S. Yu. Favorov

# LOCAL VERSIONS OF THE WIENER–LÉVY THEOREM

S. Yu. Favorov. *Local versions of the Wiener–Lévy theorem*, Mat. Stud. **57** (2022), 45–52.

Let $h$ be a real-analytic function on the neighborhood of some compact set $K$ on the plane, and let $f(y)$ be the Fourier–Stieltjes transform of a complex measure of a finite total variation without singular components on the Euclidean space. Then there exists another measure of finite total variation with the Fourier–Stieltjes transform $g(y)$ such that $g(y) = h(f(y))$ whenever the value $f(y)$ belongs to $K$.

**1. Introduction.** It is well known that for each absolutely convergent Fourier series $F(t)$ such that $F(t) \neq 0$ for all $t$ the function $1/F(t)$ also has an absolutely convergent Fourier-series expansion (the Wiener Theorem). Its natural generalization is known as the Wiener–Lévy Theorem (see, for example, [11], Ch.VI):

**Theorem 1.** *Let*

$$F(t) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi int}$$

*be an absolutely convergent Fourier series, and $h(z)$ be a holomorphic function on a neighborhood of the closure of the range of $F$. Then the function $h(F(t))$ admits an absolutely convergent Fourier series expansion as well.*

Clearly, for $h(z) = 1/z$ we get the Wiener Theorem.

The next variant of this theorem for functions on $\mathbb{R}$ is also known as the Wiener–Lévy Theorem (see for example [1], Ch.III or [8], Ch.I)

**Theorem 2.** *Let $\hat f(y)$ be the Fourier transform of some function $f \in L^1(\mathbb{R})$, and $h(z)$ be a holomorphic function on a neighborhood of the closure of the range of $\hat f$ such that $h(0) = 0$. Then there is a function $g \in L^1(\mathbb{R})$ such that its Fourier transform $\hat g(y)$ coincides with $h(\hat f(y))$.*

This theorem admits a generalization to functions from $L^1(G)$, where $G$ is an arbitrary locally compact abelian group, and one can replace a holomorphic function $h$ by any real-analytic function. Next if we replace here the absolutely continuous measure $f(x)dx$ by a pure point measure $\sum_n a_n\delta_{\lambda_n}$ ($\delta_s$, as usual, means the unit mass at the point $s$) with $\sum_n |a_n| < \infty$, we obtain the Wiener–Lévy Theorem for Dirichlet series. On the other hand, for each locally compact abelian non discrete group $G$ there exists a measure $\mu$ with a finite total variation such that values of its Fourier transform $\hat\mu$ is separated from zero on the dual group $\hat G$ (hence

2010 *Mathematics Subject Classification:* 42B05, 42B10.
*Keywords:* Wiener–Lévy theorem; Fourier transform; absolute convergent Dirichlet series; pure point measure; real-analytic function.
doi:10.30970/ms.57.1.45-52

the function $1/z$ is holomorphic on the closure of the set $\hat{\mu}(\hat{G})$), but there is no measure $\nu$ on $G$ such that its Fourier transform $\hat{\nu} = 1/\hat{\mu}$. Also, the requirement of the real-analyticity is necessary for most groups $G$ (in particular for $G = \mathbb{R}^d$) to fulfill the Wiener–Lévy Theorem (see [9], Ch.5, 6).

The local form of the Wiener–Lévy Theorem is of great interest for our study (see [8], Ch.6):

**Theorem 3.** *Let $G$ be a locally compact abelian group, let $S$ be a compact subset of the dual group $\hat{G}$, let $f \in L^1(G)$, and let $h(z)$ be a holomorphic function on a neighborhood of the closure of the set $\hat{f}(S)$. Then there is a function $g \in L^1(G)$ such that its Fourier transform $\hat{g}(y)$ coincides with $h(\hat{f}(y))$ for all $y \in S$.*

In our article, we also consider local versions of the Wiener-Lévy theorem, which are in a certain sense stronger than Theorem 3.

**2. Notations and preliminaries**. Before formulating our results, it is necessary to recall some definitions.

Denote by $M(G)$ the set of complex measures on the locally compact group $G$ with a finite total variation $\|\mu\|$, by $M_d(G)$ the set of pure point measures from $M(G)$, and by $M_{ad}$ the set of measures from $M(G)$ containing only pure point and absolutely continuous (with respect to the Haar measure) components, i.e., without singular components. The Fourier transform of $\mu \in M(G)$ is defined by the equality

$$\hat{\mu}(y) = \int_G (-x, y)\mu(dx), \qquad y \in \hat{G},$$

where $\hat{G}$ is the group of characters on $G$, and $(x, y)$ means the action of the character $y$ on $x \in G$. In particular, in the case $G = \mathbb{R}^d$ we have

$$\hat{\mu}(y) = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,y\rangle}\mu(dx), \qquad y \in \mathbb{R}^d,$$

where $\langle x, y\rangle$ means the scalar product of $x$ and $y$. If the measure $\mu$ is absolute continuous and $\mu(dx) = f(x)dx$, we will also write

$$\hat{f}(y) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,y\rangle}dx.$$

Furthermore, a complex-valued function $h$, defined on an open set $V \subset \mathbb{C}$ is said to be real-analytic on $V$ if to every point $z \in V$ there corresponds the expansion

$$h(\xi + i\eta) = \sum_{k,n=0}^{\infty} c_{k,n}(\xi - \mathbb{R}e z)^k(\eta - \operatorname{Im} z)^n, \quad \xi, \eta \in \mathbb{R}, \quad c_{k,n} \in \mathbb{C}, \tag{1}$$

which converges in some disc

$$D(z, r_z) = \{(\xi, \eta) \in \mathbb{R}^2 : \ |\xi - \operatorname{Re} z|^2 + |\eta - \operatorname{Im} z|^2 < r_z^2\}.$$

Note that series (1) converges also in the ball

$$B(z, r_z) = \{(\xi, \eta) \in \mathbb{C}^2 : \ |\xi - \operatorname{Re} z|^2 + |\eta - \operatorname{Im} z|^2 < r_z^2\},$$

and for any intersecting balls $B(z_1, r_1)$ and $B(z_2, r_2)$ the set

$$B(z_1, r_1) \cap B(z_2, r_2) \cap \mathbb{R}^2 = D(z_1, r_1) \cap D(z_2, r_2)$$

is the set of uniqueness for analytic functions of two variables. Therefore, if the function $h(z)$ is real-analytic in a neighborhood of some compact set $K \subset \mathbb{C}$, then it has a continuation to the neighborhood $\bigcup_{z\in K} B(z, r_z) \subset \mathbb{C}^2$ of $K$ as an analytic function of two complex variables $\xi$, $\eta$.

**3. Main results.**

**Theorem 4.** *Let $\mu$ be a measure from $M_{ad}(\mathbb{R}^d)$, let $h(z)$ be a real-analytic function on a neighborhood of some compact set $K \subset \mathbb{C}$. Then there is a measure $\nu \in M_{ad}(\mathbb{R}^d)$ such that for every $y \in \mathbb{R}^d$ satisfying $\hat{\mu}(y) \in K$ we have $\hat{\nu}(y) = h(\hat{\mu}(y))$.*

In particular, if $h(z) = 1/z$ or $h(z) = 1/|z|^\alpha$ for $|z| \geq \varepsilon$ and $h(z) = 0$ for $|z| \leq \varepsilon/2$, we obtain the following result:

**Corollary 1.** *For any $\mu \in M_{ad}(\mathbb{R}^d)$ and $\varepsilon > 0$, $\alpha > 0$ there are measures $\nu_\varepsilon, \nu_{\alpha,\varepsilon} \in M_{ad}(\mathbb{R}^d)$ such that in the case $|\hat{\mu}(y)| \geq \varepsilon$ we have $\hat{\nu}_\varepsilon(y) = 1/\hat{\mu}(y)$, $\hat{\nu}_{\alpha,\varepsilon}(y) = 1/|\hat{\mu}(y)|^\alpha$, and in the case $|\hat{\mu}(y)| \leq \varepsilon/2$ we have $\hat{\nu}_\varepsilon(y) = \hat{\nu}_{\alpha,\varepsilon}(y) = 0$.*

Note that the pre-image of $K$ with respect to the mapping $h$ may not be compact set in $\mathbb{R}^d$, so our result does not follow from Theorem 3.

The reasoning in the proof of Theorem 4 also provides the following statement:

**Theorem 5.** *Let $G$ be a locally compact abelian group, $\mu$ be a measure from $M_d(G)$, $K$ be an arbitrary compact set on the complex plane, and $h(z)$ be a real-analytic function on a neighborhood of $K$. Then there is a measure $\nu \in M_d(G)$ such that, for every $y \in \hat{G}$ for which $\hat{\mu}(y) \in K$, we have $\hat{\nu}(y) = h(\hat{\mu}(y))$. The support of the measure $\nu$ lies in* $\mathrm{Lin}_{\mathbb{Z}} \operatorname{supp} \mu$.

Here $\operatorname{supp} \mu$ stands for the set $\{x \in G\colon \mu(\{x\}) \neq 0\}$ if $\mu \in M_d(G)$.

For the case $G = \mathbb{R}^d$ and holomorphic $h$ on a neighborhood $V \subset \mathbb{C}$ of $K$ Theorem 5 was proved in [2]. In [2] and [4] we applied the results of this type to study Fourier quasicrystals (about this theory see, for example, [5], [6], [7]). For another application of Theorem 5 to Kahane's property of discrete sets see [3].

**4. Auxiliary lemmas and their proofs.** We will use Schwartz' space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing $C^\infty$-functions on $\mathbb{R}^d$ with the topology defined by a countable number of norms

$$N_n(\varphi) = \sup_{x\in\mathbb{R}^d} \left\{ (1+|x|)^n \max_{k_1+\dots+k_d\leq n} \left| \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} \varphi(x) \right| \right\}, \ n = 0, 1, 2, \dots$$

The Fourier transform is a continuous linear one-to-one mapping of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$, and the set of $C^\infty$-functions with compact support is dense in $\mathcal{S}(\mathbb{R}^d)$ (see [10]).

**Lemma 1.** *For every $f \in L^1(\mathbb{R}^d)$ and every $\varepsilon > 0$ there is $v \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - v\|_{L^1} < \varepsilon$ and $\hat{v}$ has a compact support.*

*Proof.* Take $f_1 \in L^1(\mathbb{R}^d)$ such that $\|f - f_1\|_{L^1} < \varepsilon/3$ and $f_1$ has a compact support. The convolution $f_2 = f_1 \star \varphi$ with a suitable $C^\infty$-function $\varphi(x) \ge 0$ with support in a small ball such that $\int_{\mathbb{R}^d} \varphi(x)dx = 1$ has the properties $\|f_2 - f_1\|_{L^1} < \varepsilon/3$ and $f_2 \in \mathcal{S}(\mathbb{R}^d)$. Therefore, $\hat{f}_2 \in \mathcal{S}(R^d)$ as well, and there is a sequence of $C^\infty$-functions with compact supports that converges to $\hat{f}_2$ in the space $\mathcal{S}(R^d)$. Let $\{v_n\}$ be the images the functions from this sequence under the inverse Fourier transform. Clearly, $v_n \to f_2$ in the space $\mathcal{S}(R^d)$, therefore,

$$\|f_2 - v_n\|_{L^1} \le \sup_{\mathbb{R}^d}(1 + |x|)^{d+1}|f_2(x) - v_n(x)| \int_{\mathbb{R}^d}(1 + |x|)^{-d-1}dx \le \\ \le C(d)\, N_{d+1}(f_2 - v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\|f_2 - v_n\|_{L^1} < \varepsilon/3$ for a suitable $v_n$, and $\|f - v_n\|_{L^1} < \varepsilon$. □

**Lemma 2.** *There is a constant $C = C(r, d)$ such that for every $v \in \mathcal{S}(\mathbb{R}^d)$ for which $\operatorname{supp} \hat{v} \subset B(0, r)$, we get*

$$\|v\|_{L^1} \le C(r, d) \left( \|\hat{v}\|_\infty + \sum_{j=1}^d \|(\partial^l/\partial y_j^l)\hat{v}\|_\infty \right),$$

*with $l = d + 1$ for odd $d$ and $l = d + 2$ for even $d$.*

*Proof.* We have

$$\hat{v}(y) = \int_{\mathbb{R}^d} v(x)e^{-2\pi i\langle x,y\rangle}dx, \quad (\partial^l/\partial y_j^l)\hat{v}(y) = (-2\pi i)^l \int_{\mathbb{R}^d} x_j^l v(x)e^{-2\pi i\langle x,y\rangle}dx, \ j = 1, \dots, d.$$

Hence, $v(x)(1 + \sum_{j=1}^d |x_j|^l)$ is the inverse Fourier transform of the function

$$\hat{v}(y) + \left(\frac{i}{2\pi}\right)^l \sum_{j=1}^d (\partial^l/\partial y_j^l)\hat{v}(y).$$

Since $\operatorname{supp} \hat{v} \subset B(0, r)$, we get

$$\|v\|_{L^1} \le C \left( \|\hat{v}\|_\infty + \sum_{j=1}^d \|(\partial^l/\partial y_j^l)\hat{v}\|_\infty \right) \int_{\mathbb{R}^d} \frac{dx}{1 + |x_1|^l + |x_2|^l + \cdots + |x_d|^l}$$

with a constant $C$ depending on $d$ and $r$. □

**Lemma 3.** *Let $T(\Theta, \tau)$ be $C^\infty$-function in variables $\Theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ and $\tau \in [0, 1]^2$, and let $T(\Theta, \tau)$ be periodic with periods 1 in each coordinate $\theta_1, \dots, \theta_N$. Then its Fourier series*

$$T(\Theta, \tau) = \sum_{k \in \mathbb{Z}^N} b_k(\tau)e^{2\pi i\langle k,\Theta\rangle}, \qquad k = (k_1, \dots, k_N),$$

*is absolutely convergent and $\sum_k |b_k(\tau)| < C$ uniformly in $\tau$.*

*Proof.* We have

$$b_k(\tau) = \int_{[0,1]^N} T(\Theta, \tau)e^{-2\pi i\langle k,\Theta\rangle}d\Theta.$$

Integrating this equality twice in parts over each variable $\theta_j$ such that $j \in J(k) = \{j\colon\ k_j \neq 0\}$, we get

$$|b_k(\tau)| \le \sup_{(\Theta,\tau)\in[0,1]^{N+2}} \left| \left( \prod_{j\in J(k)} (-4\pi^2 k_j^2)^{-1} \partial^2/\partial\theta_j^2 \right) T(\Theta,\tau) \right|.$$

Taking into account that every derivative of $T(\Theta,\tau)$ is uniformly bounded in $\tau \in [0,1]^2$, we get the estimate

$$|b_k(\tau)| \le C \min\{1, k_1^{-2}\} \cdots \min\{1, k_N^{-2}\},$$

where $C$ depends on neither $\tau$ nor $k$. This estimate implies the assertion of the lemma. □

## 5. Proofs of the main theorems.

*Proof of Theorem 4.* Let $U$ be an open set in $\mathbb{C}^2$ such that $h$ has a holomorphic continuation to $U$ as a function of two complex variables. Set $\varepsilon < (1/13)\operatorname{dist}(K, \partial U)$. Let $\varphi(|z|)$ be $C^\infty$-differentiable nonnegative function with support in $B(0,\varepsilon) \subset \mathbb{C}^2$ such that $\int_{\mathbb{C}^2} \varphi(|\zeta|) m(d\zeta) = 1$ (here $m(d\zeta)$ means the Lebesgue measure in $\mathbb{C}^2$). Consider $C^\infty$-function

$$H(z) = \int_{\operatorname{dist}(\zeta,K)<9\varepsilon} h(\zeta)\varphi(|z-\zeta|) m(d\zeta).$$

If $\operatorname{dist}(z,K) < 8\varepsilon$, we get

$$H(z) = \int_{|\zeta|\le\varepsilon} h(z-\zeta)\varphi(|\zeta|) m(d\zeta).$$

Since an average over any sphere of a holomorphic function of many variables equals the meaning of the function at the center of the sphere, we obtain that $H(z) = h(z)$ on the set $\{z\colon\ \operatorname{dist}(z,K) < 7\varepsilon\}$ and $H(z) = 0$ on the set $\{z\colon\ \operatorname{dist}(z,K) > 10\varepsilon\}$.

Let

$$\mu = f dx + \sum_n a_n \delta_{\gamma_n}, \quad f \in L^1(\mathbb{R}^d), \quad \sum_n |a_n| < \infty.$$

Using Lemma 1, take a function $v \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - v\|_{L^1} < \varepsilon$ and $\operatorname{supp} \hat{v}$ is a compact set. Pick $N < \infty$ such that $\sum_{n>N} |a_n| < \varepsilon$, and define the measure $s = \sum_{n=1}^N a_n \delta_{\gamma_n}$. Note that

$$\hat{s}(y) = \sum_{n\le N} a_n e^{-2\pi i \langle \gamma_n, y\rangle}.$$

Since $\|\mu - v dx - s\|_{L^1} < 2\varepsilon$, we see that $\|\hat{\mu}(y) - \hat{v}(y) - \hat{s}(y)\|_{L^\infty} < 2\varepsilon$. Put

$$\alpha(y) = \operatorname{Re}(\hat{v}(y) + \hat{s}(y)),\ \ \beta(y) = \operatorname{Im}(\hat{v}(y) + \hat{s}(y)).$$

Consider the function

$$F(y) = \frac{1}{(2\pi i)^2} \int_{|\alpha(y)-\zeta_1|=3\varepsilon} \int_{|\beta(y)-\zeta_2|=3\varepsilon} \frac{H(\zeta_1 + i\zeta_2) d\zeta_1 d\zeta_2}{(\zeta_1 - \operatorname{Re}\hat{\mu}(y))(\zeta_2 - \operatorname{Im}\hat{\mu}(y))}.$$

If $\hat{\mu}(y) \in K$, then $\operatorname{dist}(\alpha(y) + i\beta(y), K) < 2\varepsilon$. Therefore,

$$\mathcal{E} = \{(\zeta_1,\zeta_2)\colon\ |\zeta_1 - \alpha(y)| \le 3\varepsilon,\ |\zeta_2 - \beta(y)| \le 3\varepsilon\} \subset \{z\colon\ \operatorname{dist}(z,K) < 7\varepsilon\},$$

and $H(z)=h(z)$ in a neighborhood of $\mathcal{E}$. Using the Cauchy integral formula for the polydisk $\mathcal{E}$, we obtain

$$F(y)=h(\operatorname{Re}\hat\mu(y)+i\operatorname{Im}\hat\mu(y))=h(\hat\mu(y)).$$

Furthermore, we have for all $y\in\mathbb{R}^d$

$$F(y)=\int_0^1\int_0^1\frac{H(\alpha(y)+3\varepsilon e^{3\pi i\tau_1}+i\beta(y)+i3\varepsilon e^{2\pi i\tau_2})9\varepsilon^2e^{2\pi i(\tau_1+\tau_2)}}{(\alpha(y)+3\varepsilon e^{3\pi i\tau_1}-\operatorname{Re}\hat\mu(y))(\beta(y)+3\varepsilon e^{3\pi i\tau_2}-\operatorname{Im}\hat\mu(y))}d\tau_1\,d\tau_2.$$

Since

$$\left|\frac{\operatorname{Re}\hat\mu(y)-\alpha(y)}{3\varepsilon e^{2\pi i\tau_1}}\right|<2/3,\qquad\left|\frac{\operatorname{Im}\hat\mu(y)-\beta(y)}{3\varepsilon e^{2\pi i\tau_2}}\right|<2/3,$$

we get

$$\frac{1}{\left(1-\frac{\operatorname{Re}\hat\mu(y)-\alpha(y)}{3\varepsilon e^{2\pi i\tau_1}}\right)\left(1-\frac{\operatorname{Im}\hat\mu(y)-\beta(y)}{3\varepsilon e^{2\pi i\tau_2}}\right)}=\sum_{p,q=0}^\infty\left(\frac{\operatorname{Re}\hat\mu(y)-\alpha(y)}{3\varepsilon e^{2\pi i\tau_1}}\right)^p\left(\frac{\operatorname{Im}\hat\mu(y)-\beta(y)}{3\varepsilon e^{2\pi i\tau_2}}\right)^q,$$

therefore, the function $F(y)$ is equal to

$$\sum_{p,q=0}^\infty\left(\frac{\operatorname{Re}\hat\mu(y)-\alpha(y)}{3\varepsilon}\right)^p\left(\frac{\operatorname{Im}\hat\mu(y)-\beta(y)}{3\varepsilon}\right)^q\int_0^1\int_0^1\frac{A(y,\tau_1,\tau_2)+D(y,\tau_1,\tau_2)}{e^{2\pi i(p\tau_1+q\tau_2)}}d\tau_1\,d\tau_2,$$

with

$$A(y,\tau_1,\tau_2)=H(\hat v(y)+\hat s(y)+3\varepsilon e^{2\pi i\tau_1}+i3\varepsilon e^{2\pi i\tau_2})-H(\hat s(y)+3\varepsilon e^{2\pi i\tau_1}+i3\varepsilon e^{2\pi i\tau_2}),$$
$$D(y,\tau_1,\tau_2)=H(\hat s(y)+3\varepsilon e^{2\pi i\tau_1}+i3\varepsilon e^{2\pi i\tau_2}).$$

Define two measures on $\mathbb{R}^d$

$$\lambda_R(x)=1/2\left(\mu(x)-v(x)dx-s(x)+\overline{\mu(-x)-v(-x)dx-s(-x)}\right),$$
$$\lambda_I(x)=(1/2i)\left(\mu(x)-v(x)dx-s(x)-\overline{\mu(-x)-v(-x)dx-s(-x)}\right).$$

It is easily seen that $\|\lambda_R\|<2\varepsilon$, $\|\lambda_I\|<2\varepsilon$, and

$$\hat\lambda_R(y)=\operatorname{Re}\hat\mu(y)-\alpha(y),\ \hat\lambda_I(y)=\operatorname{Im}\hat\mu(y)-\beta(y).$$

Since the Fourier transform of convolution of measures equals the product of the Fourier transform of the measures, we get

$$[(\lambda_R/3\varepsilon)^{*p}*(\lambda_I/3\varepsilon)^{*q}]\widehat{\ }=\left(\frac{\operatorname{Re}\hat\mu(y)-\alpha(y)}{3\varepsilon}\right)^p\left(\frac{\operatorname{Im}\hat\mu(y)-\beta(y)}{3\varepsilon}\right)^q.\tag{2}$$

Also, the variation of convolution of measures does not exceed the product of variations of the measures, hence

$$\|(\lambda_R/3\varepsilon)^{*p}*(\lambda_I/3\varepsilon)^{*q}\|<(2/3)^{p+q}.\tag{3}$$

On the other hand, since $\operatorname{supp}A(y,\tau_1,\tau_2)\subset\operatorname{supp}\hat v$ and $A(y,\tau_1,\tau_2)$ is $C^\infty$ function, we see that $A(y,\tau_1,\tau_2)\in\mathcal{S}(\mathbb{R}^d)$. Therefore there exists $u_{\tau_1,\tau_2}(x)\in\mathcal{S}(\mathbb{R}^d)$ such that $\hat u_{\tau_1,\tau_2}(y)=A(y,\tau_1,\tau_2)$ for every fixed $\tau_1,\tau_2$. Then the function $A(y,\tau_1,\tau_2)$ and all its derivatives of order at most $d+2$ are bounded uniformly in $\tau_1,\ \tau_2 \in [0,1]^2$. By Lemma 2, $\|u_{\tau_1,\tau_2}\|_{L^1}$ is uniformly bounded too. Set

$$\kappa_{p,q}(x) = \int_0^1 \int_0^1 \frac{u_{\tau_1,\tau_2}(x)}{e^{2\pi i(p\tau_1+q\tau_2)}} d\tau_1\, d\tau_2 \in L^1(\mathbb{R}^d).$$

By Fubini's Theorem,

$$\sup_{p,q} \|\kappa_{p,q}\|_{L^1} < \infty, \tag{4}$$

and

$$\hat{\kappa}_{p,q}(y) = \int_0^1 \int_0^1 \frac{A(y,\tau_1,\tau_2)}{e^{2\pi i(p\tau_1+q\tau_2)}} d\tau_1\, d\tau_2. \tag{5}$$

Next, apply Lemma 3 to the function $H\left(\sum_{n\le N} a_n e^{2\pi i\theta_n} + 3\varepsilon e^{2\pi i\tau_1} + i3\varepsilon e^{2\pi i\tau_2}\right)$. We get

$$H\left(\sum_{n\le N} a_n e^{2\pi i\theta_n} + 3\varepsilon e^{2\pi i\tau_1} + i3\varepsilon e^{2\pi i\tau_2}\right) = \sum_{k\in\mathbb{Z}^N} b_k(\tau_1,\tau_2) e^{2\pi i\langle k,\Theta\rangle}, \tag{6}$$

with the condition

$$\sup_{\tau_1,\tau_2} \sum_k |b_k(\tau_1,\tau_2)| < \infty.$$

If we replace in (6) $\theta_n$ by $-\langle\gamma_n, y\rangle$ for each $n$, we get after reduction of similar terms and reindexing that

$$\sum_{k\in\mathbb{Z}^N} b_k(\tau_1,\tau_2) e^{2\pi i\langle k,\Theta\rangle} = \sum_{j\in\mathbb{N}} \tilde{b}_j(\tau_1,\tau_2) e^{2\pi i\langle\rho_j, y\rangle}, \tag{7}$$

where $\rho_j$ belong to the span over $\mathbb{Z}$ the set $\{\gamma_n\}_{n=1}^\infty$. Note that

$$\sup_{\tau_1,\tau_2} \sum_{j\in\mathbb{N}} |\tilde{b}_j(\tau_1,\tau_2)| < \infty. \tag{8}$$

Function (7) is the Fourier transform of the measure $\sum_{j\in\mathbb{N}} \tilde{b}_j(\tau_1,\tau_2)\delta_{-\rho_j}$. Set

$$\nu_{p,q} = \sum_{j\in\mathbb{N}} c_j(p,q)\delta_{-\rho_j} \quad \text{with} \quad c_j(p,q) = \int_0^1 \int_0^1 \frac{\tilde{b}_j(\tau_1,\tau_2)}{e^{2\pi i(p\tau_1+q\tau_2)}} d\tau_1\, d\tau_2.$$

It follows from (6), (8) and Fubini's Theorem that

$$\sup_{p,q} \|\nu_{p,q}\| < \infty, \tag{9}$$

and

$$\hat{\nu}_{p,q}(y) = \int_0^1 \int_0^1 \frac{D(y,\tau_1,\tau_2)}{e^{2\pi i(p\tau_1+q\tau_2)}} d\tau_1\, d\tau_2. \tag{10}$$

Finally put

$$\nu = \sum_{p,q=0}^{\infty} (\lambda_R/3\varepsilon)^{*p} * (\lambda_I/3\varepsilon)^{*q} * (\kappa_{p,q}dx + \nu_{p,q}). \tag{11}$$

We have

$$\|\nu\| \le \sum_{p,q=0}^{\infty} \|\lambda_R/3\varepsilon\|^p \|\lambda_I/3\varepsilon\|^q (\|\kappa_{p,q}\|_{L^1} + \|\nu_{p,q}\|).$$

It follows from (3), (4), and (9), that $\nu$ has a finite total variation, and, by (2), (5), and (10), that $\hat{\nu}(y) = F(y)$. □

*Proof of Theorem 5.* Let $\mu = \sum_n a_n \delta_{\gamma_n}$ with $\gamma_n \in G$ and $\sum_n |a_n| < \infty$. Then

$$\hat{\mu}(y) = \sum_n a_n(-\gamma_n, y), \qquad y \in \hat{G}.$$

Replacing in (6) $e^{2\pi i\theta_n}$ by $(-\gamma_n, y)$, we get the relation

$$\sum_{k\in\mathbb{Z}^N} b_k(\tau_1,\tau_2)e^{2\pi i\langle k,\Theta\rangle} = \sum_{j\in\mathbb{N}} \tilde{b}_j(\tau_1,\tau_2)(\rho_j, y)$$

instead of (7), where as before $\rho_j$ belong to the span over $\mathbb{Z}$ the set $\{\gamma_n\}_{n=1}^\infty$. Also, we do not use Lemmas 1 and 2, but put $A(y,\tau_1,\tau_2) \equiv 0$, $u(x,\tau_1,\tau_2) \equiv 0$, and $k_{p,q}(x) \equiv 0\ \forall p,q$. Then, repeating the reasoning in the proof of Theorem 4, we obtain the assertion of Theorem 5. □

**Remark 1.** If the function $h$ is holomorphic in a neighborhood of the compact set $K$, then there is an alternative proof of Theorem 5. Indeed, let $\Gamma$ denote the group $G$ with respect to the discrete topology. Clearly, every pure point measure $\mu \in M(G)$ is the function $f \in L^1(\Gamma)$ at the same time. Therefore $\hat{\mu}$ extends to the continuous function $\hat{f}$ on the compact group $\hat{\Gamma}$. Then $\hat{f}^{-1}(K)$ is a compact subset of $\hat{\Gamma}$, and we may apply Theorem 3. In order to obtain a statement about the support of the measure $\nu$, one has to replace the group $\Gamma$ by the group $\mathrm{Lin}_{\mathbb{Z}}\,\mathrm{supp}\,\mu$.

I am very grateful to Professor Hans Georg Feichtinger for pointing me Theorem 3, and for his attention to my work.

## REFERENCES


1. N.I. Akhiezer, Theory of Approximation, F. Ungar Pub., 1956.
2. S.Yu. Favorov, *Large Fourier quasicryals and Wiener's theorem*, Journal of Fourier Analysis and Applications, **25** (2019), №2, 377–392.
3. S.Yu. Favorov, *Local Wiener's theorem and coherent sets of frequencies*, Analysis Math., **46**, (2020), №4, 737–746.
4. S.Yu. Favorov, *Temperate distributions with locally finite support and spectrum on Euclidean spaces*, arXiv:2106.07073, to appear in: Israel Journal of Mathematics.
5. M. Baake, R. Moody, Directions in mathematical quasicrystals, eds. CRM Monograph series, 2000, V.13, AMS, Providence RI, 379 p.
6. J.C. Lagarias, *Geometric models for quasicrystals I. Delone set of finite type*, Discr. and Comp. Geometry, **21** (1999) 161–191.
7. Y. Meyer, *Guinand's measure are almost periodic distributions*, Bulletin of the Hellenic Mathematical Society, **61**, (2017) 11–20.
8. H. Reiter, J.D. Stegeman, Classical harmonic analysis and locally compact groups, Oxford University Press, Oxford, 2000.
9. W. Rudin, Fourier analysis on groups: Interscience publications, a division of John Wiley and Sons, New York, 1962.
10. W. Rudin, Functional analysis, McGraw-Hill Book Company, New York, 1973.
11. A. Zygmund, Trigonometric series, Cambridge Univesity Press, Cambridge, 2002.


Karazin's Kharkiv National University
Kharkiv, Ukraine
sfavorov@gmail.com

*Received 15.12.2021*
*Revised 23.03.2022*