \leq R+1, t \notin G_\varepsilon^0\}$,

$$Q(R, z^0) = \min\left\{ \frac{m_1(R, z^0)}{2}, m_2(R, z^0), 1 \right\}.$$

We take $R = |t_0|$. Then at least one of the numbers $|F(z^0 + t_0\mathbf{b})|$, $|\partial_{\mathbf{b}} F(z^0 + t_0\mathbf{b})|, \dots,$ $\frac{|\partial_{\mathbf{b}}^p F(z^0 + t_0\mathbf{b})|}{p!}$ is not less than $Q(R, z^0)$ (respectively, for $t_0 \in G_\varepsilon^0$ $\frac{|\partial_{\mathbf{b}}^{s_0} F(z^0 + t_0)\mathbf{b})|}{p_k^0!}$ and for $t \notin G_\varepsilon$ $|F(z^0 + t_0\mathbf{b})|$).

Hence,

$$\max\left\{ \frac{|\partial_{\mathbf{b}}^j F(z^0 + t_0\mathbf{b})|}{j!} \colon 0 \leq j \leq p \right\} \geq Q(R, z^0). \tag{14}$$

On the other hand, for $|t_0| = R$ and $j \geq p+1$ Cauchy's inequality is valid

$$\frac{|\partial_{\mathbf{b}}^j F(z^0 + t_0\mathbf{b})|}{j!} \leq \frac{1}{2\pi} \int\limits_{|\tau - t_0| = 1} \frac{|F(z^0 + \tau\mathbf{b})|}{|\tau - t_0|^{j+1}} |d\tau| \leq$$
$$\leq \max\{|F(z^0 + \tau\mathbf{b})|\colon |\tau| \leq R+1\}. \tag{15}$$

We choose a positive continuous function $L\colon \mathbb{B}^n \to \mathbb{R}_+$ such that

$$L(z^0 + t_0\mathbf{b}) \geq \max\left\{ \frac{\max\{|F(z^0 + t\mathbf{b})|\colon |\tau| \leq R+1\}}{Q(R, z^0)}, 1 \right\}.$$

From (14) and (15) with $|t_0| = R$ and $j \geq p+1$ we obtain

$$\frac{\frac{|\partial_{\mathbf{b}}^j F(z^0+t_0\mathbf{b})|}{j!L^j(z^0+t_0\mathbf{b})}}{\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t_0\mathbf{b})|}{k!L^k(z^0+t_0\mathbf{b})}\colon 0\leq k\leq p\right\}} \leq \frac{L^{-j}(z^0+t\mathbf{b})}{Q(R,z^0)L^{-p}(z^0+t\mathbf{b})}\times$$
$$\times \max\{|F(z^0+t\mathbf{b})|\colon |\tau|\leq R+1\} \leq L^{p+1-j}(z^0+t\mathbf{b}) \leq 1.$$

Since $z = z^0 + t_0\mathbf{b}$, we have

$$\frac{\left|\partial_{\mathbf{b}}^j F(z)\right|}{j!L^j(z)} \leq \max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{k!L^k(z)}\colon 0\leq k\leq p\right\}.$$

In view of arbitrariness of z the function F has bounded L-index in the direction $\mathbf{b}$. □

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